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A STUDY ON A COUPLED SYSTEM OF QUADRATIC VOLTERRA-STIELTJES INTEGRAL EQUATIONS

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Abstract. The main purpose of this paper is to investigate some existence results for a coupled system of nonlinear quadratic integral equations of Volterra-Stieltjes type in the space of continuous functions defined on a closed bounded interval. Our proof depends on the Schauder fixed point principle, such an approach allows us to obtain existence theorems under rather general assumptions.

1. Introduction and Preliminaries

It is well known that integral equations have many useful applications in describing numerous events and problems of real world, and the theory of integral equations is rapidly developing with the help of several tools of functional analysis, topology and fixed point theory (see [1, 12, 13, 14, 15, 18]). The interest in the study of quadratic Volterra-Stieltjes integral equations was initiated mainly by the papers [7, 9, 19, 20, 21, 22].

The goal of the paper is to discuss the solvability of a certain class of coupled system of quadratic Integral equations of Volterra-Stieltjes type. For the definition, background, and properties of the Stieltjes integral, we refer to Banaś [6]. The interest in the study of such coupled system of integral equations was initiated mainly by the papers (see [4, 3].

In this paper we prove some existence theorems for a coupled system of quadratic Volterra-Stieltjes integral equations containing numerous types of Volterra integral equations as special cases. We investigate solvability of those coupled system of integral equations in the space of continuous functions defined on a closed bounded interval. The main tool used in our considerations is the Schauder fixed point principle. Such a proof enables us to obtain our existence results under quite general and convenient assumptions. Moreover, we generalize our results to the indicate existence result of the coupled system of quadratic Hammerstien-Stieltjes integral equations.

The results proved in this paper generalize several ones obtained up to now for various types of nonlinear Volterra integral equations like the coupled system of quadratic Volterra integral equations of fractional order, the coupled system of quadratic Volterra-Chandrasekhar type and nonlinear equations of mixed type, also, we deduce existence theorems for coupled systems of Cauchy problems.

Throughout this paper, let $I = [0, T]$ and X be the Banach space of all ordered pairs $(x, y) \in X = C(I) \times C(I)$, with the norm

$$\|(x, y)\|_X = \max\{\|x\|_C, \|y\|_C\},$$

where

$$\|x\|_C = \sup_{t\in I} |x|,\ \ \|y\|_C = \sup_{t\in I} |y|.$$

It is clear that $(X, \|(x, y)\|_X)$ is Banach space.

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Now, we shall present some auxiliary properties of fractional calculus that will be need in this work.

Definition 1. *The Riemann-Liouville of a fractional integral of the function $f \in L^1(I)$ of order $\alpha \in R^+$ is defined by*

$$I_a^\alpha\ f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ ds.$$

and when $a = 0$, we have $I^\alpha\ f(t) = I_0^\alpha\ f(t)$.

Definition 2. *The (Caputo) fractional order derivative D^α, $\alpha \in (0,1]$ of the absolutely continuous function g is defined as*

$$D_a^\alpha\ g(t) = I_a^{1-\alpha}\ \frac{d}{dt}\ g(t)\ ,\ t \in [a,b].$$

For further properties of fractional calculus operator (See [26],[27],[28] *and* [29]*).*

2. Main results

The main object of the study in this paper is the solvability of the coupled system of nonlinear quadratic integral equations of Volterra-Stieltjes type having the form:

$$\begin{aligned} x(t) &= a_1(t) + f_1(t, y(t)) \int_0^t\ v_1(t,s,y(s))\ d_s g_1(t,s),\ t \in I \\ y(t) &= a_2(t) + f_2(t, x(t)) \int_0^t\ v_2(t,s,x(s))\ d_s g_2(t,s).\ t \in I \end{aligned} \tag{1}$$

Our goal is to show that system (1) has at least one solution in the space X. For our further purposes we denote by $\triangle$ the triangle $\triangle = \{(t,s) : 0 \le s \le t \le T\}$.
In our investigations, we assume that the following conditions are satisfied

(i) $a_i : I \to R$, $(i = 1,2)$ are continuous functions on I. There are constants a_i, where $a_i = \sup_{t \in I} |ai(t)|$.

(ii) $f_i : I \times R \to R, (i = 1,2)$ are continuous functions and there exist continuous functions $m_i(t) : I \to I$ such that

$$|f_i(t,x) - f_i(t,y)| \le m_i(t)|x - y|,$$

for all $x, y \in R$ and $t \in I$. Moreover, we put $m_i = \max\{m_i(t)\ t \in I, \}$.

(iii) $v_i(t,s,x) : \triangle \times R \to R$,(i = 1, 2) are continuous functions and there exist continuous functions $n_i(t,s) : \triangle \to I$, and continuous and nondecreasing functions $\varphi_i : R_+ \to R_+$, such that

$$|v_i(t,s,x)| \le n_i(t,s)\phi_i(|x|),$$

for all $(t,s) \in \triangle$ and $x \in R$. Moreover, we put $n_i = \max\{n_i(t,s)\ t, s \in I\}$.

(iv) Functions $s \to g_i(t,s)$ are of bounded variation on $[0,t]$ for each $t \in I$, $i = 1,2$.

(v) Functions g_i are continuous on the triangle $\triangle$ and $g_i(t,0) = 0$ for $i = 1,2$.

(vi) $g_i(t,s) = g_i : \triangle_i \to R$, $i = 1,2$ and for all t_1, $t_2 \in I$ with $t_1 < t_2$, the functions $s \to g_i(t_2,s) - g_i(t_1,s)$ are nondecreasing on I.

(vii) For any $\epsilon > 0$ there exists $\delta > 0$ such that, for all $t_1; t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \le \delta$ the following inequality holds

$$\bigvee_{s=0}^{t_1} [g_i(t_2,s) - g_i(t_1,s)] \le \epsilon,\ i = 1,2.$$

Obviously, we will assume that $g_i, (i = 1,2)$ satisfy assumptions $(iv) - (vii)$. For our purposes, we will need the following lemmas.

Lemma 1. [8] *Under assumptions (iv)-(vii), The functions $z \to \bigvee_{s=0}^{z} g_i(t,s)$ are continuous on $[0,t]$ for any any $t \in I$ (i=1,2).*

Lemma 2. [8] *Let assumptions (iv)-(vii) be satisfied. Then, for arbitrary fixed number* $0 < t_2 \in I$ *and for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $t_1 \in I$; $t_1 < t_2$ *and* $t_2 - t_1 \leq \delta$ *then* $\bigvee_{s=t_1}^{t_2} g_i(t_2, s) \leq \epsilon$, *(i=1,2).*

Further, let us observe that based on Lemma 1 we infer that there exists finite positive constants K_i, such that

$$K_i = \sup\left\{ \bigvee_{s=0}^{t} g_i(t,s) : \ t \in [0,T] \right\},$$

where $T > 0$ is arbitrarily fixed $(i = 1, 2)$.
We now introduce some functions that will be useful in our further studies:

$$W_i(\epsilon) = \sup\{ \bigvee_{s=0}^{t_2} (g_i(t_2, s) - g_i(t_1, s)) : t_1, t_2 \in I, \ t_1 < t_2; \ t_1 - t_2 \leq \epsilon, \ i = 1, 2\}.$$

In what follows let us denote by F_i the constant defined by the formula:

$$F_i = \sup\{|f_i(t, 0)| : t \in I, \ i = 1, 2\}.$$

Now, we are in position to present tha main result of the paper.

Theorem 1. *Let assumptions* (i)- (vii) *be satisfied. Then the coupled system of quadratic Volterra-Stieltjes integral equations* (1) *has at least one solution* (x, y) *belonging to the space* X.

Proof. Consider the operator A by putting

$$A(x, y)(t) = (A_1 y(t), A_2 x(t)),$$

where

$$\begin{aligned} A_1 y(t) &= a_1(t) + f_1(t, y(t)) \int_0^t \ v_1(t, s, y(s) \ d_s g_1(t, s) \\ A_2 x(t) &= a_2(t) + f_2(t, x(t)) \int_0^t \ v_2(t, s, x(s) \ d_s g_2(t, s). \end{aligned} \tag{2}$$

We prove a few results concerning the continuity and compactness of these operators in the space of continuous functions. We define the set U by

$$U = \{u = (x(t), y(t)); \ (x(t), y(t)) \in X : \|(x, y)\|_X \leq r\}.$$

For $(x, y) \in U$, and define The operator A map U into U, we have

$$\begin{aligned} |A_1 y(t)| &\leq |a_1(t)| + |f_1(t, y(t))| \int_0^t \ |v_1(t, s, y(s)| \ |d_s g_1(t, s)| \\ &\leq \|a_1\| + [m_1(t)|y(t)| + |f_1(t, 0)|] \int_0^t \ n_1(t, s) \ \varphi_1(|y(s)|) \ d_s \big(\bigvee_{s=0}^{t} g_1(t, p) \big) \\ &\leq \|a_1\| + \big[\|y\| m_1 + F_1 \big] \ n_1 \varphi_1(\|y\|) | \big(\bigvee_{s=0}^{t} g_1(t, p) \big), \\ \|A_1 y\| &\leq \|a_1\| + \big[r_1 m_1 + F_1 \big] n_1 \varphi_1(r_1) \sup_{t \in I} \big(\bigvee_{s=0}^{t} g_1(t, p) \big). \end{aligned}$$

Hence, we get

$$\|A_1 y\| \leq \|a_1\| + K_1 \big[m_1 r_1 + F_1 \big] n_1 \varphi_1(r_1).$$

From the last estimate we deduce that $r_1 = \frac{\|a_1\| + F_1 K_1 n_1 \varphi_1(r_1)}{1 - m_1 n_1 K_1 \varphi_1(r_1)}$.
By a similar way, we obtain

$$\|A_2 x\| \leq \|a_2\| + K_2 \big[m_2 r_2 + F_2 \big] n_2 \phi_2(r_2), r_2 = \frac{\|a_2\| + F_2 K_2 n_2 \varphi_2(r_2)}{1 - m_2 n_2 K_2 \varphi_2(r_2)}.$$

Therefore

$$\begin{aligned} ||Au||_X &= ||A(x, y)||_X = ||(A_1 y, A_2 x)||_X \\ &\leq \max_{t \in I} \{||A_1 y||_C, ||A_2 x||_C\} = r. \end{aligned}$$

Thus for every $u = (x, y) \in U$, we have $Au \in U$ and hence $AU \subset U$, (i.e $A : U \to U$). This means that the functions of AU are uniformly bounded on I, it is clear that the set U is nonempty, bounded, closed and convex. Now, we need to show that the set AU is relatively compact.
For $u = (x, y) \in U$, for all $\epsilon > 0$, $\delta > 0$, and for each $t_1, t_2 \in I$(without loss of generality assume that $t_1 < t_2$), such that $|t_2 - t_1| < \delta$, we have:

$$
\begin{aligned}
& |A_1y(t_2) - A_1y(t_1)| \\
= \;& |a_1(t_2) - a_1(t_1) + f_1(t_2, y(t_2)) \int_0^{t_2} v_1(t_2, s, y((s))\; d_s g_1(t_2, s) \\
- \;& f_1(t_1, y(t_1)) \int_0^{t_1} v_1(t_1, s, y(s))\; d_s g_1(t_1, s)| \\
\leq \;& |a_1(t_2) - a_1(t_1)| + |f_1(t_2, y(t_2)) - f_1(t_2, y(t_1))| \int_0^{t_2} |v_1(t_2, s, y(s))\; |d_s g_1(t_2, s)| \\
+ \;& |f_1(t_2, y(t_1)) \int_0^{t_2} v_1(t_2, s, y(s))\; d_s g_1(t_2, s) - f_1(t_1, y(t_1)) \int_0^{t_2} v_1(t_2, s, y(s))\; d_s g_1(t_2, s)| \\
+ \;& |f_1(t_1, y(t_1)) \int_0^{t_2} v_1(t_2, s, y(s))\; d_s g_1(t_2, s) - f_1(t_1, y(t_1)) \int_0^{t_2} v_1(t_2, s, y(s))\; d_s g_1(t_1, s)| \\
+ \;& |f_1(t_1, y(t_1)) \int_0^{t_2} v_1(t_2, s, y(s))\; d_s g_1(t_1, s) - f_1(t_1, y(t_1)) \int_0^{t_2} v_1(t_1, s, y(s))\; d_s g_1(t_1, s)| \\
+ \;& |f_1(t_1, y(t_1)) \int_0^{t_2} v_1(t_1, s, y(s))\; d_s g_1(t_1, s) - f_1(t_1, y(t_1)) \int_0^{t_1} v_1(t_1, s, y(s))\; d_s g_1(t_1, s)| \\
\leq \;& \omega(a_1, \epsilon) + m_1(t_2)|y(t_2) - y(t_1)| \int_0^{t_2} |v_1(t_2, s, y(s))|\; d_s\big(\bigvee_{p=0}^{s} g_1(t_2, p)\big) \\
+ \;& |f_1(t_2, y(t_1)) - f_1(t_1, y(t_1))| \int_0^{t_2} |v_1(t_2, s, y(s))|\; d_s\big(\bigvee_{p=0}^{s} g_1(t_2, p)\big) \\
+ \;& |f_1(t_1, y(t_1))| \int_0^{t_2} |v_1(t_2, s, y(s))|\; d_s\big(\bigvee_{p=0}^{s} [g_1(t_2, p) - g_1(t_1, p)]\big) \\
+ \;& |f_1(t_1, y(t_1))| \int_0^{t_2} |v_1(t_2, s, y(s)) - v_1(t_1, s, y(s))|\; d_s\big(\bigvee_{p=0}^{s} g_1(t_1, p)\big) \\
+ \;& |f_1(t_1, y(t_1))| \int_{t_1}^{t_2} |v_1(t_1, s, y(s))|\; d_s\big(\bigvee_{p=t_1}^{s} g_1(t_1, p)\big) \\
\leq \;& \omega(a_1, \epsilon) + m_1(t_2)\omega(y, \epsilon) \int_0^{t_2} n_1(t_2, s)\; \varphi_1(|y(s)|) d_s(\bigvee_{p=0}^{s} g_1(t_2, p)) \\
+ \;& \omega_{f_1}(\epsilon) \int_0^{t_2} n_1(t_2, s)\; \varphi_1(|y(s)|) d_s(\bigvee_{p=0}^{s} g_1(t_2, p)) \\
+ \;& \big[m(t_1)|y(t_1)| + |f_1(t_1, 0)|\big] \int_0^{t_2} n_1(t_2, s)\; \varphi(|y(s)|) d_s(\bigvee_{p=0}^{s} [g_1(t_2, p) - g_1(t_1, p)] \\
+ \;& \big[m(t_1)|y(t_1)| + |f_1(t_1, 0)|\big] \int_0^{t_2} \omega_{v_1}(\epsilon)\; d_s(\bigvee_{p=0}^{s} g_1(t_1, p)) \\
+ \;& \big[m(t_1)|y(t_1)| + |f_1(t_1, 0)|\big] \int_{t_1}^{t_2} n_1(t_1, s)\; \varphi(|y(s)|) d_s(\bigvee_{p=0}^{s} g_1(t_1, p))
\end{aligned}
$$

where

$$\begin{aligned}
\omega(a_i,\epsilon) &= \sup\{|a_i(t_2)-a_i(t_1)|:\ t_1,t_2\in I,\ |t_1-t_2|\le\epsilon,\ i=1,2\},\\
\omega_{f_i}(\epsilon) &= \sup\{|f_i(t_2,u)-f_i(t_1,u)|:\ t_1,t_2\in I,\ |t_1-t_2|\le\epsilon,\ u\in R,\ i=1,2\},\\
\omega_{v_i}(\epsilon) &= \sup\{|v_i(t_2,s,u(s))-v_1(t_1,s,u(s))|:\ t_1,t_2\in I,\ |t_1-t_2|\le\epsilon, u\in R,\ i=1,2\}.
\end{aligned}$$

Then, form estimate we get

$$\begin{aligned}
|A_1y(t_2)-A_1y(t_1)| &\le \omega(a_1,\epsilon)+[m_1\omega(y,\epsilon)+\omega_{f_1}(\epsilon)]n_1\phi(\|y\|)\int_0^{t_2} d_s(\bigvee_{p=0}^{s} g_1(t_2,p))\\
&+ \big[m_1\|y\|+F_1\big]\Big[n_1\phi(\|y\|)\int_0^{t_2} d_s(\bigvee_{p=0}^{s}[g_1(t_2,p)-g_1(t_1,p)]\\
&+ \omega_{v_1}(\epsilon)\int_0^{t_2} d_s(\bigvee_{p=0}^{s} g_1(t_1,p)+n_1\phi(\|y\|)\int_{t_1}^{t_2} d_s(\bigvee_{p=0}^{s} g_1(t_1,p)\Big]\\
&\le \omega(a_1,\epsilon)+\big[m_1\omega(y,\epsilon)+\omega_{f_1}(\epsilon)\big]n_1\phi(\|y\|)\bigvee_{s=0}^{t_2} g_1(t_2,p))\\
&+ \big[m_1\|y\|+F_1\big]\Big[n_1\varphi(\|y\|)\bigvee_{s=0}^{t_2}[g_1(t_2,s)-g_1(t_1,s)]\\
&+ \omega_{v_1}(\epsilon)\bigvee_{s=0}^{t_2} g_1(t_1,s)+n_1\varphi(\|y\|)\bigvee_{s=t_1}^{t_2} g_1(t_1,s)]\Big]\\
&\le \omega(a_1,\epsilon)+K_1\big[m_1\omega(y,\epsilon)+\omega_{f_1}(\epsilon)\big]n_1\varphi(r)\\
&+ \big[m_1r+F_1\big]\Big[n_1\varphi(r)W_1(\epsilon)+\omega_{v_1}(\epsilon)[g_1(t_1,t_2)-g_1(t_1,0)]\\
&+ n_1\phi(r)[g_1(t_1,t_2)-g_1(t_1,t_1)]\Big]
\end{aligned}$$

Hence, from the continuity of the functions g_1 assumption (v), we deduce that A_1 maps $C(I)$ into $C(I)$. As done above we can obtain

$$\begin{aligned}
\|A_2x(t_2)-A_2x(t_1)\| &\le \omega(a_2,\epsilon)+K_2\big[m_2\omega(x,\epsilon)+\omega_{f_2}(\epsilon)\big]n_2\varphi(r)\\
&+ \big[m_2r+F_2\big]\Big[n_2\varphi(r)W_2(\epsilon)+\omega_{v_2}(\epsilon)[g_2(t_1,t_2)-g_2(t_1,0)]\\
&+ n_2\varphi(r)[g_2(t_1,t_2)-g_2(t_1,t_1)]\Big]
\end{aligned}$$

Also, by our assumption (v), we see that A_2 maps $C(I)$ into $C(I)$.
Now, from the definition of the operator A we get

$$\begin{aligned}
Au(t_2)-Au(t_1) &= A(x,y)(t_2)-A(x,y)(t_1)\\
&= \big(A_1y(t_2),A_2x(t_2)\big)-\big(A_1y(t_1),A_2x(t_1)\big)\\
&= \big(A_1y(t_2)-A_1y(t_1),A_2x(t_2)-A_2x(t_1)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Au(t_2)-Au(t_1)\| &= \|(A_1y(t_2)-A_1y(t_1),A_2x(t_2)-A_2x(t_1))\| \\
&= \max\left\{\|A_1y(t_2)-A_1y(t_1)\|,\|A_2x(t_2)-A_2x(t_1)\|\right\} \\
&\leq \max\Big\{\omega(a_1,\epsilon)+K_1\big[m_1\omega(y,\epsilon)+\omega_{f_1}(\epsilon)\big]n_1\varphi(r) \\
&+ \big[m_1r+F_1\big]\Big[n_1\varphi(r)W_1(\epsilon)+\omega_{v_1}(\epsilon)[g_1(t_1,t_2)-g_1(t_1,0)] \\
&+ n_1\phi(r)[g_1(t_1,t_2)-g_1(t_1,t_1)]\Big], \\
&\quad \omega(a_2,\epsilon)+K_2\big[m_2\omega(x,\epsilon)+\omega_{f_2}(\epsilon)\big]n_2\varphi(r) \\
&+ \big[m_2r+F_2\big]\Big[n_2\varphi(r)W_2(\epsilon)+\omega_{v_2}(\epsilon)[g_2(t_1,t_2)-g_2(t_1,0)] \\
&+ n_2\varphi(r)[g_2(t_1,t_2)-g_2(t_1,t_1)]\Big]\Big\}.
\end{aligned}$$

This means that the class of $\{Au(t)\}$ is equi-continuous on I, then by Arzela-Ascoil theorem $\{Au(t)\}$ is relatively compact.

Now, we will show that the operator $A:U\to U$ is continuous.

Firstly, we prove that A_1 is continuous. Let $\epsilon^*>0$, the continuity of $v_i,\ i=1,2$, yields, $\exists\ \delta=\delta(\epsilon^*)$ such that $|v_i(t,s,u(s)-v_i(t,s,v(s)|<\epsilon^*$,whenever $\|u-v\|\leq\delta$, thus if $\|y-v\|\leq\delta$, we arrive at:

$$\begin{aligned}
& |(A_1y)(t)-(A_1v)(t)| \\
\leq\ & |f_1(t,y(t))\int_0^t\ v_1(t,s,y(s)\ d_sg_1(t,s)-f_1(t,v(t))\int_0^t\ v_1(t,s,v(s))\ d_sg_1(t,s)| \\
\leq\ & |f_1(t,y(t))\int_0^t\ v_1(t,s,v(s)\ d_sg_1(t,s)-f_1(t,v(t))\int_0^t\ v_1(t,s,y(s)\ d_sg_1(t,s)| \\
+\ & f_1(t,v(t))\int_0^t\ v_1(t,s,y(s)\ d_sg_1(t,s)|-f_1(t,v(t))\int_0^t\ v_1(t,s,v(s)\ d_sg_1(t,s)| \\
\leq\ & |f_1(t,y(t))-f_1(t,v(t))|\int_0^t\ |v_1(t,s,y(s)|\ d_sg_1(t,s) \\
+\ & |f_1(t,v(t))|\int_0^t\ |v_1(t,s,y(s)-v_1(t,s,v(s)|\ d_sg_1(t,s)| \\
\leq\ & m_1(t)|y(t)-v(t)|\int_0^t\ n_1(t,s)\ \varphi_1(|y(s)|)\ |d_sg_1(t,s)| \\
+\ & [m_1(t)|v(t)|+|f_1(t,0)|]\int_0^t\ |v_1(t,s,y(s)-v_1(t,s,v(s)|\ |d_sg_1(t,s)| \\
\leq\ & \big(\delta m_1n_1\varphi_1(\|y\|)+[m_1\|v\|+F_1]\epsilon^*\big)\int_0^t\ d_s\bigvee_{p=0}^{s}g_1(t,p) \\
\leq\ & \big(\delta m_1n_1\varphi_1(r)+[m_1r+F_1]\epsilon^*\big)\bigvee_{s=0}^{t}g_1(t,s), \\
\leq\ & \big(\delta m_1n_1\varphi_1(r)+[m_1r+F_1]\epsilon^*\big)K_1,
\end{aligned}$$

where

$$\epsilon=\big(\delta m_1n_1\varphi_1(r)+[m_1r+F_1]\epsilon^*\big)K_1.$$

Therefore,

$$|(A_1y)(t)-(A_1v)(t)|\leq\epsilon.$$

This means that the operator A_1 is continuous.
By a similar way as done above we can prove that for any $x, u \in C[0,T]$ and $\|x-u\| < \delta$, we have

$$|(A_2x)(t) - (A_2u)(t)| \leq \epsilon.$$

Hence A_2 is continuous operator. The operators A_i $(i = 1, 2)$ are continuous operator this imply that A is continuous operator. Since all conditions of Schauder fixed point theorem are satisfied, then A has at least one fixed point $u = (x, y) \in U$, which completes the proof. □

Corollary 1. *Let assumptions of Theorem 1 be satisfied. Then quadratic Volterra-Stieltjes functional integral equation*

$$x(t) = a(t) + f(t, y(t)) \int_0^t v(t, s, y(s))\ d_sg(t, s),\ t \in I \tag{3}$$

has at least one solution $x \in C(I)$.

Proof. Let the assumptions of Theorem 1 be satisfied, with $x = y$, $f_1 = f_2 = f$, $v_1 = v_2 = v$, and $a_1 = a_2 = a$. Then the coupled system (1) will be the Volterra-Stieltjes quadratic integral equation (3) □

3. Existence of unique solution

In this section, we study the uniqueness of the solution $(x, y) \in X$ of the coupled system of quadratic Volterra-Stieltjes integral equations (1). Assume that functions $\varphi_i : R_+ \to R_+$ have the form $\varphi_i(x) = 1 + |x|$, and the functions $n_i(t, s) \in C(I)$ denoted by $b_i = \|n_i\| = \max\{n_i(t, s)\ t, s \in I,\ i = 1, 2\}$. Then

$$|v_i(t, s, x)| \leq n_i(t, s)\ (1 + |x|).$$

Notice that this assumption is a special case of assumption (iii).
Consider now the assumptions $(ii)^*, (iii)^*$ having the form
$(ii)^*$ $f_i : I \times R \to R$ are continuous functions and there exist constants numbers m_i such that

$$|f_i(t, x) - f_i(t, y)| \leq m_i|x - y|,\ i = 1, 2.$$

for all $x, y \in R$ and $t \in I$.
$(iii)^*$ $v_i(t, s, x) : v_i = \triangle_i \times R \to R$ are continuous and satisfy the Lipschitz condition with Lipschitz constant b_i, such that

$$|v_i(t, s, x) - v_i(t, s, y)| \leq\ b_i\ |x - y|,\ i = 1, 2.$$

From this assumption, we can deduce that

$$|v_i(t, s, x)| - |v_i(t, s, 0)| \leq |v_i(t, s, x) - v_i(t, s, 0)| \leq\ b_i\ |x|,$$

which implies that

$$|v_i(t, s, x)| \leq\ |v_i(t, s, 0)| + b_i\ |x| \leq\ n_i(t, s) + b_i\ |x|,$$

where $n_i(t, s) = \sup_{t \in I} |v_i(t, s, 0)|$.

Theorem 2. *Let assumptions of Theorem 1 be satisfied with replace assumtions (ii),(iii) by $(ii)^*$, $(iii)^*$, if the following condition hold*

$$m(n + br) + [mr + F]K \leq 1.$$

Then the coupled system (1) has an unique solution $(x, y) \in X$.

Proof. Let $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$ be two solutions of the coupled system (1), we have

$$\begin{aligned}
\|(x_1, y_1) - (x_2, y_2)\|_X &= \|(x_1 - x_2, y_1 - y_2)\|_X \\
&= \max_{t \in I} \{\|x_1 - x_2\|, \|y_1 - y_2\|.\}
\end{aligned}$$

Now,

$$\begin{aligned}
& |x_1(t)-x_2(t)| \\
\leq\ & |f_1(t,y_1(t))\int_0^t\ v_1(t,s,y_1(s)\ d_sg_1(t,s)-f_1(t,y_2(t))\int_0^t\ v_1(t,s,y_2(s))\ d_sg_1(t,s)| \\
\leq\ & |f_1(t,y_1(t))\int_0^t\ v_1(t,s,y_1(s))\ d_sg_1(t,s)-f_1(t,y_2(t))\int_0^t\ v_1(t,s,y_1(s)\ d_sg_1(t,s)| \\
+\ & |f_1(t,y_2(t))\int_0^t\ v_1(t,s,y_1(s))\ d_sg_1(t,s)-f_1(t,y_2(t))\int_0^t\ v_1(t,s,y_2(s)\ d_sg_1(t,s)| \\
\leq\ & |f_1(t,y_1(t))-f_1(t,y_2(t))|\int_0^t\ |v_1(t,s,y_1(s)|\ |d_sg_1(t,s)| \\
+\ & |f_1(t,y_2(t))|\int_0^t\ |v_1(t,s,y_1(s)-v_1(t,s,y_2(s)|\ |d_sg_1(t,s)| \\
\leq\ & m_1|y_1(t)-y_2(t)|\int_0^t\ (n_1(t,s)+b_1\ |y|)d_sg_1(t,s) \\
+\ & [m_1|y_2(t)|+|f_1(t,0)|]b_1\int_0^t\ |y_1(s)-y_2(s)|\ d_sg_1(t,s) \\
\leq\ & [\|y_1-y_2\|m_1(n_1+b_1\|y_1\|)+[m_1\|y_2\|+F_1]\|y_1-y_2\|]\int_0^t\ d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right) \\
\leq\ & m_1(n_1+b_1\|y_1\|)+[m_1\|y_2\|+F_1]\|y_1-y_2\|\left(\bigvee_{s=0}^{t}g_1(t,s)\right) \\
\leq\ & m_1(n_1+b_1r_1)+[m_1r_1+F_1]K_1\|y_1-y_2\|.
\end{aligned}$$

Therefore,

$$\|x_1-x_2\|\leq m(n+br|)+[mr+F])K\|y_1-y_2\|,$$

where

$b=\max\{b_1,b_2\},\ m=\max\{m_1,m_2\}\ ,n=\max\{n_1,n_2\},\ F=\max\{F_1,F_2\}$ and $K=\max\{K_1,K_2\}$.

similarly

$$\|y_1-y_2\|\leq m(n+br)+[mr+F])K\|x_1-x_2\|.$$

Then

$$\begin{aligned}
& \|(x_1,y_1)-(x_2,y_2)\|_X=\|(x_1-x_2,y_1-y_2)\|_X \\
=\ & \max_{t\in I}\{\|x_1-x_2\|_C,\|y_1-y_2\|_C\} \\
=\ & \max_{t\in I}\{[m(n+br)+(mr+F)K]\ \|y_1-y_2\|,[m(n+br)+(mr+F)K]\ \|x_1-x_2\|\} \\
=\ & [m(n+br)+(mr+F)K]\ \max_{t\in I}\{\|x_1-x_2\|_C,\|y_1-y_2\|_C\} \\
=\ & [m(n+br)+(mr+F)K]\ \|(x_1,y_1)-(x_2,y_2)\|_X.
\end{aligned}$$

Which implies that

$$[1-m(n+br)+(mr+F)K]\ \|(x_1,y_1)-(x_2,y_2)\|_X\leq 0.$$

Therefore

$$\|(x_1,y_1)-(x_2,y_2)\|_X=0.$$

This means that

$$(x_1,y_1)=(x_2,y_2)\implies x_1=x_2,\ y_1=y_2.$$

This proves the uniqueness of the soloution of the coupled system (1). □

4. A Coupled system of the nonlinear quadratic Volterra-Hammerstein-Stieltjes integral equations

This section, as an application, deals with the existence of continuous solution for the coupled system of quadratic Hammerstein-Stieltjes functional integral equations

$$\begin{aligned} x(t) &= a_1(t) + f_1(t, y(t)) \int_0^t \; k_1(t,s)h_1(s, y(s)) \; d_s g_1(t,s), \; t \in I \\ y(t) &= a_2(t) + f_2(t, x(t)) \int_0^t \; k_2(t,s)h_2(s, y(s))) \; d_s g_2(t,s), \; t \in I \end{aligned} \tag{4}$$

Consider the following assumption:
Assumption (1): Let $h_i : I \times R \to R$ and $k_i : I \times I \to R_+$ assume that h_i, k_i satisfy the following assumptions:

(i) $h_i(t, y(t))$ $(i = 1, 2)$ are continuous functions.

(ii) There exist continuous functions $m_i^*(t)$ and continuous and nondecreasing functions $\phi_i : R_+ \to R_+$, such that
$$|h_i(t, x)| \le m_i^*(t)\varphi_i(|x|),$$
for all $t \in I$ and $x \in R$, $i = 1, 2$. Moreover, we put $M_i^* = \max\{m_i^*(t), \; t \in I, \; i = 1, 2\}$.

(iii) Functions $k_i(t,s) = k_i : \triangle \to R$ are continuous on the triangle $\triangle$, such that $K_i^* = \sup_t\{k_i(t,s)\}$, where K_i^* are positive constants.

Definition 3. *By a solution for the coupled system (4), we mean the pair of functions* $(x, y) \in X$.

Being now for the existence of continuous solutions of (4), we have the following theorem.

Theorem 3. *Let assumptions (1) and (i)-(vi) of Theorem 1 be satisfied. Then the coupled system of quadratic Hammerstien-Stieltjes integral equations (4) has at least one continuous solution* $(x, y) \in X$.

Proof. let $v_i(t, s, y(s))) = k_i(t,s)h_i(s, y(s))$. Then from assumption (1), we find that assumptions of Theorem 1 are satisfied and the result follows. □

5. A coupled systems of quadratic Volterra Integral Equations of fractional order

In this section, we will consider a coupled systems of quadratic Volterra integral equations of fractional order, which has the form

$$\begin{aligned} x(t) &= a_1(t) + f_1(t, y(t)) \int_0^t \; \frac{(t-s)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \; v_1(t, s, y(s) \; ds, \; t \in I \\ y(t) &= a_2(t) + f_2(t, x(t)) \int_0^t \; \frac{(t-s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \; v_2(t, s, x(s) \; ds, \; t \in I \end{aligned} \tag{5}$$

where $t \in I = [0, T]$ and $\alpha_i \in (0, 1)$, and $\Gamma(\alpha_i)$, $i = 1, 2$, refers to gamma functions. Let us mention that (5) represents the so-called a coupled systems of Volterra quadratic integral equations of fractional order. Recently, such a type this type has been widely investigated in some papers [1, 12, 13, 15, 16, 17]
Here, we show that a coupled systems of fractional orders (5) can be treated as a special case of a coupled systems of quadratic Volterra-Stieltjes integral equations (2) studied in Section 2.

In fact, we can consider functions $g_i(t,s) = g_i : \triangle \to R, \; i = 1, 2$, defined by the formula

$$g_i(t,s) = \frac{t^{\alpha_i} - (t-s)^{\alpha}}{\Gamma(\alpha_i + 1)}.$$

We can see that functions $g_i, i = 1, 2$, satisfy assumptions (iv)-(vii) in Theorem 1, see [8, 11].
Now, we state the following existence results for couple system of quadratic Volterra integral equations of fractional order (5).

Theorem 4. *Let assumptions* (i)*-*(vii) *of Theorem 1 be satisfied. Then a coupled systems of fractional orders* (5) *has at least one solution* $(x, y) \in X$.

Corollary 2. *Let assumptions of Theorem 4 be satisfied (with $x = y$, $v_1 = v_2 = v$,$f_1 = f_2 = f$, $a_1 = a_2 = 0$ and $\alpha_1 = \alpha_2 = \alpha$). Then the fractional-order quadratic integral equation*

$$x(t) = a(t) + f(t, y(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; v(t, s, y(s)) \; ds, \;\; t \in I$$

has at least one solution in $x \in C(I)$.

Corollary 3. *Let assumptions of Theorem 4 be satisfied, with $f_1(t, y(t)) = f_2(t, x(t)) = 1$. Then a coupled system of the fractional-order quadratic integral system*

$$\begin{aligned} x(t) &= a_1(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; v_1(t, s, y(s))) \; ds, \;\; t \in I \\ y(t) &= a_2(t) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \; v_2(t, s, x(s))) \; ds, \;\; t \in I \end{aligned} \tag{6}$$

has at least one solution in $(x, y) \in X$.

Now, letting $\alpha_1, \; \alpha_2 \to 1$, we obtain

Corollary 4. *Let assumptions of Theorem 4 be satisfied. Then the coupled system of the initial value problems*

$$\begin{aligned} \frac{x(t)}{dt} &= v_1(t, t, y(t)), \;\; t \in I, \;\; x(0) = x_0, \\ \frac{y(t)}{dt} &= v_2(t, t, x(t)), \;\; t \in I, \;\; y(0) = y_0, \end{aligned} \tag{7}$$

has at least one solution in $(x, y) \in X$.

Proof. Let assumptions of Theorem 4 be satisfied (with $f_1(t, y(t)) = f_2(t, x(t)) = 1$, $a_1(t) = x_0$, $a_2(t) = y_0$ and letting $\alpha, \; \beta \to 1$. Then a coupled system of the fractional-order quadratic integral equations

$$\begin{aligned} x(t) &= x_0 + \int_0^t \; v_1(t, s, y(s)) \; ds, \;\; t \in I, \\ y(t) &= y_0 + \int_0^t \; v_2(t, s, x(s)) \; ds, \;\; t \in I, \end{aligned} \tag{8}$$

has at least one solution in X which is equivalent to the coupled system of the initial value problems (7). □

Corollary 5. *Let assumptions of Theorem 4 be satisfied. Then the coupled system of fractional-order differential equations*

$$\begin{aligned} D^{\alpha_1} x(t) &= v_1(t, t, y(t)), \;\; t \in I \\ D^{\alpha_2} y(t) &= v_2(t, t, x(t)), \;\; t \in I \end{aligned} \tag{9}$$

with the initial conditions

$$I^{1-\alpha_1} x(t)|_{t=0} = I^{1-\alpha_2} y(t)|_{t=0} = 0, \;\; \alpha_1, \alpha_2 \in (0, 1], \tag{10}$$

has at least one solution in $(x, y) \in X$.

Proof. let us proof the coupled system of the initial value problems (9) and (10) is equivalent to the coupled system of quadratic integral system

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} \; v_1(t, s, y(s)) \; ds, \;\; t \in I, \;\; \alpha_1 \in (0, 1) \\ y(t) &= \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \; v_2(t, s, x(s)) \; ds, \;\; t \in I, \;\; \alpha_2 \in (0, 1) \end{aligned} \tag{11}$$

By operating $I^{1-\alpha_1}$ and $I^{1-\alpha_2}$ respectively on each equation of coupled system (11), and applying properties of fractional operator [29], we obtain

$$\begin{aligned} I^{1-\alpha_1} x(t) &= I^1 v_1(t, s, y(s)), \;\; I^{1-\alpha_1} x(t)|_{t=0} = 0 \\ I^{1-\alpha_2} y(t) &= I^1 v_2(t, s, x(s)), \;\; I^{1-\alpha_2} y(t)|_{t=0} = 0. \end{aligned}$$

Also,

$$\frac{d}{dt}I^{1-\alpha_1}x(t) = v_1(t,t,y(t)),\ t \in I,\ \alpha_1 \in (0,1)$$
$$\frac{d}{dt}I^{1-\alpha_2}y(t) = v_2(t,t,x(t)),\ t \in I,\ \alpha_2 \in (0,1).$$

Conversely, by integrating the coupled system of initial value problems (9) and (10), we have

$$I^{1-\alpha_1}x(t) - \ I^{1-\alpha_1}x(t)|_{t=0} = I^1 v_1(t,t,y(t))$$
$$I^{1-\alpha_2}y(t) - \ I^{1-\alpha_2}y(t)|_{t=0} = I^1 v_2(t,t,x(t)).$$

Operating by I^{α_1} and I^{α_2} respectively on each equation and differentiating, we have (11). Thus, the equivalence hold.

Let assumptions of Theorem 4 be satisfied (with $f_1(t,y(t)) = f_2(t,x(t)) = 1,\ a_1(t) = a_2(t) = 0$. Then there exists at least one solution in X for the coupled system (9 and 10). □

6. Further discussions and example

This section is devoted to discuss some details appearing in our existence results proved in the previous section. At the beginning we consider a few special cases of the system (1). First of all let us notice that the classical nonlinear coupled systems of Volterra quadratic integral equations

$$\begin{aligned} x(t) &= a_1(t) + \int_0^t \ v_1(t,s,y(s))\ ds, \\ y(t) &= a_2(t) + \int_0^t \ v_2(t,s,x(s))\ ds, \end{aligned} \tag{12}$$

is a special case of system (1) if we put $f_i(t,x) = 1$ and $g_i(t,s) = s$. This implies that equation (12) can be investigated with help of Theorem 1. Observe that some assumptions of that Theorem are automatically satisfied for system (12). Indeed, assumptions (iv)-(vii) are trivially satisfied. Assumption with $f_i(t,0) = 1,\ (i = 1.2)$. Thus we have to assume hypotheses (i)- (vii). Let us notice that in the case of system (12) we have $F_i = 1$ where F_i is the constants defined in Section 3.

It is worthwhile mentioning that system (1) contains also other special cases which are important in applications. For example, the following integral equation

$$x(t) = f(t,x(t)) \int_0^t \ v(t,s,x(s))\ ds \tag{13}$$

plays a significant role in describing some problems connected with traffic theory and biology (see [10]). Obviously this equation is a special case of system (1), if we put $x = y,\ v_1 = v_2 = v$ $,f_1 = f_2 = f,\ a_1 = a_2 = 0$ and $g_i(t,s) = s$.

Next, let us consider the couple system of quadratic integral equations of Volterra type having the form

$$\begin{aligned} x(t) &= a_1(t) + y(t) \int_0^t \ \frac{t}{t+s}\ v_1(t,s,y(s)))\ ds \\ y(t) &= a_2(t) + x(t) \int_0^t \ \frac{t}{t+s}\ v_2(t,s,x(s)))\ ds. \end{aligned} \tag{14}$$

This system represents the couple system of Chandrasekhar's quadratic integral equations.

We now show that couple system (14) is a special case of couple system (1). To do this let us consider the function $g_i : \triangle \to R\ (i = 1,2)$, defined by the formula

$$g_i(t,s) = \begin{cases} \ t\ \ln \frac{t+s}{t},\ \text{for } t > 0 \text{ and } s \geq 0, \\ \\ \quad 0 \text{ for } t = 0 \text{ and } s \geq 0. \end{cases} \tag{15}$$

Then we see that system (14) can be written in the form (1).

In order to discuss the existence result for system (14), let us first notice that $f_1(t,y) = y$ and $f_2(t,x) = x$. This suggests that the existence result concerning couple system (14) is contained in

Theorem 1, where we have to assume hypotheses (i) and (vii).
Using standard methods of mathematical analysis it can be seen that functions $g_i(t,s)$ defined by (15) satisfies assumptions (iv)-(vii). Moreover, observe that assumption (ii) is satisfied with $m_i(t) = 1$ and $f_i(t,0) = 0$.
If we let $x = y$, $v_1 = v_2 = v$ and $a_1 = a_2 = a$, then the coupled system (14) will be counterpart of the famous Chandrasekhar quadratic integral equation of form

$$x(t) = a(t) + x(t) \int_0^t \frac{t}{t+s} \, v(t,s,x(s)) \, ds, \tag{16}$$

which has numerous applications (cf. [2, 24]).
Further, let us recall that if the function $g_i(t,s)$ satisfies assumptions (iv)-(vii).Then it represents the distribution function of a two dimensional random variable [5, 25]. The converse implication is also almost valid [10, 25]. Particularly, we may consider the function as being the distribution function of a two dimensional random variable of continuous type. Such functions has the form

$$g_i(t,s) = \int_0^t \left(\int_0^s p_i(z,y)dy\right) dz, \ i = 1,2, \tag{17}$$

where $p_i(z,y)$ are the so-called density functions [5]. Obviously this functions satisfy assumptions (iv)-(vii). Moreover, we have

$$d_s g_i(t,s) = \int_0^t p_i(s,y)dy, \ i = 1,2, \tag{18}$$

and in this case couple system (1) has the form

$$\begin{aligned} x(t) &= a_1(t) + f_1(t,y(t)) \int_0^t v_1(t,s,y(s)) \left(\int_0^t p_1(s,y)dy\right) ds \\ y(t) &= a_2(t) + f_2(t,x(t)) \int_0^t v_2(t,s,x(s)) \left(\int_0^t p_2(s,y)dy\right) ds. \end{aligned} \tag{19}$$

If we let $x = y$, $v_1 = v_2 = v$,$f_1 = f_2 = f$ and $a_1 = a_2 = a$. Then the coupled system (19) will be

$$x(t) = a(t) + f(t,x(t)) \int_0^t v(t,s,x(s)) \left(\int_0^t p_1(s,y)dy\right) ds. \tag{20}$$

Let us mention that this equation is the usual quadratic nonlinear Volterra integral equation.
In what follows, we provide example illustrating our obtained results.
Example 1. Consider the following couple system of quadratic integral Volterra type

$$\begin{aligned} x(t) &= \frac{e^t}{e+e^{t+1}} + \frac{1}{\Gamma(2/3)} \sin\left(\frac{t^2+y(t)}{1+t^2}\right) \int_0^t \frac{\sqrt{|y(s)|}}{(4+t^2+s^2)(t-s)^{1/3}} \, ds, \\ y(t) &= t^2 \exp(-t) + \frac{\exp(-t)+x(t)}{\Gamma(1/2)} \int_0^t \frac{\sin(s^2+t)+\sqrt[3]{x^2(s)}}{(t-s)^{1/2}} \, ds. \end{aligned} \tag{21}$$

Observe that the above system is a special case of (5). Indeed, we put $\alpha_1 = 2/3$, $\alpha_2 = 1/2$ and

$$a_1(t) = \frac{e^t}{e+e^{t+1}}, \ a_2(t) = t^2 \exp(-t),$$

$$f_1(t,y) = \sin\left(\frac{t^2+y(t)}{1+t^2}\right), \ f_2(t,x) = \exp(-t) + x(t),$$

$$v_1(t,s,y) = \frac{\sqrt{|y(s)|}}{4+^2 t+s^2}, \ v_2(t,s,x) = \sin(s^2+t) + \sqrt[3]{x^2(s)}.$$

Assume that functions $f_i(t,s), \ i = 1,2$ satisfy (5) and $K_i = \frac{1}{\Gamma(\alpha_i+1)}$. Then we can easily verify that the assumptions of Theorem 1 are satisfied.
In fact, functions $a_i, (i = 1,2)$ are continuous on I, with $\|a_1\| = 1/e$ and $\|a_2\| = e$. Thus assumption (i) is satisfied. Further, notice that f_i are continuous on $I \times R$ and satisfies the Lipschitz condition

with the constants $m_i = 1$. Moreover, $F_1 = \sin(1)$ and $F_2 = 1$. Next, let us note that functions $v_1 = v_1(t, s, y)$ and $v_2 = v_2(t, s, x)$ are continuous on the set $\triangle \times R$ and the following inequality hold

$$|v_1(t, s, y)| \leq \frac{\sqrt{|y(s)|}}{4 + t^2 + s^2} \leq \frac{1}{4}\sqrt{|y|},$$

$$|v_2(t, s, y)| \leq \frac{tx^{2/3}(s)}{1 + t + s^2} \leq x^{2/3}.$$

This yields that the estimate from assumption (iii) is satisfied with $\phi_1(r_1) = \frac{1}{4}\sqrt{r_1}$ and $\phi_2(r_2) = r_2^{2/3}$. Finally, let us pay attention to the fact that inequalities from of Theorem 1 has the form

$$1/e + \frac{1}{\Gamma(5/3)}[r_1 + \sin(1)]\frac{1}{4}\sqrt{r_1} \leq r_1, \tag{22}$$

$$4/e^2 + (1/\sqrt{\pi})(r_1 + 1)r_2^{2/3} \leq r_2. \tag{23}$$

Keeping in mind that $\Gamma(5/3) > 0.8856$, $\Gamma(1/2) = \sqrt{\pi}$ [23] and $sin1 = 0.8415\ .\ .\ .$, it is easily seen that the number $r_0 = 1$ satisfies inequality (22) and (23) Thus, based on Theorem 4 we deduce that the couple system of quadratic integral Volterra type (5) has at least one solution belonging the space X.

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