Journal of Fractional Calculus and Applications
Vol. 13(2) July 2022, pp. 172-189.
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

LIE SYMMETRY ANALYSIS OF FORTH-ORDER TIME FRACTIONAL KDV EQUATION

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Abstract. In present paper, Lie group analysis is applied to consider vector field and symmetry reduction on forth-order time fractional KdV equation, power series solution and the convergence are investigated. Stability analysis of trivial solution to the reduction differential equation is showed by constructing appropriate Lyapunov function. Conservation laws of the equation are well constructed with a detailed derivation making use of Noether's operator.

1. Introduction

Conservative or non-conservative form of differential equation most applies to fluid dynamics (including continuity equation, momentum equation, energy equation), from the view of micro body, the conservative control equation is equivalent to non-conservative equation, satisfactory results can be obtained by using both conservative and non-conservative shock assembly method. Theoretical fluid mechanics and mathematical derivations pay little attention to this, the people who care about is computational fluid dynamics, in the process of research the people found that there are great differences between conservative and non-conservative fluid control equation in calculating some special flow fields, since it is proved by practice that the equation in conservative form is more convenient and stable than that of non-conservative form in numerical calculation; the control equation in conservation form can be expressed by same general equation, this will help to the application of simplified and program structure of the organization. More precisely for calculating shock wave, experience shows that the control equation in conservation form should be used, the calculated flow fields are usually smooth and stable. If the non-conservation form is used, the obtained results of the calculated flow fields could usually show unsatisfactory spatial oscillation in upstream and downstream of the shock wave, the shock wave may appear in improper position, even the calculation results may become unstable [26]. In view of these, we tend to study some problems for the conservative form of differential equation.

2010 *Mathematics Subject Classification.* 26A33, 35K55, 35L65.
Key words and phrases. Time fractional KdV equation, symmetry reduction, exact solution, Lyapunov function, conservation laws.
Submitted Jan. 25, 2022. Revised March 10, 2022.

Time fractional KdV equation in conservative form has been applied to describe a wide range of physics phenomena of the evolution and interaction to nonlinear wave, it possesses infinitely symmetry and bi-Hamiltonian formulation. Since KdV equation in sense of the fractional derivative holds advantage not only on the time instant but on the previous history, and so it has obtained popularity and importance as generalizations of integer order calculus, which could be successfully modeled in aerodynamics, continuum mechanics, solitons and turbulence et al [2, 17, 18, 23]. Many known nonlinear analysis methods have been successfully used to study the number of properties for time fractional KdV equation, for instance, the symmetry group of scaling transformations are determined, containing among particular cases the diffusion wave equation, for its group invariant solution, a fractional ordinary differential equation (FODE) with the new independent variable is derived [4]. Sahadevan and Bakkyaraj [25] derived Lie point symmetries to time fractional generalized Burgers and KdV equations, each of equation has been transformed into nonlinear FODE with new independent variable. Wang et al [28] studied the invariance properties of the time fractional generalized KdV equation using Lie group analysis. Djordjevic and Atanackovic [7] analyzed self-similar solutions to a nonlinear fractional diffusion equation and fractional Burgers/KdV equation by using Lie-group scaling transformation, both equations are reduced to nonlinear FODE, and solved the resulting ordinary differential equations numerically. Liu [16] made complete group classifications on the fractional KdV type of equation, investigated the symmetry reduction and exact solutions, and so forth. Taking the advantage of modified Riemann-Liouville calculus approach that the initial condition for fractional differential equation takes on the traditional form as for integer order differential equation, fractional partial differential equation (FPDE) could extent Lie symmetry analysis to derive it infinitesimals. Stability and boundedness of solution are also very important in the theory and applications of differential equation, such as the well-posedness of the problem for determining solution, even the regularity of a weak solution. Although the stability of solution for nonlinear differential equation of higher order with multiple deviating arguments have not been widely discussed, the basic reason is the difficulty for constructing the appropriate Liapunov function, see [5, 6, 10, 14, 27] and the references therein. From the above studying results, we know that Lie symmetry analysis [19, 29, 3, 11] can provide an effective procedure for explicit solution of a wide and general class of differential equation representing real physical problems, and it helps to study the group theoretical property. Accompanying, conservation laws of the equation are important for investigating integrability and linearization mappings, establishing group invariant solution of existence and uniqueness. The purpose of this article is further present Lie symmetry analysis to investigate the existence of nontrivial solution, stability of trivial solution and conservative laws of the time fractional KdV equation.

Depending on the complexity and application of fractional calculus, many scholars have pushed forward and developed fractional calculus, various definitions of fractional differentiation [12, 21]. Now we state the modified Riemann-Liouville fractional calculus of order α as follows.

Definition 1. *A function $f(x,t)$, $(x,t) \in \Omega \times T \subset \mathbb{R} \times \mathbb{R}^+$, is said to be in the space C_γ, $\gamma \in \mathbb{R}$ with respect to t if there exists a real number p $(> \gamma)$, such that $f(x,t) = t^p f_1(x,t)$, where $f_1 \in C(\Omega \times T)$. Obviously, $C_\gamma \subset C_\delta$ if $\delta \leq \gamma$.*

Definition 2. *Assume that* $f \in C_\gamma(\Omega \times T)$ $(\gamma \geq -1)$, *we use the equality for the integral with respect to* $(\mathrm{d}t)^\alpha$

$$I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t f(x,\tau)(\mathrm{d}\tau)^\alpha, \quad 0 < \alpha < 1.$$

Definition 3. *Jumarie's* α*-order derivative for* $f \in C_\gamma(\Omega \times T)$ $(\gamma \geq -1)$ *is defined as*

$$\partial_t^\alpha f(x,t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha-1}(f(x,\tau) - f(x,0))\mathrm{d}\tau, \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha}(f(x,\tau) - f(x,0))\mathrm{d}\tau, 0 < \alpha < 1, \\ \big(\frac{\partial^{\alpha-n}}{\partial t^{\alpha-n}} f(x,t)\big)^{(n)}, n \leq \alpha < n+1, n \geq 1. \end{cases}$$

Leibnitz' formula of the fractional differential takes the form

$$\partial_t^\alpha \big(f(x,t)g(x,t)\big) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \partial_t^{\alpha-n} f(x,t) \partial_t^n g(x,t),$$

where $\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$. Faà di Bruno formula is given as

$$\partial_t^m f(g(x,t)) = \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{k}{r} \frac{1}{k!} (-g)^r \frac{\partial^m g^{k-r}}{\partial t^m} \frac{\mathrm{d}^k f(g)}{\mathrm{d}g^k}.$$

This paper is organized as follows: In Section 2, the vector field of forth-order time fractional KdV equation is presented by using Lie symmetry analysis, based on optimal dynamical system, all of similarity reductions to the equation are obtained. Exact series solution and convergence of the reduction equation are investigated in 3. Stability analysis of the equilibrium for reduction differential equation system is discussed in 4. Section 5 local conservation laws for the proposed equation are constructed. Finally, conclusions will be given in Section 6.

2. Symmetry analysis and similarity reduction

We consider FPDE

$$\partial_t^\alpha u + F(u, u_x, u_{2x}, u_{3x}, u_{4x}, \ldots) = 0, \quad \alpha > 0, \tag{1}$$

where $u_{nx} \in C_\gamma(\Omega \times T)$ $(\gamma \geq -1)$, subscripts denote partial derivatives. Let us assume that Eq. (1) is invariant under one parameter ϵ continuous transformation

$$\begin{aligned} &\bar{t} = t + \epsilon\tau(x,t,u) + O(\epsilon^2), \quad \bar{x} = x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ &\bar{u} = u + \epsilon\eta(x,t,u) + O(\epsilon^2), \quad \partial_{\bar{t}}^\alpha \bar{u} = \partial_t^\alpha u + \epsilon\zeta_\alpha^0 + O(\epsilon^2), \\ &\partial_{\bar{x}}\bar{u} = \partial_x u + \epsilon\zeta_1^1 + O(\epsilon^2), \quad \partial_{2\bar{x}}\bar{u} = \partial_{2x} u + \epsilon\zeta_2^1 + O(\epsilon^2), \\ &\partial_{3\bar{x}}\bar{u} = \partial_{3x} u + \epsilon\zeta_3^1 + O(\epsilon^2), \quad \ldots, \end{aligned} \tag{2}$$

where τ, ξ and η are the infinitesimals, ζ_α^0, ζ_1^1, ζ_2^1, ... are the extended infinitesimals of orders $\alpha, 1, 2, \ldots$, respectively

$$\begin{aligned} &\zeta_1^1 = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \quad \zeta_2^1 = D_x(\zeta_1^1) - u_{2x} D_x(\xi) - u_{xt} D_x(\tau), \\ &\zeta_3^1 = D_x(\zeta_2^1) - u_{3x} D_x(\xi) - u_{2xt} D_x(\tau), \quad \ldots, \end{aligned}$$

where D_x denotes the total derivative operator, is defined as $D_x = \partial_x + u_x\partial_u + u_{2x}\partial_{u_x} + \cdots$ with infinitesimal generator $X = \tau\partial_t + \xi\partial_x + \eta\partial_u$. Since the lower terminal of the integral in time fractional modified Riemann-Liouville derivative is fixed and, therefore it should be invariant with respect to the transformation (2), such invariance condition yields

$$\tau(x,t,u)|_{t=0} = 0. \tag{3}$$

α-th extended infinitesimal related to the modified Riemann-Liouville derivative with (3) reads

$$\zeta_\alpha^0 = D_t^\alpha(\eta) + \xi D_t^\alpha u_x - D_t^\alpha(\xi u_x) + D_t^\alpha\big(D_t(\tau)u\big) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}u, \tag{4}$$

in which total fractional derivative operator D_t^α is given as

$$D_t^\alpha = \partial_t^\alpha + (\partial_t^\alpha u)\partial_u + (\partial_t^\alpha u_t)\partial_{u_t} + (\partial_t^\alpha u_x)\partial_{u_x} + (\partial_t^\alpha u_{2x})\partial_{u_{2x}} + \dots.$$

Using Leibnitz' formula, (4) can be presented by

$$\zeta_\alpha^0 = D_t^\alpha \eta - \alpha D_t \tau \partial_t^\alpha u - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n \xi D_t^{\alpha-n} u_x - \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1}\tau D_t^{\alpha-n} u. \tag{5}$$

Making use of Faà di Bruno formula along Leibnitz formula for the modified Riemann-Liouville derivative with $\varphi(x,t) = 1$, then could be read the first term $D_t^\alpha \eta$ in (5) as

$$D_t^\alpha \eta = \partial_t^\alpha \eta + \eta_u \partial_t^\alpha u - u\partial_t^\alpha \eta_u + \sum_{n=1}^{\infty} \binom{\alpha}{n} \partial_t^n \eta_u D_t^{\alpha-n} u + \omega,$$

where $\omega = \sum\limits_{n=2}^{\infty}\sum\limits_{m=2}^{n}\sum\limits_{k=2}^{m}\sum\limits_{r=0}^{k-1} \binom{\alpha}{n}\binom{n}{m}\binom{k}{r} \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r \partial_t^m u^{k-r} \partial_t^{n-m}(\partial_u^k \eta)$.
As a consequence the α-th extended infinitesimal presented in (5) becomes

$$\begin{aligned}\zeta_\alpha^0 = {} & \partial_t^\alpha \eta + (\eta_u - \alpha\tau_t)\partial_t^\alpha u - u\partial_t^\alpha \eta_u + \sum_{n=1}^{\infty}\Big(\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1} D_t^{n+1}\tau\Big) \\ & \times D_t^{\alpha-n} u - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n} u_x + \omega.\end{aligned} \tag{6}$$

For invariance of (1) under the transformation (2), we obtain that

$$\partial_{\bar{t}}^\alpha \bar{u} + F(\bar{x}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{2\bar{x}}, \bar{u}_{3\bar{x}}, \bar{u}_{4\bar{x}}, \dots) = 0, \tag{7}$$

for any solution $\bar{u} = u(\bar{x}, \bar{t})$ of (1). Taking into account the higher order of (1), expanding (7) about $\epsilon = 0$, using the infinitesimals and their extensions (2), equating the coefficients of ϵ, neglecting the higher powers of ϵ, we give a revised invariant equation of (1)

$$\begin{aligned}\zeta_\alpha^0 + \xi\partial_x F + \tau\partial_t^\alpha F + \eta\partial_u F + \zeta_1^1 \partial_{u_x} F + \zeta_2^1 \partial_{u_{2x}} F + \zeta_3^1 \partial_{u_{3x}} F + \zeta_4^1 \partial_{u_{4x}} F \\ + \dots = 0.\end{aligned} \tag{8}$$

For solving (8) with (1), we could explicitly determine the infinitesimals τ, ξ and η, we notice that ω given in (7) vanishes when the infinitesimal η is linear about u.

Definition 4. *A solution $u = \upsilon(x,t)$ is said to be invariant solution of Eq. (1) if and only if*
(i) $u = \upsilon(x,t)$ is invariant surface, i.e. $X\upsilon = 0$,
(ii) $u = \upsilon(x,t)$ satisfies (1).

Forth-order time fractional KdV equation in conservative form is

$$\partial_t^\alpha u + au^5 + bu^3u_x + cuu_x^2 + du^2u_{2x} + eu_xu_{2x} + fuu_{3x} + u_{4x} = 0, \tag{9}$$

with $0 < \alpha < 1$, a, b, c, d, e, f are invariant under the transformation (2), and so a transformed equation is read

$$\partial_{\bar{t}}^\alpha \bar{u} + a\bar{u}^5 + b\bar{u}^3\bar{u}_{\bar{x}} + c\bar{u}\bar{u}_{\bar{x}}^2 + d\bar{u}^2\bar{u}_{2\bar{x}} + e\bar{u}_{\bar{x}}\bar{u}_{2\bar{x}} + f\bar{u}\bar{u}_{3\bar{x}} + \bar{u}_{4\bar{x}} = 0. \tag{10}$$

Making use of the transformation (2) in (10), we obtain invariant equation of (9)

$$\begin{aligned}&\zeta_\alpha^0 + (bu^3 + 2cuu_x + eu_{2x})\zeta_1^1 + (du^2 + eu_x)\zeta_2^1 + fu\zeta_3^1 + \zeta_4^1 \\ &\quad + \eta(5au^4 + 3bu^2u_x + cu_x^2 + 2duu_{2x} + fu_{3x}) = 0,\end{aligned} \tag{11}$$

which depend on variables u_x, u_t, u_{2x}, u_{xt}, u_{2xt}, u_{3x}, u_{4x} ..., and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$ for $n = 1, 2, \ldots$ are considered to be independent. Such structure of (11) allows one to reduce it into a system. Substituting ζ_k^1 $(k = 1, 2, \ldots, 4)$ and ζ_α^0 into (11), equating various powers of derivatives of u to zero, then we obtain over determined system

$$\begin{cases} \xi_u = \xi_t = \tau_u = \tau_x = \eta_{uu} = 0, \\ 4\xi_x - \alpha\tau_t = 0, \\ \binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}\tau = 0 \quad \text{for} \quad n = 1, 2, \ldots, \\ \partial_t^\alpha\eta - u\partial_t^\alpha\eta_u + bu^3\eta_x + du^2\eta_{2x} + fu\eta_{3x} + \eta_{4x} + 5a\eta u^4 = 0, \\ bu^3(\alpha\tau_t - \xi_x) + 2cu\eta_x + du^2(2\eta_{xu} - \xi_{2x}) + e\eta_{2x} + fu(3\eta_{2xu} - \xi_{3x}) \\ \quad +(4\eta_{3xu} - \xi_{4x}) + 3b\eta u^2 = 0. \end{cases} \tag{12}$$

Solving the system (12) consistently, we get the explicit forms of infinitesimals $\xi = a_1x + b_1$, $\tau = \frac{4a_1}{\alpha}t$, $\eta = -a_1u$, where $a_1 \neq 0, b_1$ are constants, and infinitesimal operators $X_1 = x\partial_x + \frac{4t}{\alpha}\partial_t - u\partial_u$, $X_2 = \partial_t^\alpha$, $X_3 = \partial_x$, then the underlying Lie algebra of (9) is two dimensional with the extended basis $\left\{X_1, X_2, X_3, X_3 - \frac{v}{\Gamma(1+\alpha)}X_2\right\}$, v is nonzero constant coefficient, it is easy to check that the symmetry generator founds the closed Lie algebra. Further, we deal with the symmetry reduction and exact solution of (9), consider the similarity reduction and group-invariant solution based on an optimal dynamical system. From the optimal system of group-invariant solution of the equation, other such solution to the equation can be derived.

For the generator X_1, similarity transformation could be obtained by solving an associated characteristic equation $\frac{\mathrm{d}x}{x} = \frac{\alpha \mathrm{d}t}{4t} = -\frac{\mathrm{d}u}{u}$.

Theorem 1. *The similarity transformation $u = t^{-\frac{\alpha}{4}}\varphi(z)$ along with similarity variable $z = xt^{-\frac{\alpha}{4}}$ reduces (9) into nonlinear differential equation with variable z*

$$P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}\varphi + a\varphi^5 + b\varphi^3\varphi' + c\varphi\varphi'^2 + d\varphi^2\varphi'' + e\varphi'\varphi'' + f\varphi\varphi^{(3)} + \varphi^{(4)} = 0. \tag{13}$$

Proof. Consider the first term $\partial_t^\alpha u$ of (9), others are easy to be obtained by the similarity transformation. For $u \in C_\gamma(\Omega \times T)$ $(\gamma \geq -1)$, $\varphi \in C(\Omega \times T)$, we could choose appropriate γ such that $-\frac{\alpha}{4} > \gamma$ by Definition 1. Let $n - 1 \leq \alpha \leq n$, $n = 1, 2, \ldots$, thus the time fractional derivative term for the similarity transformation is become as

$$\partial_t^\alpha u = \partial_t^n\Big(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{-\frac{\alpha}{4}}\varphi(xs^{-\frac{\alpha}{4}})\mathrm{d}s\Big).$$

Set $v = \frac{t}{s}$, there is $\mathrm{d}s = -\frac{t}{v^2}\mathrm{d}v$, we have

$$\partial_t^\alpha u = \partial_t^n \Big(t^{n-\frac{5\alpha}{4}} \frac{1}{\Gamma(n-\alpha)} \int_1^\infty (v-1)^{n-\alpha-1} v^{-(n+1-\frac{5\alpha}{4})} \varphi(zv^{\frac{\alpha}{4}}) \mathrm{d}v \Big),$$

following the definition of Erdélyi-Kober fractional integral operator, we present

$$\partial_t^\alpha u = \partial_t^n \Big(t^{n-\frac{5\alpha}{4}} \big(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4}, n-\alpha} \varphi \big)(z) \Big). \tag{14}$$

In order to simplify (14), there could consider $t\partial_t \phi(z) = tx(-\frac{\alpha}{4}) t^{-\frac{\alpha}{4}-1} \frac{\mathrm{d}}{\mathrm{d}z}\phi(z) = -\frac{\alpha}{4} z \frac{\mathrm{d}}{\mathrm{d}z}\phi(z)$, $(\phi \in C^1(0,\infty))$, and so

$$\begin{aligned}\partial_t^n \Big(t^{n-\frac{5\alpha}{4}} \big(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4}, n-\alpha} \varphi \big)(z) \Big) &= \partial_t^{n-1} \Big(\partial_t \Big(t^{n-\frac{5\alpha}{4}} \big(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4}, n-\alpha} \varphi \big)(z) \Big) \Big) \\ &= \partial_t^{n-1} \Big(t^{n-\frac{5\alpha}{4}-1} \Big(n - \frac{5\alpha}{4} - \frac{\alpha}{4} z \frac{\mathrm{d}}{\mathrm{d}z} \Big) \Big(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4}, n-\alpha} \varphi \Big)(z) \Big).\end{aligned}$$

Repeating on similar procedure for $n-1$ times, yields

$$\partial_t^n \Big(t^{n-\frac{5\alpha}{4}} \big(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4}, n-\alpha} \varphi \big)(z) \Big) = t^{-\frac{5\alpha}{4}} \prod_{j=0}^{n-1} \Big(1 - \frac{5\alpha}{4} + j - \frac{\alpha}{4} z \frac{\mathrm{d}}{\mathrm{d}z} \Big) \Big(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4}, n-\alpha} \varphi \Big)(z),$$

there can be written as

$$\partial_t^n \Big(t^{n-\frac{5\alpha}{4}} \big(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4}, n-\alpha} \varphi \big)(z) \Big) = t^{-\frac{5\alpha}{4}} \Big(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} \varphi \Big)(z).$$

Hence, we obtain

$$\partial_t^\alpha u = t^{-\frac{5\alpha}{4}} \Big(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} \varphi \Big)(z).$$

□

For the generator X_2, we get constant solution of (9) is $u(x,t) = k$.

For the generator X_3, we present stationary solution $u = \varphi(z)$, where $z = x$. (9) is translated into the conservation form differential equation

$$a\varphi^5 + b\varphi^3\varphi' + c\varphi\varphi'^2 + d\varphi^2\varphi'' + e\varphi'\varphi'' + f\varphi\varphi^{(3)} + \varphi^{(4)} = 0.$$

For the generator $X_3 - \frac{v}{\Gamma(1+\alpha)} X_2$, we obtain travelling wave solution $u = \varphi(z)$, where $z = x - \frac{v}{\Gamma(1+\alpha)} t^\alpha$, $v > 0$ is regarded as the wave velocity. (9) is translated into

$$a\varphi^5 + b\varphi^3\varphi' + c\varphi\varphi'^2 + d\varphi^2\varphi'' + e\varphi'\varphi'' + f\varphi\varphi^{(3)} + \varphi^{(4)} - v\varphi' = 0. \tag{15}$$

3. Power series solution

For the conservation form FPDE, to seek the exact solution, we mean there that could be obtained from the corresponding FODE, exact solution of Eq. (9) is obtained actually in preceding Section 2, we want to detect explicit solution expressed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures, and so on. The existing research results tell that power series can be used to solve differential equation [1, 22, 15], we will apply power series to consider the solution of the reduction FODE. Once we obtain solution of the reduction FODE, solution of (9) is successfully obtained. Now we set $\varphi(z) = \sum\limits_{n=0}^{\infty} c_n z^n$, substituting $\varphi(z)$ into (13), it yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\Gamma(1-\frac{\alpha}{4}-\frac{n\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4}-\frac{n\alpha}{4})} c_n z^n + a \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j} c_i c_{j-i} c_{l-j} c_{m-l} c_{n-m} z^n \\
&\quad + b \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l} (n+1-m) c_j c_{l-j} c_{m-l} c_{n+1-m} z^n \\
&\quad + c \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m} (m+1)(n+1-m) c_l c_{m+1} c_{n+1-m} z^n \\
&\quad + d \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m} (n+1-m)(n+2-m) c_l c_{m-l} c_{n+2-m} z^n \\
&\quad + e \sum_{n=0}^{\infty}\sum_{m=0}^{n} (m+1)(n+1-m)(n+2-m) c_{m+1} c_{n+2-m} z^n \\
&\quad + f \sum_{n=0}^{\infty}\sum_{m=0}^{n} (n+1-m)(n+2-m)(n+3-m) c_m c_{n+3-m} z^n \\
&\quad + \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4) c_{n+4} z^n = 0,
\end{aligned}
$$

by comparing coefficients, for $n = 0$, one has that $c_4 = -\frac{1}{24}\big(\frac{\Gamma(1-\frac{\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4})} c_0 + a c_0^5 + b c_0^3 c_1 + c c_0 c_1^2 + 2d c_0^2 c_2 + 2e c_1 c_2 + 6f c_0 c_3\big)$. For $n \geq 1$, we could provide recursion formula

$$
\begin{aligned}
c_{n+4} = &-\frac{1}{(n+1)(n+2)(n+3)(n+4)} \Big(\frac{\Gamma(1-\frac{\alpha}{4}-\frac{n\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4}-\frac{n\alpha}{4})} c_n \\
&\quad + a \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j} c_i c_{j-i} c_{l-j} c_{m-l} c_{n-m} \\
&\quad + b \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l} (n+1-m) c_j c_{l-j} c_{m-l} c_{n+1-m} \\
&\quad + c \sum_{m=0}^{n}\sum_{l=0}^{m} (m+1)(n+1-m) c_l c_{m+1} c_{n+1-m} \\
&\quad + d \sum_{m=0}^{n}\sum_{l=0}^{m} (n+1-m)(n+2-m) c_l c_{m-l} c_{n+2-m} \\
&\quad + e \sum_{m=0}^{n} (m+1)(n+1-m)(n+2-m) c_{m+1} c_{n+2-m} \\
&\quad + f \sum_{m=0}^{n} (n+1-m)(n+2-m)(n+3-m) c_m c_{n+3-m} \Big),
\end{aligned}
\tag{16}
$$

for the chosen constants c_i $(i = 0, 1, 2, 3)$, others of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (16) in unique manner. This implies that for (13), there exists power series solution $\varphi(z)$ with the coefficients given by (16), it is easy to check that the obtained series solution $\varphi(z)$ of Eq. (13) is analytic, so series solution of (13) can be written as

$$\begin{aligned}
\varphi(z) &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \sum_{n=1}^{\infty} c_{n+4} z^{n+4} \\
&= c_0 + c_1 z + c_2 z^2 + c_3 z^3 \\
&\quad - \frac{1}{24}\Big(\frac{\Gamma(1-\frac{\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4})} c_0 + a c_0^5 + b c_0^3 c_1 + c c_0 c_1^2 + 2d c_0^2 c_2 + 2e c_1 c_2 + 6f c_0 c_3\Big) z^4 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)} \Big(\frac{\Gamma(1-\frac{\alpha}{4}-\frac{n\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4}-\frac{n\alpha}{4})} c_n \\
&\quad + a \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j} c_i c_{j-i} c_{l-j} c_{m-l} c_{n-m} \\
&\quad + b \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l} (n+1-m) c_j c_{l-j} c_{m-l} c_{n+1-m} \\
&\quad + c \sum_{m=0}^{n}\sum_{l=0}^{m} (m+1)(n+1-m) c_l c_{m+1} c_{n+1-m} \\
&\quad + d \sum_{m=0}^{n}\sum_{l=0}^{m} (n+1-m)(n+2-m) c_l c_{m-l} c_{n+2-m} \\
&\quad + e \sum_{m=0}^{n} (m+1)(n+1-m)(n+2-m) c_{m+1} c_{n+2-m} \\
&\quad + f \sum_{m=0}^{n} (n+1-m)(n+2-m)(n+3-m) c_m c_{n+3-m}\Big) z^{n+4}.
\end{aligned}$$

Then series solution of (9) is given as

$$\begin{aligned}
u(x,t) &= c_0 t^{-\frac{\alpha}{4}} + c_1 x t^{-\frac{\alpha}{2}} + c_2 x^2 t^{-\frac{3\alpha}{4}} + c_3 x^3 t^{-\alpha} \\
&\quad - \frac{1}{24}\Big(\frac{\Gamma(1-\frac{\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4})} c_0 + a c_0^5 + b c_0^3 c_1 + c c_0 c_1^2 + 2d c_0^2 c_2 + 2e c_1 c_2 + 6f c_0 c_3\Big) x^4 t^{-\frac{5\alpha}{4}} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)} \Big(\frac{\Gamma(1-\frac{\alpha}{4}-\frac{n\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4}-\frac{n\alpha}{4})} c_n \\
&\quad + a \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j} c_i c_{j-i} c_{l-j} c_{m-l} c_{n-m} \\
&\quad + b \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l} (n+1-m) c_j c_{l-j} c_{m-l} c_{n+1-m}
\end{aligned}$$

$$
\begin{aligned}
&+ c\sum_{m=0}^{n}\sum_{l=0}^{m}(m+1)(n+1-m)c_l c_{m+1}c_{n+1-m}\\
&+ d\sum_{m=0}^{n}\sum_{l=0}^{m}(n+1-m)(n+2-m)c_l c_{m-l}c_{n+2-m}\\
&+ e\sum_{m=0}^{n}(m+1)(n+1-m)(n+2-m)c_{m+1}c_{n+2-m}\\
&+ f\sum_{m=0}^{n}(n+1-m)(n+2-m)(n+3-m)c_m c_{n+3-m}\Big)x^{n+4}t^{-\frac{(n+5)\alpha}{4}}.
\end{aligned}
$$

From the preceding discussion, we could provide the convergence of series solution $\varphi(z)$ of (13), in fact, from (16), it has

$$
\begin{aligned}
|c_{n+4}| \le \Big(&\frac{|\Gamma(1-\frac{\alpha}{4}-\frac{n\alpha}{4})|}{|\Gamma(1-\frac{5\alpha}{4}-\frac{n\alpha}{4})|}|c_n| + |a|\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j}|c_i||c_{j-i}||c_{l-j}||c_{m-l}||c_{n-m}|\\
&+ |b|\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}|c_j||c_{l-j}||c_{m-l}||c_{n+1-m}| + |c|\sum_{m=0}^{n}\sum_{l=0}^{m}|c_l||c_{m+1}||c_{n+1-m}|\\
&+ |d|\sum_{m=0}^{n}\sum_{l=0}^{m}|c_l||c_{m-l}||c_{n+2-m}| + |e|\sum_{m=0}^{n}|c_{m+1}||c_{n+2-m}|\\
&+ |f|\sum_{m=0}^{n}|c_m||c_{n+3-m}|\Big), \quad (n=1,2,\ldots).
\end{aligned} \tag{17}
$$

Taking into account the property of Γ function, it is no difficulty to find that $\frac{|\Gamma(1-\frac{\alpha}{4}-\frac{n\alpha}{4})|}{|\Gamma(1-\frac{5\alpha}{4}-\frac{n\alpha}{4})|} < 1$ for n. Hence (17) is written as

$$
\begin{aligned}
|c_{n+4}| \le M\Big(&|c_n| + \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j}|c_i||c_{j-i}||c_{l-j}||c_{m-l}||c_{n-m}|\\
&+ \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}|c_j||c_{l-j}||c_{m-l}||c_{n+1-m}| + \sum_{m=0}^{n}\sum_{l=0}^{m}|c_l||c_{m+1}||c_{n+1-m}|\\
&+ \sum_{m=0}^{n}\sum_{l=0}^{m}|c_l||c_{m-l}||c_{n+2-m}| + \sum_{m=0}^{n}|c_{m+1}||c_{n+2-m}| + \sum_{m=0}^{n}|c_m||c_{n+3-m}|\Big),
\end{aligned}
$$

where $M = \max\{1, |a|, |b|, |c|, |d|, |e|, |f|\}$. Introduce a power series $A(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, set $a_i = |c_i|$, $(i = 0, 1, \ldots, 4)$ and

$$
\begin{aligned}
a_{n+4} = M\Big(&a_n + \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j}a_i a_{j-i}a_{l-j}a_{m-l}a_{n-m}\\
&+ \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}a_j a_{l-j}a_{m-l}a_{n+1-m} + \sum_{m=0}^{n}\sum_{l=0}^{m}a_l a_{m+1}a_{n+1-m}
\end{aligned}
$$

$$+\sum_{m=0}^{n}\sum_{l=0}^{m}a_l a_{m-l}a_{n+2-m}+\sum_{m=0}^{n}a_{m+1}a_{n+2-m}+\sum_{m=0}^{n}a_m a_{n+3-m}\Big),$$

it is easily see that $|c_n|\le a_n$, $(n=0,1,\ldots)$, $A(z)$ is the majorant series of (17). Further, we show that series $A(z)$ exists positive convergence radius, note that by formal calculation, it yields

$$\begin{aligned}
A(z)&=a_0+a_1z+a_2z^2+a_3z^3+M\sum_{n=0}^{\infty}a_n z^{n+4}\\
&+M\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j}a_i a_{j-i}a_{l-j}a_{m-l}a_{n-m}z^{n+4}\\
&+M\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}a_j a_{l-j}a_{m-l}a_{n+1-m}z^{n+4}\\
&+M\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}a_l a_{m+1}a_{n+1-m}z^{n+4}\\
&+M\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}a_l a_{m-l}a_{n+2-m}z^{n+4}\\
&+M\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_{m+1}a_{n+2-m}z^{n+4}\\
&+M\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_m a_{n+3-m}z^{n+4}.
\end{aligned}$$

Consider the implicit functional system with respect to the independent variable z

$$\begin{aligned}
\mathcal{A}(z,A)&=A-a_0-a_1z-a_2z^2-a_3z^3-Mz^4A-Mz^4(A^5-a_0A^4)\\
&-Mz^3(A^4-a_0A^3)-Mz^2(A^3-a_0A^2)-Mz(A^2-2a_0A\\
&-a_1zA+a_0a_1z+a_0^2)-Mz(A^2-a_0A-a_1A-a_2^2z^2A\\
&-a_1a_2z^3+a_1a_2z^2-a_0a_1z+a_0a_1)+\cdots.
\end{aligned}$$

It is easy to check that $\mathcal{A}(z,A)$ is analytic in a neighborhood of $(0,a_0)$, $\mathcal{A}(0,a_0)=0$ and $\frac{\partial}{\partial A}\mathcal{A}(0,a_0)\neq 0$, by the implicit function theorem [24], we could obtain that the series $A(z)$ is analytic in the neighborhood of the point $(0,a_0)$ and with the positive convergence radius, this implies that the power series (17) converge in the neighborhood of the point $(0,a_0)$.

Remark 1. *Since*

$$\begin{aligned}
\bar{c}_{n+4}=-\frac{1}{(n+1)(n+2)(n+3)}&\Big(\frac{\Gamma(1-\frac{\alpha}{4}-\frac{n\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4}-\frac{n\alpha}{4})}c_n\\
&+a\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j}c_i c_{j-i}c_{l-j}c_{m-l}c_{n-m}
\end{aligned}$$

$$
\begin{aligned}
&+ b \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{j=0}^{l} (n+1-m) c_j c_{l-j} c_{m-l} c_{n+1-m} \\
&+ c \sum_{m=0}^{n} \sum_{l=0}^{m} (m+1)(n+1-m) c_l c_{m+1} c_{n+1-m} \\
&+ d \sum_{m=0}^{n} \sum_{l=0}^{m} (n+1-m)(n+2-m) c_l c_{m-l} c_{n+2-m} \\
&+ e \sum_{m=0}^{n} (m+1)(n+1-m)(n+2-m) c_{m+1} c_{n+2-m} \\
&+ f \sum_{m=0}^{n} (n+1-m)(n+2-m)(n+3-m) c_m c_{n+3-m} \Big),
\end{aligned}
$$

where $\bar{c}_{n+4}$ is the coefficient of $\partial_x u$, we can also obtain the convergence of series solution $\partial_x u$ via the above similar argument.

Substituting $\varphi(z)$ into (15), it yields

$$
\begin{aligned}
&a \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{j} c_i c_{j-i} c_{l-j} c_{m-l} c_{n-m} z^n \\
&\quad + b \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{j=0}^{l} (n+1-m) c_j c_{l-j} c_{m-l} c_{n+1-m} z^n \\
&\quad + c \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{m} (m+1)(n+1-m) c_l c_{m+1} c_{n+1-m} z^n \\
&\quad + d \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{m} (n+1-m)(n+2-m) c_l c_{m-l} c_{n+2-m} z^n \\
&\quad + e \sum_{n=0}^{\infty} \sum_{m=0}^{n} (m+1)(n+1-m)(n+2-m) c_{m+1} c_{n+2-m} z^n \\
&\quad + f \sum_{n=0}^{\infty} \sum_{m=0}^{n} (n+1-m)(n+2-m)(n+3-m) c_m c_{n+3-m} z^n \\
&\quad + \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4) c_{n+4} z^n - v \sum_{n=0}^{\infty} (n+1) c_{n+1} z^n = 0,
\end{aligned}
$$

also for $n \geq 1$, we obtain recursion formula

$$
\begin{aligned}
c_{n+4} = &-\frac{1}{(n+1)(n+2)(n+3)(n+4)} \Big(a \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{j} c_i c_{j-i} c_{l-j} c_{m-l} c_{n-m} \\
&+ b \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{j=0}^{l} (n+1-m) c_j c_{l-j} c_{m-l} c_{n+1-m}
\end{aligned}
$$

$$
\begin{aligned}
&+ c\sum_{m=0}^{n}\sum_{l=0}^{m}(m+1)(n+1-m)c_l c_{m+1}c_{n+1-m}\\
&+ d\sum_{m=0}^{n}\sum_{l=0}^{m}(n+1-m)(n+2-m)c_l c_{m-l}c_{n+2-m}\\
&+ e\sum_{m=0}^{n}(m+1)(n+1-m)(n+2-m)c_{m+1}c_{n+2-m}\\
&+ f\sum_{m=0}^{n}(n+1-m)(n+2-m)(n+3-m)c_m c_{n+3-m}\\
&- v(n+1)c_{n+1}\Big),
\end{aligned}
$$

for the chosen constants c_i $(i=0,1,2,3)$, others of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from c_{n+4}. This implies that for (15), there exists power series wave solution $\varphi(z)$ with the coefficient given by c_{n+4}, the series wave solution of (9) is successfully obtained

$$
\begin{aligned}
&u(x,t)\\
&= c_0 + c_1\big(x-\frac{v}{\Gamma(1+\alpha)}t^{\alpha}\big) + c_2\big(x-\frac{v}{\Gamma(1+\alpha)}t^{\alpha}\big)^2 + c_3\big(x-\frac{v}{\Gamma(1+\alpha)}t^{\alpha}\big)^3\\
&- \frac{1}{24}\big(ac_0^5 + bc_0^3c_1 + cc_0c_1^2 + 2dc_0^2c_2 + 2ec_1c_2 + 6fc_0c_3 - vc_1\big)\big(x-\frac{v}{\Gamma(1+\alpha)}t^{\alpha}\big)^4\\
&- \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}\Big(a\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{j}c_i c_{j-i}c_{l-j}c_{m-l}c_{n-m}\\
&+ b\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{l}(n+1-m)c_j c_{l-j}c_{m-l}c_{n+1-m}\\
&+ c\sum_{m=0}^{n}\sum_{l=0}^{m}(m+1)(n+1-m)c_l c_{m+1}c_{n+1-m}\\
&+ d\sum_{m=0}^{n}\sum_{l=0}^{m}(n+1-m)(n+2-m)c_l c_{m-l}c_{n+2-m}\\
&+ e\sum_{m=0}^{n}(m+1)(n+1-m)(n+2-m)c_{m+1}c_{n+2-m}\\
&+ f\sum_{m=0}^{n}(n+1-m)(n+2-m)(n+3-m)c_m c_{n+3-m}\Big)(x-\frac{v}{\Gamma(1+\alpha)}t^{\alpha})^{n+4}.
\end{aligned}
$$

Also the above obtained series wave solution of (15) is convergence on the variable z, it can be tackled in similar way of (13).

4. Stability analysis

In this section, we will discuss stability of trivial solution of (9) by constructing appropriate Lyapunov function, the stability analysis of solution reflects in well-posedness of the problem for determining solution, even the regularity of weak solution.

Definition 5. *An equilibrium point $\bar{x}$ of $\dot{x} = f(x)$ (i.e. $f(\bar{x}) = 0$), $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$, is local stable, if for every $R > 0$ there exists $r > 0$, such that $\|x(0) - \bar{x}\| < r$ yields $\|x(t) - \bar{x}\| < R$, $t \geq 0$. If local stable and $\|x(0) - \bar{x}\| < r$ yield $\lim_{t\to\infty} x(t) = \bar{x}$, then called local asymptotic stable. If asymptotic stable for all $x(0) \in \mathbb{R}^n$, then called global asymptotic stable.*

Lemma 1. [13] *Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, such that*

(1) $V(0) = 0$, $V(x) > 0$, for all $x \neq 0$,

(2) $\dot{V}(x) \leq 0$, for all x,

(3) $V(x) \to \infty$ as $\|x\| \to \infty$,

(4) The only solution of $\dot{x} = f(x)$, $\dot{V}(x) = 0$ is $x(t) = 0$.

Then $x = 0$ is global asymptotic stable.

$X_3 - \frac{v}{\Gamma(1+\alpha)} X_2$ in Section 2 has been obtained the travelling wave solution, then (9) is translated as

$$\varphi^{(4)} + \varphi\varphi^{(3)} + (\varphi^2 + \varphi')\varphi'' + (\varphi^3 + \varphi\varphi' - 1)\varphi' + \varphi^5 = 0, \tag{18}$$

it could be equivalent to the following system

$$\begin{cases} \varphi' = \lambda, \\ \lambda' = \mu, \\ \mu' = \nu, \\ \nu' = -f(\varphi, \lambda, \mu)\nu - g(\varphi, \lambda)\mu - h(\varphi, \lambda)\lambda - k(\varphi). \end{cases} \tag{19}$$

Theorem 2. *If the functions $f(\varphi, \lambda, \mu)$ $(f(\varphi, \lambda, \mu) > 0)$, $g(\varphi, \lambda)$ are continuous and continuous first order partial derivatives, the functions $h(\varphi, \lambda)$, $k(\varphi)$ are continuous, there exist the constants $a > 0$, $b > 0$, $c > 0$ such that $a^2 - 4c > 0$ and satisfy*

(1) $k(0) = 0, k_1(\varphi) = \frac{k(\varphi)}{\varphi} > c > 0$ $(\varphi \neq 0), g(\varphi, \lambda) - a > 0$;

(2) $abf(\varphi, \lambda, \mu) - b^2 - cf^2(\varphi, \lambda, \mu) > 0$;

(3) $\big(ab - 2cf(\varphi, \lambda, \mu)\big)\nu + b^2\mu > (h(\varphi, \lambda) - b)\lambda > 0$;

(4) $\big(k(\varphi) - c\varphi + \mu(g(\varphi, \lambda) - a)(b\mu + a\nu) > 0$;

(5) $g'_\varphi(\varphi, \lambda)\lambda \leq 0$;

(6) $f'_\varphi(\varphi, \lambda, \mu)\lambda + f'_\lambda(\varphi, \lambda, \mu)\mu \leq 0$.

Then a trivial solution of (18) satisfies the system (19) is the global asymptotic stable.

Proof. Set $f(\varphi, \lambda, \mu) = \varphi$, $g(\varphi, \lambda) = \varphi^2 + \lambda$, $h(\varphi, \lambda) = \varphi^3 + \varphi\lambda - 1$, $k(\varphi) = \varphi^5$, the functions f, g, h, k satisfy the conditions of theorem, we structure Lyapunov function V for the system (19) by analogy

$$\begin{aligned} V(\varphi, \lambda, \mu, \nu) = {} & \frac{1}{4}a^2(2c\varphi + b\lambda + a\mu)^2 + \frac{1}{2}c\big((a^2 - 4c)a + 2b^2\big)\lambda^2 + \frac{1}{4}(a^2 - 4c) \\ & \times (b\lambda + a\mu)^2 + \frac{1}{2}a(2c\lambda + b\mu + a\nu)^2 + 2a^2c\int_0^\varphi (k(\xi) - c\xi)\mathrm{d}\xi \\ & + 2a^2c\int_0^\lambda (g(\varphi, \eta) - a)\lambda\mathrm{d}\eta + ab\int_0^\mu (af(\varphi, \lambda, \zeta) - b)\zeta\mathrm{d}\zeta. \end{aligned}$$

Further,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}z} &= -\frac{a^2}{b}\big(abf(\varphi,\lambda,\mu)-b^2-cf^2(\varphi,\lambda,\mu)\big)\nu^2-\frac{a^2c}{b}\big(h(\varphi,\lambda)\lambda+f(\varphi,\lambda,\mu)\nu\big)^2\\ &\quad-\frac{a^2}{b}\lambda(h(\varphi,\lambda)-b)\big((ab-2cf(\varphi,\lambda,\mu))\nu+b^2\mu-c(h(\varphi,\lambda)-b)\lambda\big)\\ &\quad-a^2\big(k(\varphi)-c\varphi+\mu(g(\varphi,\lambda)-a)\big)(b\mu+a\nu)+2a^2c\int_0^{\lambda}g'_{\varphi}(\varphi,\eta)\lambda\eta\mathrm{d}\eta\\ &\quad+a^2b\int_0^{\mu}\big(f'_{\varphi}(\varphi,\lambda,\zeta)\lambda+f'_{\lambda}(\varphi,\lambda,\zeta)\zeta\big)\zeta\mathrm{d}\zeta\\ &\le-\frac{a^2}{b}\big(abf(\varphi,\lambda,\mu)-b^2-cf^2(\varphi,\lambda,\mu)\big)\nu^2-\frac{a^2c}{b}\big(h(\varphi,\lambda)\lambda+f(\varphi,\lambda,\mu)\nu\big)^2\\ &\quad-\frac{a^2}{b}\lambda(h(\varphi,\lambda)-b)\big((ab-2cf(\varphi,\lambda,\mu))\nu+b^2\mu-c(h(\varphi,\lambda)-b)\lambda\big)\\ &\quad-a^2\big(k(\varphi)-c\varphi+\mu(g(\varphi,\lambda)-a)\big)(b\mu+a\nu)\le 0,\end{aligned}$$

and $\Omega \triangleq \{(\varphi,\lambda,\mu,\nu):\frac{\mathrm{d}V}{\mathrm{d}z}=0\}$ do not include other trajectories except the trivial solution. In fact, assume that there exist the trajectories $\varphi=\varphi(z)$, $\lambda=\lambda(z)$, $\mu=\mu(z)$, $\nu=\nu(z)$, by the system and the set Ω, we obtain that $\varphi=\lambda=\mu=\nu=0$. Moreover, it is not difficult to get that $V(x)\to\infty$ as $\|x\|\to\infty$ holds. So, by Lemma 1 the trivial solution of (9) is global asymptotic stable.

In view of the function $k_1(\varphi)=\frac{k(\varphi)}{\varphi}>c>0$ and $abf(\varphi,\lambda,\mu)-b^2-cf^2(\varphi,\lambda,\mu)>0$, we get that the Liapunov function V is positive definite and has infinite lower bound. □

5. Local conservation laws

In this section, we will construct local conservation laws on the basis of the vector field for the given equation, the locality means that the left-hand side of (9) depends locally on u with respect to x,t, i.e., any value is completely determined by the value of u in sufficiently small region of x,t. Since the local conservation laws usually express mathematically as a partial differential equation which gives the relation between quantity and "transport" of that quantity. It states that the conserved quantity at a point or within a volume can only change by quantity which flows in or out of the volume [8]. Moreover, Noether's theorem states that there is a one-to-one correspondence between each one of them and the differentiable symmetry of nature. For example, the conservation of energy follows from the time-invariance of physical systems, and the conservation of angular momentum arises from the fact that physical systems behave the same regardless of how they are oriented in space. The vector $C=(C^t,C^x)$ is called conserved vector [20] for (9) satisfies

$$\partial_t(C^t)+\partial_x(C^x)|_{(9)}=0, \tag{20}$$

where C^t and C^x are named the conserved density and conserved flux, respectively, and neither one involves derivatives with respect to t, called conservation laws. The density-flux pairs are polynomials in u and derivatives of u with respect to x, i.e. $C^t=C^t(u,u_x,\ldots)$, $C^x=C^x(u,u_x,\ldots)$. For polynomial form C^t and C^x, integration of (20) yields

$$\int_{-\infty}^{\infty} C^t \mathrm{d}x = constant$$

provided that C^x vanishes at ∞ [9]. Formal Lagrangian for (9) can be written the form

$$H = v(x,t)\big(\partial_t^\alpha u + au^5 + bu^3u_x + cuu_x^2 + du^2u_{2x} + eu_xu_{2x} + fuu_{3x} + u_{4x}\big), \quad (21)$$

where $v(x,t)$ is new dependent variable. Due to the formal Lagrangian, action integral is given as

$$\iint\limits_{\Omega\times[0,T]} H(x,t,v,u,\partial_t^\alpha u, u_x, u_{2x}, \ldots)\mathrm{d}x\mathrm{d}t, \quad H \in L^1(\Omega\times\mathbb{R}^m;\mathbb{R}).$$

The adjoint equation of (9) is defined by $F = \delta_u H$, where

$$\delta_u = \partial_u - (\partial_t^\alpha)^* \frac{\partial}{\partial(\partial_t^\alpha u)} - \partial_x\frac{\partial}{\partial u_x} + \partial_{2x}\frac{\partial}{\partial u_{2x}} - \partial_{3x}\frac{\partial}{\partial u_{3x}} + \partial_{4x}\frac{\partial}{\partial u_{4x}} + \cdots,$$

in which $(\partial_t^\alpha)^*$ is adjoint operator of ∂_t^α. Taking into consideration the fractional derivative, we have

$$(\partial_t^\alpha u)^* = (-1)^n J_T^{n-\alpha}(\partial_t^n u) = {}^C(\partial_T^\alpha u),$$

where ${}^C(\partial_T^\alpha)$ is right-hand side multivariate Caputo derivative operator, $J_T^{n-\alpha}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\times\int_t^T u(x,\mu)(\mu-t)^{n-\alpha-1}\mathrm{d}\mu$, $n=[\alpha]+1$, $u\in L^1(\Omega\times[0,T];\mathbb{R})$. Substituting (21) into F, the adjoint equation of (9) is

$$(\partial_t^\alpha v)^* + au^5 - bu^3v_x - 2cuu_xv_x + du^2v_{2x} - ev_xu_{2x} + eu_xv_{2x} - fuv_{3x} + v_{4x} = 0.$$

So we have the adjoint equation of (9) as Euler-Lagrange equation

$$\delta_u H = 0.$$

In view of the time fractional derivative, we introduce Nother operators N^t and N^x as follows

$$N^t := \tau I + \sum_{k=0}^{n-1}(-1)^k\partial_t^{\alpha-1-k}(W)\partial_t^k\frac{\partial}{\partial(\partial_t^\alpha u)} - (-1)^n J\big(W, \partial_t^n\frac{\partial}{\partial(\partial_t^\alpha u)}\big),$$

where I is identity operator, Lie characteristic function $W = \eta - \tau u_t - \xi u_x$, $J(u,v) = \frac{1}{\Gamma(n-\alpha)}\iint\limits_{[0,t]\times[t,T]}\frac{u(x,\mu)v(x,\nu)}{(\mu-\nu)^{\alpha+1-n}}\mathrm{d}\mu\mathrm{d}\nu$, $u,v\in L^1(\Omega\times[0,T];\mathbb{R})$,

$$\begin{aligned} N^x :&= \xi I + W\big(\frac{\partial}{\partial u_x} - \partial_x\frac{\partial}{\partial u_{2x}} + \partial_{2x}\frac{\partial}{\partial u_{3x}} - \partial_{3x}\frac{\partial}{\partial u_{4x}}\big) \\ &+ \partial_x(W)\big(\frac{\partial}{\partial u_{2x}} - \partial_x\frac{\partial}{\partial u_{3x}} + \partial_{2x}\frac{\partial}{\partial u_{4x}}\big) \\ &+ \partial_{2x}(W)\big(\frac{\partial}{\partial u_{3x}} - \partial_x\frac{\partial}{\partial u_{4x}}\big) + \partial_{3x}(W)\frac{\partial}{\partial u_{4x}}. \end{aligned}$$

Acting Nother operators N^t and N^x on H, (20) becomes

$$\partial_t(N^tH) + \partial_x(N^xH)|_{(9)} = 0. \quad (22)$$

By the extended vector field, (9) yield local conservation laws (22) with the row conserved vector

$$\begin{aligned} C^t &= v\partial_t^{\alpha-1}(W) + J(W, \partial_t v), \\ C^x &= W\big(bu^3 v + 2(c-d)uu_x v - du^2 v_x - (e+2f)u_x v_x - fu_{2x}v - fuv_{2x} - v_{3x}\big) \\ &\quad + W_x\big(du^2 v + (e-f)u_x v - fuv_x + v_{2x}\big) + W_{2x}(fuv - v_x) + W_{3x}v. \end{aligned} \tag{23}$$

Based on Lie point extended symmetry generator, we obtain the conserved vector of (9).

Lie characteristic function $W_1 = -x\partial_x u - \frac{4t}{\alpha}\partial_t u - u$ is obtained by the generator X_1, substituting W_1 into (23), the first row conserved vector (C^t, C^x) is presented, where

$$\begin{aligned} C^t &= -v\partial_t^{\alpha-1}(x\partial_x u) - \frac{4vt}{\alpha}\partial_t^\alpha u - \frac{4vt^{2-\alpha}}{\alpha\Gamma(3-\alpha)}\partial_t u - v\partial_t^{\alpha-1}u - J\big(x\partial_x u + \frac{4t}{\alpha}\partial_t u + u, \\ &\quad \partial_t v\big), \\ C^x &= -\big(x\partial_x u + \frac{4t}{\alpha}\partial_t u + u\big)\big(bu^3 v + 2(c-d)uu_x v - du^2 v_x - (e+2f)u_x v_x - fu_{2x}v \\ &\quad - fuv_{2x} - v_{3x}\big) - \big(2\partial_x u + x\partial_{2x}u + \frac{4t}{\alpha}\partial_{xt}u\big)\big(du^2 v + (e-f)u_x v - fuv_x + v_{2x}\big) \\ &\quad - \big(3\partial_{2x}u + x\partial_{3x}u + \frac{4t}{\alpha}\partial_{2xt}u\big)(fuv - v_x) - \big(4\partial_{3x}u + x\partial_{4x}u + \frac{4t}{\alpha}\partial_{3xt}u\big)v. \end{aligned}$$

Lie characteristic function $W_2 = -\partial_t^\alpha u$ is obtained by the generator X_2, substituting W_2 into (23), the second row conserved vector (C^t, C^x) is given, where

$$\begin{aligned} C^t &= -v\partial_t^{2\alpha-1}u - J(\partial_t^\alpha u, \partial_t v), \\ C^x &= -\partial_t^\alpha u\big(bu^3 v + 2(c-d)uu_x v - du^2 v_x - (e+2f)u_x v_x - fu_{2x}v - fuv_{2x} - v_{3x}\big) \\ &\quad - \partial_{xt}^\alpha u\big(du^2 v + (e-f)u_x v - fuv_x + v_{2x}\big) - \partial_{2xt}^\alpha u(fuv - v_x) - (\partial_{3xt}^\alpha u)v. \end{aligned}$$

Lie characteristic function $W_3 = -\partial_x u$ is obtained by the generator X_3, substituting W_3 into (23), the third row conserved vector (C^t, C^x) is provided, where

$$\begin{aligned} C^t &= -v\partial_t^{\alpha-1}(\partial_x u) - J(\partial_x u, \partial_t v), \\ C^x &= -\partial_x u\big(bu^3 v + 2(c-d)uu_x v - du^2 v_x - (e+2f)u_x v_x - fu_{2x}v - fuv_{2x} - v_{3x}\big) \\ &\quad - \partial_{2x}u\big(du^2 v + (e-f)u_x v - fuv_x + v_{2x}\big) - \partial_{3x}u(fuv - v_x) - (\partial_{4x}u)v. \end{aligned}$$

Lie characteristic function $W_4 = -\partial_x u + \frac{v}{\Gamma(1+\alpha)}\partial_t^\alpha u$ is obtained by the generator $X_3 - \frac{v}{\Gamma(1+\alpha)}X_2$, substituting W_3 into (23), the forth row conserved vector (C^t, C^x) is given, where

$$\begin{aligned} C^t &= -v\partial_t^{\alpha-1}(\partial_x u) + \frac{v^2}{\Gamma(1+\alpha)}\partial_t^{2\alpha-1}u - J\big(\partial_x u - \frac{v}{\Gamma(1+\alpha)}\partial_t^\alpha u, \partial_t v\big), \\ C^x &= -\big(\partial_x u - \frac{v}{\Gamma(1+\alpha)}\partial_t^\alpha u\big)\big(bu^3 v + 2(c-d)uu_x v - du^2 v_x - (e+2f)u_x v_x - fu_{2x}v \\ &\quad - fuv_{2x} - v_{3x}\big) - \big(\partial_{2x}u - \frac{v}{\Gamma(1+\alpha)}\partial_{xt}^\alpha u\big)\big(du^2 v + (e-f)u_x v - fuv_x + v_{2x}\big) \\ &\quad - \big(\partial_{3x}u - \frac{v}{\Gamma(1+\alpha)}\partial_{2xt}^\alpha u\big)(fuv - v_x) - \big(\partial_{4x}u - \frac{v}{\Gamma(1+\alpha)}\partial_{3xt}^\alpha u\big)v. \end{aligned}$$

Remark 2. *In view of the obtained solution of (9) in Section 3, we note that* u *and* $\partial_x u$ *are convergence by Remark 1, respectively, the integral* $\int_{-\infty}^{\infty} C^t \mathrm{d}x$ *provides local conserved quantity, this state that local conservation laws are important for investigating integrability mapping and establishing existence of the KdV equation.*

6. Conclusions

The presented analysis illustrates Lie symmetry approach to study forth-order time fractional KdV equation in conservative form, and the geometric vector field to equation is presented, Lie symmetry analysis shows that the underlying symmetry algebra of each of the equation is two dimensional, the reduction of dimension in the symmetry algebra is due to the fact that KdV equation is not invariant under the translation symmetry. We have shown that the equation can be transformed into differential equation with independent variable, then the series solution is obtained, the global asymptotical stability of trivial solution for forth order nonlinear differential equation is considered by constructing appropriate Lyapunov function, conservation laws of the equation are constructed making use of Noether's operator. Our results witness that the symmetry analysis is very efficient and powerful way in finding the nature properties (i.e. the existence of nontrivial solution, stability of trivial solution, and conservative laws of the system) of solution for the proposed equation.

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