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A NOTE ON THE LOCAL STABILITY THEORY FOR CAPUTO FRACTIONAL PLANAR SYSTEM

MARVIN HOTI

Abstract. In this manuscript a local stability theory framework is proposed for Caputo dynamical systems. It is shown that under suitable conditions that a hyperbolic equilibrium point is a stable spiral or unstable spiral.

1. Introduction

The introduction of fractional calculus (fractional differential and integral equations) to fields such as mathematical biology has not gone without notice lately. The application of fractional derivative to mathematical biology is thought to be a more accurate operator as opposed to the integer order derivative. Commonly, the Riemann-Liouville and Caputo fractional derivative/integral operators are used in application, see [1, 2, 3, 4, 5, 6, 10, 13, 14, 15, 16, 20, 18, 19].

Many authors have made great contributions in the theoretical realm of fractional calculus, however it still remains unclear why the fractional approach is superior to classical, integer, order calculus. That is to say, it is not well understood what the physical meaning of fractional calculus is, and thus it is not easy to justify what role they play in application, such as in the field of mathematical biology; in the modelling of predator-prey systems or compartmental epidemic models. In this paper, using the results obtained in [24, 25] we attempt to shed some light into this problem. Furthermore, we consider the classical local stability theory and extend it to the fractional case. The current literature is lacking a solid foundation of a theoretical framework for the local stability theory of equilibrium points of Caputo fractional dynamical systems.

The first part of this paper heavily uses the results obtained in [24, 25] to set up a clear definition of fractional integral and derivative equations of the Riemann-Liouville type and Caputo type. Then, using the results obtained in [24], we attempt to establish a clear picture of the physical interpretation of the fractional operators. It is argued in this paper, that if the conclusion stated in [24] is correct. Then, time is not homogeneous in nature and the kernel of the integral operators can be chosen to fit (or consider) a different time scale. Thus, implying that fractional derivatives or fractional integrals can take many forms, and their application is

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then to model the evolution of a system in different time scales. Lastly, we consider the local stability theory for both hyperbolic equilibrium points. It is shown in this manuscript that, under suitable conditions, the origin $(0, 0)$ of a planar Caputo dynamical system is a spiral (stable/unstable). More accurately, it is shown that the stability depends on the order of fractional derivative,

$$\alpha \in \mathbb{Q}^* := \{\alpha \in \mathbb{Q} | \alpha = \frac{1}{m}, m = 2, 3, 4, ..., N...\}.$$

2. Preliminaries

The well known Riemann-Liouville (left-side) fractional integral operator of order $\beta \in (0, \infty)$ is given by Volterra integral operator of the form

$$I^{\beta}_{0^+} u(x) := \frac{1}{\Gamma(\beta)} \int_0^x \frac{u(y)}{(x-y)^{1-\beta}} dy \quad \text{for each } x \in [0, b], \tag{1}$$

where $u : [0, b] \to \mathbb{R}$ is a suitable measurable function such that the Lebesgue integral on the right hand side of(1) exists for almost every ($a.e.$) $x \in [0, b]$, and $\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx$ is the standard euler gamma function. Let $n \in \mathbb{N}$. The notation introduced in [25] expresses the operators I^n_{0+} and $I^{n-\alpha}_{0+}$ as nth order Riemann-Liouville integral operator and the nth order Riemann-Liouville fractional integral operator with fraction α, respectively. Additionally, the operator D^α defined by

$$D^\alpha u(x) := (I^{1-\alpha}_{0^+} u)^{'}(x) \quad \text{for each } x \in [0, b] \tag{2}$$

is said to be first the Riemann-Liouville fractional differential operator with fraction α. Adopting the notion introduced in [25], we can write the modified Caputo fractional differential operator with fraction α as,

$$D^\alpha_* u(x) := (I^{1-\alpha}_{0^+} (u - u_0))^{'}(x) \quad \text{for each } x \in [0, b] \tag{3}$$

The advantage, as pointed out in [25], in using the aforementioned notation to express the Riemann-Liouville fractional differential operator with fraction $\alpha \in (0, 1)$ and Caputo fractional differential operator with fraction $\alpha \in (0, 1)$ is to avoid employing the ceiling or floor functions, as is generally done in the literature.

We now introduce the two different types of time, similarly as done so in [24]. Time can be thought off as *homogeneous* or *non-homogeneous*. In order to explain this notion carefully we will introduce the following: let t be the last measured instance of real measurable time. Let, τ be the time, $0 < \tau < t$, and set

$$g_t(\tau) := \frac{1}{\Gamma(1-\alpha)} [t^\alpha - (t-\tau)^\alpha] \tag{4}$$

to represent the *non-homogeneous* timescale. Note, that if we take $\alpha = 1$, then $g_t(\tau) = \tau$ and we have the *homogeneous* timescale. Then, the *non-homogeneous* timescale, is a representation of a "deformed" timescale. That is to say, between each time τ the **actual** amount of time that has elapsed is given by g_t. Thus, we can see that in the deformed timescale the amount of *actual* time that elapses between two time instances, τ_1 and τ_2, is not always the same. As opposed to the *homogeneous* timescale in which the amount of *actual* that elapses between any two time instances is always the same, traditionally 1.

We remark that from (4), we have that

$$I^{\alpha}_{0^+}u(t) = \int_0^t u(\tau)dg_t(\tau), \tag{5}$$

where, instead of integrating over t, we integrated over the function g_t. This is appropriate here, since we assume that the actual timescale is given by g_t instead of just t. In fact, if we take $\alpha = 1$, then (5) reduces to

$$I^{1}_{0^+}u(t) = \int_0^t u(\tau)d\tau,$$

the classical, integer, case. It is important to mention here that, the fractional parameter α is responsible for formation of the timescale, and ultimately the behavior of time. Thus, impacting the evolution dynamics of systems. Indeed, if we consider *non-homogeneous* time, then between two time instances, τ_1 and τ_2, a process could have evolved to a certain stage, but the time that would have elapsed could have been slower than what would have elapsed on a *homogeneous* timescale. Thus, the process would be evolving faster, than originally thought to, relative to the *homogeneous* timescale. This observation is important in the justification of fractional calculus. However, this observation does not clearly distinguish if the overall qualitative dynamics are impacted by the change in timescale. Before we address this topic, it is important to point out that this concept of different timescale leads to an issue regarding fractional integral operators.

In equation (5), it is seen that the Riemann-Liouville fractional integral equation is integrated over a different timescale, the one introduced in (4). However, no justification is given as to why the timescale proposed in (4) is correct, or universally true. Therefore, it is reasonable to assume that another timescale could be a more accurate representation of the natural world. This conclusion then leads to the following observation-Riemann-Liouville fractional integral equations would only have any practical meaning, if the process that they are being used to describe (or model) assumes a *non-homogeneous* timescale that is given in (4). In the next section we introduce the preliminary definitions and results that are needed to proceed to discuss Caputo Fractional Dynamical Systems.

3. Local Stability Theory

In this section we present some background theory, without loss of generality we can take the equilibrium point to be the origin $(0, 0)$, where an equilibrium point is considered to be a constant solution. Here, and in the following sections, we adopt cD^{α}_0 to be the Caputo fractional operator. Additionally, since we are studying the flow from the perspective of dynamical systems, it is sufficient to consider the regular Caputo derivative, and not the modified one. Consider the Caputo planar system below

$$\begin{cases} cD^{\alpha}_0 x(t) = f(x(t), y(t)), \\ cD^{\alpha}_0 y(t) = g(x(t), y(t)), \end{cases} \tag{6}$$

subject to the initial condition:

$$(x(0), y(0)) = (x_0, y_0)$$

where $\alpha \in (0, 1)$, $f, g \in C^1(\mathbb{R}^2)$, and we are looking for solutions x, and y such that x and y are absolutely continuous.

Since, $f, g \in C^1(\mathbb{R}^2)$, it is well known that for any $(x_0, y_0) \in \mathbb{R}^2$ the initial value problem (6) has a unique solution, see [12].

We denote by $A(x, y)$ the Jacobian matrix of f and g at (x, y), that is,

$$A(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{7}$$

and by $|A(x, y)|$ and $\operatorname{tr}(A(x, y))$ the determinant and trace of $A(x, y)$, respectively.

Definition 3.1. A point $(x^*, y^*) \in \mathbb{R}^2$ is called an equilibrium point of (6) if $f(x^*, y^*) = g(x^*, y^*) = 0$.

Below we define the linearized system of (6) about the equilibrium point (x^*, y^*).

Definition 3.2. Let A be the matrix defined in (7) is evaluated at the equilibrium point (x^*, y^*). Then,

$$cD_0^\alpha X = A^* X, \tag{8}$$

where $X = (x, y)^T$, is the linearization of system (6) at the equilibrium point (x^*, y^*).

Definition 3.3. Let $\alpha > 0$, and $t \in \mathbb{R}^+$. The function E_α, defined by

$$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + \beta)}, \tag{9}$$

whenever the series converges is called the Mittag-Leffler function.

Definition 3.4. Two functions are asymptotically equivalent if

$$\lim_{t \to \infty} \frac{f(t)}{g(t)} = 1.$$

Written as $f \approx_\infty g$.

Below we introduce a special case of the result obtained in [11], chapter 1.

Lemma 3.1. *If* $0 < \alpha < 1$, $\beta = 1$ *and* $\mu \in \mathbb{R}$ *such that*

$$\alpha \frac{\pi}{2} < \mu < \alpha\pi. \tag{10}$$

Then, $E_\alpha(z) \approx_\infty \frac{1}{\alpha} e^{z^{1/\alpha}}$, *as* $|z| \to \infty$ *and* $|arg(z)| \leq \mu$.

We introduce a new result that will become useful in determining the qualitative behaviour of hyperbolic equilibrium points.

Lemma 3.2. *Let* $c \in \mathbb{C}$ *such that* $c = a + ib$, *where* $a, b \neq 0$. *Let* $0 < \alpha < 1$, *such that* $\alpha \in \mathbb{Q}^* := \{\alpha \in \mathbb{Q} | \alpha = \frac{1}{m}, m = 2, 3, 4, \cdots, \mathbb{N}, \cdots\}$. *Then,*

$$E_\alpha(ct^\alpha) \approx_\infty \eta e^{\omega t}[\cos(\psi t) + i \sin(\psi t)], \quad \text{for } |\arg(c)| \leq \mu, \tag{11}$$

where μ *is defined in* (10), $\eta := \frac{1}{\alpha}$, *and*

$$\omega + i\psi := \binom{\eta}{0} a^\eta + \binom{\eta}{1} a^{\eta-1}(ib) + ... + (ib)^\eta, \quad \text{for some } \omega,\ \psi \in \mathbb{R}. \tag{12}$$

Proof. Let $c \in \mathbb{C}$, such that $c = a+ib$, where $b \neq 0$. Then, there exists a $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$ such that a $|\arg(c)| \leq \mu$. Let $0 < \alpha < 1$, such that $\alpha \in \mathbb{Q}^*$. By Lemma 3.1, we have

$$E_\alpha(ct^\alpha) \approx_\infty \frac{1}{\alpha} e^{(ct^\alpha)^{1/\alpha}}, \quad \text{as } t \to \infty \text{ and } |arg(c)| \leq \mu.$$

Since, $0 < \alpha < 1$, such that $\alpha \in \mathbb{Q}^*$. Then, $\eta \in \mathbb{Z}_+$ and,

$$\begin{aligned}
(1/\alpha)e^{(ct^\alpha)^{1/\alpha}} &= \alpha_1 e^{(a+ib)^\eta t} \\
&= \eta e^{\left(\sum_{j=0}^{\eta} \binom{n}{j} a^\eta (ib)^{\eta-j}\right)t} \\
&= \eta e^{\left(\binom{\eta}{0} a^\eta + \binom{\eta}{1} a^{\eta-1}(ib) + \ldots + (ib)^\eta\right)t} \\
&= \eta e^{\left(\binom{\eta}{0} a^\eta + \binom{\eta}{1} a^{\eta-1}(ib) + \ldots + (ib)^\eta\right)t} \\
&= \eta e^{(\omega+i\psi)t} = \eta e^{\omega t}\left(\cos(\psi t) + i \sin(\psi t)\right), \qquad (13)
\end{aligned}$$

where $\omega + i\psi = \binom{\eta}{0} a^\eta + \binom{\eta}{1} a^{\eta-1}(ib) + \ldots + (ib)^\eta$, for some ω, $\psi \in \mathbb{R}$. Hence,

$$E_\alpha(ct^\alpha) \approx_\infty \eta e^{\omega t}[\cos(\psi t) + i \sin(\psi t)], \quad \text{for } |\arg(c)| \leq \mu.$$

□

Remark 3.1. Lemma 3.2 is new. It shows that the Mittag-Leffler function behaves in an oscillatory manner, asymptotically. This result is crucial for the main result of this section.

Lemma 3.3. *Let $\alpha \in (0, 1)$, and $t \in \mathbb{R}^+$. Then, E^1_α represents the first derivative of the Mittag-Leffler function defined by (9), and*

$$E^1_\alpha(t) = \sum_{j=0}^{\infty} \frac{(j+1)t^j}{\Gamma(1+\alpha+\alpha j)}. \qquad (14)$$

Lemma 3.4. *Let $\alpha > 0$, and $t \in \mathbb{R}^+$. Then, $E_\alpha \in AC[a, b]$ for all $t \in [a, b]$.*

Proof. We will show that

$$E_\alpha(t) = E_\alpha(a) + \int_a^t E^1_\alpha(s)ds \quad \text{for all } a \leq t \leq b.$$

Indeed,

$$\begin{aligned}
\int_a^t E_\alpha^1(s)ds &= \int_0^t \left(\sum_0^\infty \frac{(j+1)s^j}{\Gamma(1+\alpha+\alpha j)} \right) ds \\
&= \int_0^t \left(\frac{1}{\Gamma(1+\alpha)} + \frac{2s}{\Gamma(1+2\alpha)} + \frac{3s^2}{\Gamma(1+3\alpha)} + ... + \frac{(j+1)s^j}{\Gamma(1+(j+1)\alpha)} + ... \right) ds \\
&= \left(\frac{t}{\Gamma(1+\alpha)} \right)\Big|_a^t + \left(\frac{t^2}{\Gamma(1+2\alpha)} \right)\Big|_a^t + \left(\frac{t^3}{\Gamma(1+3\alpha)} \right)\Big|_a^t + ... + \left(\frac{t^j}{\Gamma(1+\alpha j)} \right)\Big|_a^t + ... \\
&= \sum_{j=0}^\infty \frac{t^j}{\Gamma(\alpha j+\beta)} - \sum_{j=0}^\infty \frac{a^j}{\Gamma(\alpha j+\beta)} \\
&= E_\alpha(t) - E_\alpha(a).
\end{aligned}$$

Hence,

$$E_\alpha(t) = E_\alpha(a) + \int_a^t E_\alpha^1(s)ds \quad \text{for all } a \le t \le b.$$

and by definition, $E_\alpha \in AC[a,b]$ for all $a \le t \le b$.

□

Remark 3.2. Lemma 3.4 is new. It shows that the Mittag-Leffler function, $E_\alpha \in AC[a,b]$ for all $a \le t \le b$. This property of the Mittag-Leffler function defined in (9) has not been mentioned by any authors in the past. However, it is well known that the solution space of the Caputo derivative is in AC. Additionally, the solution to the initial value problem (6) where f, and g are linear is given by a linear combination of the Mittag-Leffler function defined in (9). Thus, for the solution to the linear Caputo dynamical system to be well-defined, it must satisfy $E_\alpha \in AC[a,b]$ for all $a \le t \le b$.

Below we introduce a couple of well known results regarding the stability theory for Caputo dynamical systems. The following Lemma is a special case ($n = 2$) of Lemma 3.2 in[9].

4. Hyperbolic Equilibrium Points

In this section we introduce a couple of well-known results for the local stability theory for (6), and we propose a new result, Theorem 4.1, for the local stability of (6) for the case $\alpha \in \mathbb{Q}^*$.

Below we introduce a couple of well known results regarding the stability theory for Caputo dynamical systems. The following Lemma is a special case ($n = 2$) of Lemma 3.2 in[9].

Lemma 4.1. *Let (x^*, y^*) be an equilibrium point of* (6) *and A be defined as in* (7)*. Let λ_1 and λ_2 be the eigenvalues of A. Then, the following assertions hold.*

(1) *The equilibrium point (x^*, y^*) is locally asymptotically stable if and only if $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$.*

(2) *The equilibrium point (x^*, y^*) is stable if and only if $|arg(\lambda_{1,2})| \ge \frac{\alpha\pi}{2}$ and the eigenvalues with $|arg(\lambda_{1,2})| = \frac{\alpha\pi}{2}$ have the same geometric multiplicity and algebraic multiplicity.*

(3) *The equilibrium point (x^*, y^*) is unstable if and only if $|arg(\lambda_{1,2})| < \frac{\alpha\pi}{2}$.*

The following result follows from Lemma 3, where the conditions are expressed in terms of $tr(A(x^*, y^*))$, and $|A(x^*, y^*)|$.

Lemma 4.2. *If (x^*, y^*) is a equilibrium point of (6), then the following assertions hold.*

(i) *If $|A(x^*, y^*)| < 0$, then (x^*, y^*) is unstable. (6).*
(ii) *If $|A(x^*, y^*)| > 0$, $tr(A(x^*, y^*)) > 0$ and $(tr(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| \geq 0$, then (x^*, y^*) is unstable.*
(iii) *If $|A(x^*, y^*)| > 0$, $tr(A(x^*, y^*)) < 0$, then (x^*, y^*) is Locally Asymptotically stable.*

The following result was obtained in [6]. It states that the local stability of hyperbolic equilibrium points of (6) are topologically equivalent to that of system (8).

Lemma 4.3. *If the origin $(0, 0)$ is a hyperbolic equilibrium point of (6), then vector field $(f(x, y), g(x, y))$ is topologically equivalent with its linearization vector field given by the linear system $cD_0^\alpha X = AX$ in the neighborhood of the origin $(0, 0)$.*

Remark 4.1. If (x^*, y^*) is an equilibrium point of (8) such that $(x^*, y^*) \neq (0, 0)$. Then, the equilibrium point,(x^*, y^*), can be translated to the origin.

With the basis of stability introduced above, we now present the main result of this section.

Theorem 4.1. *Let (x^*, y^*) be an equilibrium point of (6). Let λ_1 and λ_2 be eigenvalues of the matrix A defined in (7) and suppose that $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$, with $b \neq 0$. Then, if $|A(x^*, y^*)| > 0$, $tr(A(x^*, y^*)) > 0$ and $(tr(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| < 0$, then (x^*, y^*) is an unstable focus of (6) for $\alpha \in \mathbb{Q}^* \cap (\alpha^*, 1)$; stable focus of (6) for $\alpha \in \mathbb{Q}^* \cap (0, \alpha^*)$.*

Proof. By Lemma 4.3 we can study (8) to determine the qualitative behavior of the equilibrium point (x^*, y^*). If $(x^*, y^*) \neq (0, 0)$, then we can use the following substitution

$$x_1 = x - x^*, \quad y_1 = y - y^*,$$

to translate the equilibrium point (x^*, y^*) to the origin. In fact, $cD_0^\alpha x_1(t) = cD_0^\alpha(x_1(t) - x^*)$, and $cD_0^\alpha y_1(t) = cD_0^\alpha(y_1(t) - y^*)$. Since, x^* and y^* are both constant.

Define $|\arg(\lambda_{1,2})|$ to be the argument of the eigenvalues λ_1, and λ_2, which are equal since the eigenvalues are complex conjugates. Additionally, since $b \neq 0$, then there exists a $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$ such that $|\arg(\lambda_{1,2})| \leq \mu$. Moreover, $\lambda_1 \neq \lambda_2$, thus the general solution, $X(t) := (x(t), y(t))$, to (8) can be expressed as follows

$$X(t) = c_1 u_1 E_\alpha(\lambda_1 t^\alpha) + c_2 u_2 E_\alpha(\lambda_2 t^\alpha), \tag{15}$$

where $c_1, c_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{R}^2$ are the eigenvectors corresponding to λ_1 and λ_2, respectively.

Furthermore, by a direct application of Lemma 3.2, we have that

$$X(t) \approx_\infty c_1 u_1 \Big[\eta e^{\omega t} (\cos(\psi t) + i \sin(\psi t)) \Big] \\ + c_2 u_2 \Big[\eta e^{\omega t} (\cos(\psi t) - i \sin(\psi t)) \Big] \tag{16}$$

Thus, from (16) we can see that the solutions, can be expressed strictly as real valued solutions, similarly as the classical case. In addition, the solutions behave in an oscillatory manner, qualitatively.

Next, recall that the eigenvalues for a planar system can can be represented as

$$\lambda_1 = \frac{tr(A(x,y)) + \sqrt{tr(A(x,y))^2 - 4\det(A(x,y))}}{2},$$
$$\lambda_2 = \frac{tr(A(x,y)) - \sqrt{tr(A(x,y))^2 - 4\det(A(x,y))}}{2}.$$

(1) If $tr(A(x,y)) > 0$, and $tr(A(x,y))^2 - 4\det(A(x,y)) < 0$, then the eigenvalues are complex conjugates. Note that the eigenvalues are independent of α, so $|\arg(\lambda_{1,2})|$ is fixed. Specifically, the term $\alpha^* = \frac{2}{\pi}|\arg(\lambda_{1,2})|$ is fixed.

(i) Suppose that $\alpha \in \mathbb{Q}^* \cap (\alpha^*, 1)$ then $|\arg(\lambda_{1,2})| < \frac{\alpha\pi}{2} \le \mu < \alpha\pi$. From (16) the solution has a oscillatory behaviour. This together with Lemma 4.2 allows us to conclude that (x^*, y^*) is an unstable focus.

(ii) Suppose that $\alpha \in \mathbb{Q}^* \cap (0, \alpha^*)$ then there exists a $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$ such that $\frac{\alpha\pi}{2} < |\arg(\lambda_{1,2})| \le \mu < \alpha\pi$. From (16) the solution has a oscillatory behavior. This together with Lemma 4.2 allows us to conclude that (x^*, y^*) is a stable focus. □

Remark 4.2. In Theorem 4.1 we provide a new method for determining the qualitative behavior of solutions near equilibrium point (x^*, y^*). In particular, we use the asymptotic expansion properties of the Mittag-Leffler functions to achieve the results. The case where $\alpha = \alpha^*$ is not treated in Theorem 4.1, thus all that can be concluded for $\alpha = \alpha^*$, is that the the equilibrium point (x^*, y^*) is stable., see Lemma 4.1. In fact, the condition $\alpha = \alpha^*$, has been misrepresented in the literature, see [19]. The authors in [19] claimed that the equilibrium point (x^*, y^*) is a stable node, if $\alpha = \alpha^*$. This claim is also not correct, in fact this would require an additional constraint on the equilibrium point. namely, that it is locally asymptotically stable, from Lemma 4.1 this is not the case. Additionally, Theorem (4) (f) in [19] is not correct. In fact, the author states that if all the eigenvalues are complex and satisfy $|\arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$, then the equilibrium point (x^*, y^*) is a stable focus of (6). However, this can only be concluded in its entirety for the linear system (8), provided that the equilibrium point is hyperbolic (8) has no zero eigenvalues). Indeed, consider the case when the complex eigenvalues have a zero real part, then $|\arg(\lambda_{1,2})| = \frac{\pi}{2} \ge \frac{\alpha\pi}{2}$. However, since the eigenvalues zero real parts, then this equilibrium point is a non hyperbolic equilibrium point, and the linearization Lemma 4.3 does not apply.

5. Conclusion

To summarize, in this manuscript we present a couple of new results for the local stability theory of both hyperbolic equilibrium points. In section 4, we show that the solution of a hyperbolic equilibrium point, in a neighbourhood of the equilibrium point, behave in a spiral manner. This result is new, and it extends the behaviour observed in the classical dynamical systems, as demonstrated in [21], to the Caputo kind. This extension seems natural due to the continuity property of the Caputo derivative. However, as shown in Theorem 4.1, the result depends heavily on α, and as such we can only conclude this for $\alpha \in \mathbb{Q}^*$. The reason for this, is due to the use of Lemma 3.2 in the proof of Theorem 4.1.

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Marvin Hoti

Faculty of Mathematics, Ryerson University, Toronto, Canada

E-mail address: marvin.hoti@ryerson.ca