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EXISTENCE AND UNIQUENESS OF SOLUTION TO A FRACTIONAL EULER-LAGRANGE EQUATION WITH BOTH RIEMANN-LIOUVILLE AND CAPUTO DERIVATIVES

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Abstract. Existence and uniqueness of mild and strong solutions to the fractional nonlinear boundary value problem, with Riemann-Liouville and Caputo derivatives,

$${}_t^C D_b^\alpha\ {}_0^R D_t^\beta x(t) = f(t, x(t)),\ x(b) = x_b,\ {}_0 I_t^{1-\beta} x(t)|_{t=0} = 0,\ \alpha, \beta \in (0,1),\ t \in [0,b]$$

will be discussed. Continuous dependence of solution on the boundary condition will be proved. An example will be given to illustrate our results.

1. Introduction

The fractional calculus started to be successfully applied in many fields involving the dynamics of complex systems.

In this case the fractional proposed models may involved left and right fractional derivatives. We mention here the fractional Euler-Lagrange equations [2, 6] as a particular and important case of fractional equations with two types of derivatives. These new types of equations required special attention because of their complex form, therefore the study of the existence, uniqueness and continuous dependence is still an open problem in the field of fractional calculus. (see [1, 3, 4, 7, 10, 11, 12]).

Our goal is giving the definition of mild and strong solutions with discussing the existence and uniqueness of them to the problem

$${}_t^C D_b^\alpha\ {}_0^R D_t^\beta x(t) = f(t, x(t)),\ x(b) = x_b,\ {}_0 I_t^{1-\beta} x(t)|_{t=0} = 0,\ \alpha, \beta \in (0,1), t \in J := [0,b]. \tag{1}$$

The rest of this paper is organized as follows: In section 2, principal definitions and theorems used in this paper will be given. The main results are collected in section 3, The existence results follow from Schauder fixed point theorem and the uniqueness of solutions is a consequence of the Banach contraction principle. Finally our conclusions are given in section 4.

2. Preliminaries

Here we collect the definitions and results which will be used in the sequel.

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Left and right Riemann-Liouville fractional integrals to the function $f \in L_1(J)$, the set of all integrable functions on the interval $J=[a,b]$, with $\alpha>0$ are defined by (see [9])

$$
\begin{aligned}
{}_aI_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau,\\
{}_tI_b^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}f(\tau)d\tau.
\end{aligned}
$$

About fractional derivative there are a lot of definitions. In our problem, we use two of them: the first is the left Riemann-Liouville fractional derivative which is defined by

$$
{}_a^RD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}D^n\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau, \tag{2}
$$

where $n-1<\alpha<n$ and $D=\frac{d}{dt}$. And the second is the right Caputo fractional derivative which is defined by

$$
{}_t^CD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}(-D)^n f(\tau)d\tau \tag{3}
$$

In the following lemma we give some properties of these fractional operators' spaces (for detail see [8, 9]).

Lemma 2.1. *we have that*

1. *If $f\in AC(J)$, the set of all absolutely continuous functions from J into $\mathbb{R}$, then ${}_aI_t^\alpha f\in AC(J)$;*
2. *If $f\in C(J)$, the set of all continuous functions from J into $\mathbb{R}$, then ${}_aI_t^\alpha f\in C(J)$;*
3. *If f is a Riemann integrable function, then ${}_aI_t^\alpha f(t)$ exists for all $t\in J$;*
4. *If $f\in L_1(J)$, then ${}_aI_t^\alpha f(t)$ exists almost every where and ${}_aI_t^\alpha f(t)\in L_1(J)$;*
5. *The left Riemann-Liouville fractional derivative of $f(t)$ exists if ${}_aI_t^{1-\alpha}f\in AC(J)$ which gives ${}_a^RD_t^\alpha f\in L_1(J)$,*
6. *The right Caputo fractional derivative of $f(t)$ exists if $f\in AC(J)$ which gives ${}_a^CD_t^\alpha f\in L_1(J)$.*

Finally, we recall the Schauder fixed point theorem as follows. (see [5, 11]):

Theorem 2.2. *Let X be a Banach space, and E be a nonempty, closed, convex subset of X. If $T:E\to E$ is a continuous mapping, such that $T(E)$ is a relative compact subset of X, then T has at least one fixed point in E.*

3. Main Results

Consider the nonhomogeneous linear problem

$$
{}_t^CD_b^\alpha\ {}_0^RD_t^\beta x(t) = y(t),\ x(b)=x_b,\ {}_0I_t^{1-\beta}x(t)|_{t=0}=0,\ \alpha,\beta\in(0,1),\ t\in J \tag{4}
$$

Firstly, on the basis of the following lemma, we transform problem (4) to a fixed point problem. Then we give the definition of mild and strong solution to (1), and then the existence and uniqueness results for the obtained fixed point problem is discussed.

Lemma 3.1. *If $x(t)$ is a solution of (4) with $y\in C(J)$, then $x(t)$ is a solution of the integral equation*

$$
x(t) = \frac{t^\beta}{b^\beta}\left(x_b - \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)d\tau ds\right) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)d\tau ds \tag{5}
$$

The converse is satisfied if $y\in AC(J)$.

Proof. Operating by ${}_tI_b^\alpha$ on both sides we get

$$
{}_0^RD_t^\beta x(t) =_t I_b^\alpha y(t)+c_1,\ c_1:=\ {}_0^RD_t^\beta x(t)|_{t=0}. \tag{6}
$$

Operating on both sides by ${}_0I_t^{\beta}$ with using the given conditions we get our problem in the form

$$x(t)=\frac{t^{\beta}}{b^{\beta}}\left(x_b-\int_0^b\frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)d\tau ds\right)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)d\tau ds. \tag{7}$$

Conversely, if $y\in AC(J)$ then ${}_tI_b^{\alpha}y(t)\in AC(J)$ which gives that $x\in AC(J)$. It is easy to see that $x(b)=x_b$ and ${}_0I_t^{1-\alpha}x(t)|_{t=0}=0$. Now, operating by ${}_0^RD_t^{\beta}$ on both sides of (5) we get

$${}_0^RD_t^{\beta}x(t)=\frac{\Gamma(\beta+1)}{b^{\beta}\Gamma(\beta+1-\alpha)}\left(x_b-\int_0^b\frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)d\tau ds\right)+\ {}_tI_b^{\alpha}y(t). \tag{8}$$

Operating by ${}_t^CD_b^{\alpha}$ on both sides gives

$${}_t^CD_b^{\alpha}\ {}_0^RD_t^{\beta}x(t)=y(t)$$

which completes the proof. □

Now we can define mild and strong solution to (1) as follows:

Definition 3.1. Let $J=[0,b]$, the mild solution to (1) on J is a function $x\in C(J)$ which satisfies

$$\begin{aligned} x(t) &= \frac{t^{\beta}}{b^{\beta}}\left(x_b-\int_0^b\frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,x(\tau))d\tau ds\right)\\ &\quad+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,x(\tau))d\tau ds. \end{aligned} \tag{9}$$

Moreover if the solution to the integral equation 9 satisfies ${}_0^RD_t^{\beta}x\in AC(J)$ and (1) holds on J, it is called a strong solution to problem (1).

Now for the integral equation (9) we define

$$\begin{aligned} Fx(t) &= \frac{t^{\beta}}{b^{\beta}}\left(x_b-\int_0^b\frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,x(\tau))d\tau ds\right)\\ &\quad+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,x(\tau))d\tau ds \end{aligned} \tag{10}$$

on the space $C(J)$ with the sup-norm.

Consider the following hypothesis:

- (H1) The function $f:J\times\mathbb{R}\to R$ is measurable in $t\in J$ for each $x\in\mathbb{R}$ and continuous in $x\in R$ for almost all $t\in J$;
- (H2) $|f(t,z)|\le a_1(t)+a_2(t)|z|$, where $a_1,a_2\in L_1(J)$ are nonnegative functions;

Let

$$M_1=\left|x_b-\int_0^b\frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,x(\tau))d\tau ds\right|,$$

$$M_2=\sup_{t\in J}\left(\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}a_1(\tau)d\tau ds\right)$$

and

$$M_3=\sup_{t\in J}\left(\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^b\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}a_2(\tau)d\tau ds\right).$$

Theorem 3.2. *Let the function f satisfies (H1)-(H2), then if $M_3<1$ there exists at least one solution to the integral equation (9).*

Proof. We prove this theorem in some steps Let $B_r = \{x \in C(J) : \ \|x\| \leq r\}$ where $r \geq \frac{M_1+M_2}{1-M_3}$, Clearly, $\bar{B}_r$ is nonempty, closed, bounded, and convex.
Step 1: F is a uniformly bounded in $\bar{B}_r$.
Let $x \in \bar{B}_r$ and use (H1) we obtain

$$\begin{aligned}
|F(x(t))| &= \left|\frac{t^\beta}{b^\beta}\left(x_b - \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds\right)\right. \\
&\quad \left. + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds\right| \\
&\leq \frac{t^\beta}{b^\beta} M_1 + \left|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds\right| \\
&\leq M_1 + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} (a_1(\tau) + a_2(\tau)|x(\tau)|) d\tau ds \\
&\leq M_1 + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} a_1(\tau) ds \\
&\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} a_2(\tau)|x(\tau)| d\tau ds,
\end{aligned}$$

and where $\|Fx\| = \sup_{t\in J} |F(x(t))|$, we get that

$$\|Fx\| \leq M_1 + M_2 + M_3\|x\| \leq r$$

which proves that $F : \bar{B}_r \to \bar{B}_r$ is uniformly bounded
Step 2: $F(\bar{B}_r)$ is relatively compact.
let $t_1 < t_2 \in (0, b)$ and $x \in \bar{B}_r$, then we have

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= \left|\frac{t_2^\beta - t_1^\beta}{b^\beta}\left[x_b - \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds\right]\right. \\
&\quad + \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds \\
&\quad \left. - \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds\Big]\right| \\
&= \frac{t_2^\beta - t_1^\beta}{b^\beta} M_1 + \left|\int_0^{t_1} \left(\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\right) \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds\Big]\right| \\
&\leq \frac{t_2^\beta - t_1^\beta}{b^\beta} M_1 + M_5 \frac{b^\alpha}{\Gamma(\alpha+1)} \left[\frac{t_2^\beta - t_1^\beta}{\Gamma(\beta+1)} + \frac{2(t_2-t_1)^\beta}{\Gamma(\beta+1)}\right],
\end{aligned}$$

Which proves that all the functions in $F(\bar{B}_r)$ are equicontinuous. According to Arzela-Ascoli theorem, $F(\bar{B}_r)$ is a relatively compact set.
Step 3: F is a continuous operator
Let x_n be a sequence in $\bar{B}_r$ converges to $x \in \bar{B}_r$, then from (H1)

$$f(t, x_n(t)) \to f(t, x(t)) \text{ as } n \to \infty$$

and $\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x_n(\tau))$ is a sequence of measurable functions such that

$$\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x_n(\tau)) \to \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) \text{ as } n \to \infty, \ a.e. \ \tau \in [s, b]$$

with

$$|\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x_n(\tau))| \leq \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} (a_1(\tau) + a_2(\tau)|x_n(\tau)|) \in L_1(J)$$

Now applying Lebesgue-dominated convergence theorem to obtain

$$\int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x_n(\tau)) d\tau \to \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau \text{ as } n \to \infty.$$

Applying Lebesgue-dominated convergence theorem again to $\int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x_n(\tau)) d\tau$, which satisfies its conditions on $[0, b]$, to obtain as $n \to \infty$

$$\int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x_n(\tau)) d\tau ds \to \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds$$

and

$$\int_0^t \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x_n(\tau)) d\tau ds \to \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau)) d\tau ds$$

Which prove that $\lim_{n\to\infty} Fx_n(t) = Fx(t)$. Thus F is a continuous operator. Now from Step 1 - Step 3 we can apply the Schauder fixed point theorem, which gives that F has a fixed point in $\bar{B}_r$ which is the required solution. □

Consider the following hypothesis

(H3) $|f(t, z_1) - f(t, z_2)| \leq h(t)|z_1 - z_2|$, where $h \in C(J)$ is a nonnegative function.

Corollary 3.3. *Let the function f satisfies (H1) and (H3), then if $M_3 < 1$ there exists at least one solution to the integral equation (9) (where $a_2(t) = h(t)$).*

Proof. Taking $a_1(t) = |f(t, 0)$ and $a_2(t) = h(t)$ we get by using (H3) that

$$|f(t,x)| = |f(t,0) + f(t,x) - f(t,0)| \leq |f(t,0)| + |f(t,x) - f(t,0)| \leq |f(t,0)| + h(t)|x| = a_1(t) + a_2(t)|x|$$

thus (H2) is satisfied, which completes the proof. □

Remark 1. It is noted that the solution to the integral equation (9) given in the previous theorems is only a continuous function which proves that it is only a mild solution to (1) not a strong one.

Example 3.1. Consider the problem

$${}_t^C D_b^\alpha \, {}_0^R D_t^\beta x(t) = \frac{A}{2} t \sin(x(t)) + t^4, \ x(b) = x_b, \ {}_0 I_t^{1-\beta} x(t)|_{t=0} = 0, \ \alpha, \beta \in (0,1), \ A = constant. \quad (11)$$

It is obvious that $f(t, x(t)) = \frac{A}{2} t sin(u(t)) + t^4$ satisfies the conditions of Theorem 3.2 with $a_1(t) = t^4$ and $a_2(t) = \frac{A}{2} t$. Then the problem (11) has a mild solution.

Now for uniqueness of solution we get the following theorem which depends on the Banach contraction principle.

Theorem 3.4. *Let the function f satisfies (H1) and (H3), then if $M_3 < 1$ and $\frac{\|h\|(2\beta+\alpha)b^{\beta+\alpha}}{(\beta+\alpha)\Gamma(\beta+1)\Gamma(\alpha+1)} < 1$, then there exists a unique solution to the integral equation (9).*

Proof. From Corollary 3.3 and step 1 in Theorem 3.2, we get that $F : \bar{B} \to \bar{B}$. Now for any $x, y \in \bar{B}$ we have

$$\begin{aligned} |Fx(t) - Fy(t)| &\leq \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau ds \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau ds \\ &\leq \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)|x(\tau) - y(\tau)| d\tau ds \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)|x(\tau) - y(\tau)| d\tau ds \\ &\leq \|h\| \|x - y\| \left(\int_0^b \frac{(b-s)^{\beta+\alpha-1}}{\Gamma(\beta)\Gamma(\alpha+1)} ds + \frac{b^\alpha}{\Gamma(\alpha+1)} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \right) \\ &\leq \frac{\|h\|(2\beta+\alpha)b^{\beta+\alpha}}{(\beta+\alpha)\Gamma(\beta+1)\Gamma(\alpha+1)} \|x - y\|. \end{aligned}$$

Thus we get

$$\|Fx - Fy\| \leq \frac{\|h\|(2\beta+\alpha)b^{\beta+\alpha}}{(\beta+\alpha)\Gamma(\beta+1)\Gamma(\alpha+1)} \|x - y\|.$$

Thus the operator F is a contraction mapping on a Banach space $\bar{B}$ then applying Banach fixed point theorem we get the result. □

Finally, in the following theorem we prove the continuous dependence of the solution on the boundary condition.

Theorem 3.5. *Let the function f satisfies (H1) and (H3), with $M_3 < 1$ and $\frac{\|h\|(2\beta+\alpha)b^{\beta+\alpha}}{(\beta+\alpha)\Gamma(\beta+1)\Gamma(\alpha+1)} \neq 1$, then the solution to the integral equation (9) depends continuously on the given boundary condition x_b.*

Proof. Let x and y are two solutions of (9) with boundary condition x_b and y_b respectively, then we have

$$\begin{aligned} |x(t) - y(t)| &\leq |x_b - y_b| + \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau ds \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau ds \\ &\leq |x_b - y_b| + \int_0^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)|x(\tau) - y(\tau)| d\tau ds \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^b \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)|x(\tau) - y(\tau)| d\tau ds \\ &\leq |x_b - y_b| + \frac{\|h\|(2\beta+\alpha)b^{\beta+\alpha}}{(\beta+\alpha)\Gamma(\beta+1)\Gamma(\alpha+1)} \|x - y\|. \end{aligned}$$

which implies

$$\|x - y\| \leq \frac{1}{1 - \frac{\|h\|(2\beta+\alpha)b^{\beta+\alpha}}{(\beta+\alpha)\Gamma(\beta+1)\Gamma(\alpha+1)}} |x_b - y_b| \tag{12}$$

which completes the proof. □

4. Conclusion

The fractional variational principles played an important role in fractional control problems as well as in physics. The Euler-Lagrange equations contain different types of fractional derivative. In this paper we discuss one of them which contains both left Riemann-Liouville and right Caputo fractional derivatives, we prove the existence of mild solution and give the conditions to obtain a unique solution with proving that this mild solution depends continuously on the boundary conditions.

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