PRICING CORPORATE DEFAULTABLE BOND USING DECLARED FIRM VALUE

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ABSTRACT. We study the pricing problem for corporate defaultable bond from the viewpoint of the investors outside the firm that could not exactly know about the information of the firm. We consider the problem for pricing of corporate defaultable bond in the case when the firm value is only declared in some fixed discrete time and unexpected default intensity is determined by the declared firm value. Here we provide a partial differential equation model for such a defaultable bond and give its pricing formula. Our pricing model is derived to solving problems of partial differential equations with random constants (default intensity) and terminal values of binary types. Our main method is to use the solving method of a partial differential equation with a random constant in every subinterval and to take expectation to remove the random constants.

1. INTRODUCTION

There are two main approaches to pricing defaultable corporate bonds; one is the *structural approach* and the other one is the *reduced form approach*.

In the structural method, we think that the default event occurs when the firm value is not enough to repay debt, that is, when the firm value reaches a certain lower threshold (called *default barrier*) from the above. Such a default can be expected and thus we call it *expected default*.

In the reduced-form approach, the default is treated as an unpredictable event governed by default intensity process. In this case, the default event can occur without any correlation with the firm value and such a default is called *unexpected default*. In the reduced-form approach, if the default probability in time interval $[t, t + \Delta t]$ is $\lambda \Delta t$, then λ is called a *default intensity*.

If an investor knows all information about the firm value and default barrier in every time, then it is better for him to use the structural approach. If an investor can not exactly know about the firm value or default barrier, then he needs to use reduced form model.

Nowadays, the use of unified models of structural approach and reduced-form approach is a trend [2, 3, 4, 5, 7, 9]. For example, using unified models of structural

approach and reduced-form approach, Realdon [9] studied a pricing of corporate bonds in the case with constant default intensity and gave pricing formulae of the bond using PDE method. Cathcart et al. [4] studied a pricing of corporate bonds in the case when the default intensity is a linear function of the interest rate. They gave a semi-analytical pricing formula. Cathcart et al.[5] presented a valuation model that combines features of both the structural and reduced-form approaches for modeling default risk. In [5] the default intensity is a linear function of the state variable and the interest rate and they found that term structures of credit spreads generated using the middle-way approach were more in line with empirical observations. Other authors studied the pricing model of defaultable bonds in which the default intensity is given as a stochastic process [2, 3, 7, 9, 10]. In [7], the authors provided analytical pricing formula of corporate defaultable bond with both expected and unexpected defaults in the case when stochastic default intensity follows one of 3 special cases of Wilmott model [10]. Bi et al. [3] provided the similar result with [7] in the case when stochastic default intensity follows CIR-like model. Ballestra et al. [2] proposed a new model to price defaultable bonds which incorporates features of both structural and reduced-form models of credit risk where default intensity is described by an additional stochastic differential equation coupled with the process of the firm's asset value, and provided a closed-form approximate solution to their model.

In the papers [2, 3, 4, 5, 7, 9], they tried to express the price of the bond in terms of the firm value or the related signal variable to the firm value and the values of default intensity and defaul barrier at any time in the whole lifetime of the bond.

On the other hand, every company announces its management data once in a certain term (for example, every quarter or every six months) and the announced data reflect the firm's financial circumstances. It is difficult for investors outside of the firm to know the firm's financial data except for these discrete announcing dates.

According to this circumstance, in this paper we study the pricing problem for defaultable corporate bond from the viewpoint of the investors outside the firm that could only know the time-discretely announced information of the firm. We assume that we only know the firm value and the default barrier at several fixed discrete announcing dates and we dont know about any information of the firm value in another time. We assume that the default intensity between the adjoined two announcing dates is determined by its announced firm value at the former announcing date and it is not changed in that time interval. And we assume that the firm value follows a geometric Brownian motion. (This problem was studied in [8] but it included an error in deriving the pricing formula.) Such an approach is a kind of study of defaultable bond under *insufficient information* about the firm and it is interesting to note that Agliardi et al. [1] studied bond pricing problem under *imprecise information* with the technique of fuzzy mathematics.

In this paper, when pricing corporate defaultable bond, we use the reduced form approach on every time interval between the adjoined two announcing dates and use the structural approach at the announcing dates. And although we take the unexpected default intensity as a constant, but we assume that the unexpected default intensity between the adjoined two announcing dates depends on its announced firm value at the former announcing date. Thus we try to use all available information we can get. We suppose such an approach would comparatively be

reasonable.

Characteristics of our model are 1) the starting point is the viewpoint of investors outside of the firm that could not exactly know the firm value and default information; 2) our model is one of structural-reduced form unified model.

In our model, the short rate follows a generalized Hull-White model. The default event occurs in expected manner when the firm value reaches a certain lower threshold - the default barrier at one of the announcing dates or in unexpected manner at the first jump time of a Poisson process with intensity, respectively. Then our pricing problem is derived to a solving problem of PDE with random constant default intensity and terminal value of binary type in every subinterval between the two adjoined announcing dates.

Our main method to solve this problem is to use the solving method of a partial differential equation with a random constant in every subinterval between the two adjoined announcing dates and to take expectation to remove the random constant.

The remainder of the article is organized as follows: in section 2 we provide our modeling on corporate bond problem and give the pricing formula. In section 3 we prove the pricing formula.

2. Modeling and the Pricing Formula

Assumptions

1) A firm issues a corporate bond with maturity T and maturity face value 1.

2) Let $0 = t_0 < t_1 < \cdots < t_N - 1 < t_N = T$. At every time t_i, the firm value $V_i = V(t_i)$ is revaluated and announced. The firm value $V(t)$ follows a geometric Brown motion

$$dV(t) = (\mu - b)V(t)dt + s_V V(t)dW_1(t)$$

under the risk neutral martingale measure. (μ, b and s_V are constants and $W_1(t)$ is an 1-dimensional standard Wiener process.) The firm continuously pays out dividend in rate b for a unit of firm value.

3) The unexpected default probability in the interval $[t, t+\Delta t] \cap [t_i, t_{i+1})$ is $\lambda_i \Delta t$, and the default intensity λ_i is a known deterministic function of the firm value V_i at the time t_i. For example, if we can assume that $\lambda_i = \lambda(V_i) = \ln(1 + \frac{1}{V_i})$, then λ_i goes to 0 when V goes to infinity. This can be compatible with the real situations.

4) Short rate satisfies the following condition under the risk neutral martingale measure:

$$dr_t = a_r(r,t)dt + s_r(t)dW_2(t), \ a_r(r,t) = a_1(t) - a_2(t)r. \tag{1}$$

Here $W_2(t)$ is an 1-dimensional standard Wiener process.

5) The unexpected default recovery R_{ud} is given by $R_u \cdot Z(r,t)$ (exogenous recovery). Here recovery rate $0 \le R_u \le 1$ is a constant and $Z(r,t)$ is the price of the default free zero coupon bond.

6) The expected default barrier is only given at the time t_i. Expected default event occurs when $V(t_i) \le K_i$. Here K_i is a constant and the expected default recovery R_{ed} is given by $R_e \cdot Z(r,t)$, where recovery rate $0 \le R_e \le 1$ is a constant.

Method of Modeling

For simplicity, we assume that $N = 2$. From assumption 3), after the time t_1 the unexpected default intensity $\lambda_1 = \lambda(V_1)$ in the subinterval $(t_1, T]$ is a *known constant* and there is no any expected default in the open interval (t_1, T). And at the time $t = T$ expected default event occurs when $V_2 < K_2$. So under the

condition that the firm value V_1 at the time t_1 is already known, the bond price $C_1(r, t; V_2|V_1)$ in the interval $(t_1, T]$ (when we regard the firm value V_2 at the time $t_2 = T$ as a known quantity) can be seen as a derivative of short rate with the constant default intensity $\lambda(V_1)$ and it satisfies the following reduced-form model [10]:

$$\frac{\partial C_1}{\partial t} + \frac{s_r^2(t)}{2}\frac{\partial^2 C_1}{\partial r^2} + a_r(r, t)\frac{\partial C_1}{\partial r} - rC_1 + \lambda(V_1)(R_u Z(r, t) - C_1) = 0, \quad (2)$$

$$C_1(r, t_2) = C_1(r, t_2; V_2|V_1) = \begin{cases} 1 & \text{if } V_2 > K_2, \\ R_e & \text{if } V_2 \le K_2. \end{cases} \quad (3)$$

Here $Z(r, t)$ is default free zero coupon bond price and V_2 at the time $t < T$ is, in fact, an unknown random parameter. We solve the problem (2), (3) to get the function $C_1(r, t; V_2|V_1)$.

From the assumption 2) we can get the distribution of V_2 under the condition that V_1 is known, and thus taking expectation in $C_1(r, t; V_2|V_1)$ on V_2 we can get our bond price $C(r, t; V_1)$ in the interval $(t_1, T]$.

In the interval $[0, t_1]$ the unexpected default intensity $\lambda(V_0)$ is a known constant and at the time $t = t_1$ expected default event occurs when $V_1 < K_1$. So for every fixed firm value V_1 (at the time t_1) the bond price $C_0(r, t; V_1|V_0)$ in the interval $[0, t_1]$ satisfies the following reduced-form model:

$$\frac{\partial C_0}{\partial t} + \frac{s_r^2(t)}{2}\frac{\partial^2 C_0}{\partial r^2} + a_r(r, t)\frac{\partial C_0}{\partial r} - rC_0 + \lambda(V_0)(R_u Z(r, t) - C_0) = 0, \quad (4)$$

$$C_0(r, t_1) = C_0(r, t_1; V_1|V_0) = \begin{cases} C(r, t_1; V_1) & \text{if } V_1 > K_1, \\ R_e Z(r, t_1) & \text{if } V_1 \le K_1. \end{cases} \quad (5)$$

Here V_1 at the time $t < t_1$ is in fact an unknown random parameter, too.

If we solve the problem (4), (5) to get the function $C_0(r, t; V_1|V_0)$ and take expectation on V_1, then we can get our bond price $C(r, t; V_0)$ in the interval $[0, t_1)$.

V_2 in the problem (4), (5) and V_1 in the problem (2), (3) are random constants independent on the variables r and t of our equation. Thus the problem (2), (3) and the problem (4), (5) are terminal problems of partial differential equations with random parameters. And the terminal value conditions (3) and (5) are the functions of binary type that alternatively take two values on conditions.

The bond price $C(r, t; V_0)$ in the interval $[0, t_1)$ depends on not only the short rate r and t but also the initial firm value V_0 (at $t = 0$) and default barriers K_1, K_2.

The Pricing Formula
We have the following pricing formula in the time interval $[0, t_1)$:

$$C(r, t; V_0, K_1, K_2) =$$

$$= Z(r, t)\{e^{-\lambda(V_0)(t_1-t)}[R_u N_2(\alpha_1, \alpha_2 : A) + R_u R_e N_2(\alpha_1, -\alpha_2 : \tilde{A}) + I_{22} + I_{24}]$$
$$+ [1 - e^{-\lambda(V_0)(t_1-t)}]R_u N(\alpha_1) + [R_u + (1 - R_u)e^{-\lambda(V_0)(t_1-t)}]R_e N(-\alpha_1)\}. \quad (6)$$

Here $Z(r,t)$ is the price of risk free bond given in the next section and

$$N_2(a,b:A) = \frac{\sqrt{\det A}}{2\pi} \int_{-\infty}^a \int_{-\infty}^b e^{-\frac{1}{2}\xi^\perp \cdot A\xi} dxdy, \ \xi = (x,y)^\perp,$$

$$\alpha_1 = \frac{1}{s_V\sqrt{t_1}} \left[\ln \frac{V_0}{K_1} + (\mu - b - \frac{s_V^2}{2})t_1 \right],$$

$$\alpha_2 = \frac{1}{s_V\sqrt{t_2 - t_1}} \left[\ln \frac{V_0}{K_2} + (\mu - b - \frac{s_V^2}{2})t_2 \right],$$

$$A = \begin{pmatrix} \frac{t_2}{t_2-t_1} & \sqrt{\frac{t_1}{t_2-t_1}} \\ \sqrt{\frac{t_1}{t_2-t_1}} & 1 \end{pmatrix}, \ \tilde{A} = \begin{pmatrix} \frac{t_2}{t_2-t_1} & -\sqrt{\frac{t_1}{t_2-t_1}} \\ -\sqrt{\frac{t_1}{t_2-t_1}} & 1 \end{pmatrix},$$

$$N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-\frac{1}{2}x^2} dx,$$

$$I_{22} = (1 - R_u)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_1} F(x)N\left(\alpha_2 + x\sqrt{\frac{t_1}{t_2 - t_1}}\right) e^{-\frac{x^2}{2}} dx,$$

$$I_{24} = (1 - R_u)R_e\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_1} F(x)N\left(-\alpha_2 - x\sqrt{\frac{t_1}{t_2 - t_1}}\right) e^{-\frac{x^2}{2}} dx,$$

Here

$$F(x) = \exp\left\{ -(t_2 - t_1)\lambda\left(V_0 e^{\left(\mu - b - \frac{s_V^2}{2}\right)t_1 + s_V x\sqrt{t_1}} \right) \right\},$$

A proof of the formula (6) is provided in appendix.

3. APPENDIX: PROOF OF THE PRICING FORMULA

Here we prove the formula (6).

Under the assumption 4) in the domain $K = \{(r,t)|r \in \mathbf{R}, t \in [0,T]\}$, the price $Z(r,t)$ of risk free bond satisfies the following problem:

$$\frac{\partial Z}{\partial t} + \frac{s_r^2(t)}{2} \frac{\partial^2 Z}{\partial r^2} + a_r(r,t)\frac{\partial Z}{\partial r} - rZ = 0, \ Z(r,T) = 1. \tag{7}$$

The solution is given by

$$Z(r,t) = e^{A(t) - B(t)r}, \tag{8}$$

Here $A(t)$ and $B(t)$ are differently given dependant on the specific models (including *Vasicek, Ho-Lee and Hull-White* models) of short rate [10]. For example, if the short rate satisfies the *Vasicek* model, that is, if the coefficients $a_1(t), a_2(t), s_r(t)$ in (1) are all constants a_1, a_2, s_r, then $A(t)$ and $B(t)$ are given as follows [10]:

$$B(t) = \frac{1 - e^{-a_2(T-t)}}{a_2}, \ A(t) = -\int_t^T [a_2 B(u) - \frac{1}{2}s_r^2 B^2(u)]du.$$

Solving the problem (2) **and** (3)

In (2) and (3) we use the unknown function transformation $C_1(r,t) = u_1(t)Z(r,t)$ and consider the equation (7) and the relation (8), then we have the following equation with $u_1(t)$ as an unknown function:

$$\frac{du_1}{dt} - \lambda(V_1)u_1 + \lambda(V_1)R_u = 0, \ (t_1 < t < t_2 = T),$$

$$u_1(T) = \begin{cases} 1 & \text{if } V_2 > K_2, \\ R_e & \text{if } V_2 \leq K_2. \end{cases}$$

It is an initial value problem of an ordinary differential equation and the solution is easily given by

$$u_1(t) = \begin{cases} R_u + (1 - R_u)e^{-\lambda(V_1)(t_2 - t)} & \text{if } V_2 > K_2, \\ R_u + (R_e - R_u)e^{-\lambda(V_1)(t_2 - t)} & \text{if } V_2 \leq K_2. \end{cases}$$

Thus the solution to (2) and (3) is given by

$$C_1(r, t; V_2 | V_1) = \begin{cases} Z(r,t)[R_u + (1 - R_u)e^{-\lambda(V_1)(t_2 - t)}] & \text{if } V_2 > K_2, \\ Z(r,t)[R_u + (R_e - R_u)e^{-\lambda(V_1)(t_2 - t)}] & \text{if } V_2 \leq K_2. \end{cases} \quad (9)$$

The price of the Bond in the time interval $(t_1, T]$

From the assumption 2) we have

$$V_t = V_s \exp\left[(\mu - b - \frac{s_V^2}{2})(t - s) + s_V(W_{1t} - W_{1s})\right],$$

$$\text{Prob}\{W_{1t} - W_{1s} \in A\} = \int_A \frac{1}{\sqrt{2\pi(t - s)}} \exp\left[-\frac{x^2}{2(t - s)}\right] dx.$$

Thus we have

$$\text{Prob}\{V_2 > K_2\} = \int_{-\frac{1}{s_V}[\ln\frac{V_1}{K_2} + (\mu - b - \frac{s_V^2}{2})(t_2 - t_1)]}^{\infty} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left[-\frac{x^2}{2(t_2 - t_1)}\right] dx$$

$$= \int_{-\infty}^{\frac{1}{s_V\sqrt{t_2 - t_1}}[\ln\frac{V_1}{K_2} + (\mu - b - \frac{s_V^2}{2})(t_2 - t_1)]} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] dx.$$

If in the above expression we use the cumulated distribution function $N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}x^2} dx$ of standard normal distribution and the notation of

$$d_-(x/K, \mu, T - t) = \frac{\ln\frac{x}{K} + (\mu - b - \frac{s_V^2}{2})(T - t)}{s_V\sqrt{T - t}},$$

then we can get $\text{Prob}\{V_2 > K_2\} = N[d_-(V_1/K_2, \mu, t_2 - t_1)]$ and similarly we have $\text{Prob}\{V_2 \leq K_2\} = N[-d_-(V_1/K_2, \mu, t_2 - t_1)]$.

We take expectation in (9) to remove the random constant V_2, then we have the price $C(r, t : V_1)$ of our bond in the interval $[t_1, T]$:

$$C(r, t : V_1) = Z(r,t)\left[R_u + (1 - R_u)e^{-\lambda(V_1)(t_2 - t)}\right]\left\{N\left[d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right] + \right.$$

$$\left. + R_e N\left[-d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right]\right\}$$

In particular, if we denote

$$f(V_1) = \left[R_u + (1 - R_u)e^{-\lambda(V_1)(t_2 - t_1)}\right]\left\{N\left[d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right] + \right.$$

$$\left. + R_e N\left[-d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right]\right\} \quad (10)$$

then at the time t_1 we have

$$C(r, t_1 : V_1) = Z(r, t_1)f(V_1).$$

Solving of (4) **and** (5)

Now we know the price $C(r, t_1 : V_1)$ of our bond at the time t_1 and thus the problem (4) and (5) on the interval $[0, t_1]$ is written as follows:

$$\frac{\partial C_0}{\partial t} + \frac{s_r^2(t)}{2}\frac{\partial^2 C_0}{\partial r^2} + a_r(r,t)\frac{\partial C_0}{\partial r} - rC_0 + \lambda(V_0)(R_u Z(r,t) - C_0) = 0, \ (11)$$

$$C_0(r,t_1) = \begin{cases} Z(r,t_1)f(V_1) & \text{if } V_1 > K_1, \\ R_e Z(r,t_1) & \text{if } V_1 \leq K_1. \end{cases} \tag{12}$$

V_0 is known in the interval $[0, t_1]$, so $\lambda(V_0)$ is known constant. But V_1 is a random parameter in the interval $[0, t_1)$. For every fixed V_1, when $V_1 \leq K_1$, we use the same method as the above to get the solution of (11) and (12):

$$C_0(r,t; V_1 \leq K_1|V_0) = Z(r,t)[R_u + (R_e - R_u)e^{-\lambda(V_0)(t_1-t)}], \ 0 \leq t < t_1. \tag{13}$$

Similarly, when $V_1 > K_1$, we can get

$$C_0(r,t; V_1 > K_1|V_0) = Z(r,t)[R_u + (f(V_1) - R_u)e^{-\lambda(V_0)(t_1-t)}], \ 0 \leq t < t_1. \tag{14}$$

The price of the bond in time interval $[0, t_1]$

We add (13) and (14) after taking expectation on V_1 to remove it in (13) and (14), then we have the price in the time interval $[0, t_1]$. As (13) does not include V_1 and we already knew

$$\text{Prob}\{V_1 \leq K_1\} = N[-d_-(V_0/K_1, \mu, t_1)],$$

as the above, we easily get the expectation of (13):

$$E(C_0(r,t; V_1 \leq K_1|V_0)) =$$

$$= R_e Z(r,t)\left[R_u + (1 - R_u)e^{-\lambda(V_0)(t_1-t)}\right] N\left[-d_-\left(\frac{V_0}{K_1}, \mu, t_1\right)\right]. \tag{15}$$

Now we calculate the expectation $E(C_0(r,t; V_1 > K_1|V_0))$ of (14). Unlike (13), $C_0(r,t; V_1 > K_1|V_0)$ is a function of V_1, and so we denote

$$g(V_1) = g(r,t,V_0; V_1) := C_0(r,t; V_1 > K_1|V_0).$$

From the assumption 2),

$$V_1 = V_0 \exp\left[\left(\mu - b - \frac{s_V^2}{2}\right)t_1 + s_V W_{1t_1}\right] \tag{16}$$

and thus $g(V_1)$ is written as

$$g(r,t,V_0; V_1) = g\left(r,t,V_0; V_0 e^{\left(\mu - b - \frac{s_V^2}{2}\right)t_1 + s_V W_{1t_1}}\right).$$

And we note that $\text{Prob}\{W_{1t} \in A\} = \int_A \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{x^2}{2t}\right]dx$, and $V_1 > K_1 \iff W_{1t} > -\frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + \left(\mu - b - \frac{s_V^2}{2}\right)t_1\right]$. Thus we have

$$E(C_0(r,t; V_1 > K_1|V_0)) =$$

$$= \frac{1}{\sqrt{2\pi t_1}}\int_{-\infty}^{\frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + (\mu - b - \frac{s_V^2}{2})t_1\right]} g\left(r,t,V_0; V_0 e^{(\mu - b - \frac{s_V^2}{2})t_1 + s_V x}\right) e^{-\frac{x^2}{2t_1}}dx \tag{17}$$

In order to calculate the integral of (17), we need to get the representation of the function

$$G(x) = g\left(r, t, V_0; V_0 e^{(\mu - b - \frac{s_V^2}{2})t_1 + s_V x}\right)$$

in the integrand in (17). From the definition of $g(r, t, V_0; V_1)$ and (14) we have

$$g(r, t, V_0; V_1) = R_u Z(r, t)\left[1 - e^{-\lambda(V_0)(t_1 - t)}\right] + Z(r, t)e^{-\lambda(V_0)(t_1 - t)}f(V_1)$$

In (??), if we write $f(V_1)$ as $f(V_1) = g_{21}(V_1) + g_{22}(V_1) + g_{23}(V_1) + g_{24}(V_1)$, then we have

$$g(r, t, V_0; V_1) = g_1(r, t, V_0) + Z(r, t)e^{-\lambda(V_0)(t_1 - t)}[g_{21}(V_1) + g_{22}(V_1) + g_{23}(V_1) + g_{24}(V_1)]$$

Here

$$g_1(r, t, V_0) = R_u Z(r, t)\left[1 - e^{-\lambda(V_0)(t_1 - t)}\right],$$

$$g_{21}(V_1) = R_u N\left[d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right],$$

$$g_{22}(V_1) = (1 - R_u)e^{-\lambda(V_1)(t_2 - t_1)}N\left[d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right], \qquad (18)$$

$$g_{23}(V_1) = R_u R_e N\left[-d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right],$$

$$g_{24}(V_1) = R_e(1 - R_u)e^{-\lambda(V_1)(t_2 - t_1)}N\left[-d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right)\right].$$

In (16) we denote $x = W_{1t_1}$, then $V_1 = V_0 \exp\left[\left(\mu - b - \frac{s_V^2}{2}\right)t_1 + s_V x\right]$ and so we can write

$$e^{-\lambda(V_1)(t_2 - t_1)} = \exp\left\{-(t_2 - t_1)\lambda\left(V_0 e^{(\mu - b - \frac{s_V^2}{2})t_1 + s_V x}\right)\right\}.$$

In particular, if the function $\lambda(V)$ is given by $\lambda(V) = \ln(1 + \frac{1}{V})$, then we have

$$e^{-\lambda(V_1)(t_2 - t_1)} = \left[\frac{V_0 e^{(\mu - b - \frac{s_V^2}{2})t_1 + s_V x}}{1 + V_0 e^{(\mu - b - \frac{s_V^2}{2})t_1 + s_V x}}\right]^{t_2 - t_1}. \qquad (19)$$

Now we represent $d_-(V_1/K_2, \mu, t_2 - t_1)$ in (18) as a function of x. If we denote

$$\alpha_2 = \frac{\ln\frac{V_0}{K_2} + \left(\mu - b - \frac{s_V^2}{2}\right)t_2}{s_V\sqrt{t_2 - t_1}}, \qquad (20)$$

then we have

$$d_-\left(\frac{V_1}{K_2}, \mu, t_2 - t_1\right) = \frac{\ln\frac{V_1}{K_2} + (\mu - b - \frac{s_V^2}{2})(t_2 - t_1)}{s_V\sqrt{t_2 - t_1}} = \alpha_2 + \frac{x}{\sqrt{t_2 - t_1}}.$$

Using this we get the representations of g_{21}, g_{22}, g_{23} and g_{24} in (18) in terms of x (these are still writen as g_{2i}):

$$g_{21}(x) = R_u \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_2} e^{-\frac{1}{2}\left(y + \frac{x}{\sqrt{t_2-t_1}}\right)^2} dy, \tag{21}$$

$$g_{22}(x) = (1 - R_u) \exp\left[-(t_2 - t_1)\lambda(V_0 e^{(\mu-b-\frac{s_V^2}{2})t_1 + s_V x})\right] N\left(\alpha_2 + \frac{x}{\sqrt{t_2-t_1}}\right), \tag{22}$$

$$g_{23}(x) = R_u \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\alpha_2} e^{-\frac{1}{2}\left(y - \frac{x}{\sqrt{t_2-t_1}}\right)^2} dy, \tag{23}$$

$$g_{24}(x) = R_e(1 - R_u) \exp\left[-(t_2 - t_1)\lambda\left(V_0 e^{(\mu-b-\frac{s_V^2}{2})t_1 + s_V x}\right)\right] N\left(-\alpha_2 - \frac{x}{\sqrt{t_2-t_1}}\right). \tag{24}$$

Now we calculate (17).

$$E(C_0(r, t; V_1 > K_1 | V_0)) =$$

$$= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + \left(\mu-b-\frac{s_V^2}{2}\right)t_1\right]} g\left(r, t, V_0; V_0 e^{(\mu-b-\frac{s_V^2}{2})t_1 + s_V x}\right) e^{-\frac{x^2}{2t_1}} dx$$

$$= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + \left(\mu-b-\frac{s_V^2}{2}\right)t_1\right]} g_1(r, t, V_0) e^{-\frac{x^2}{2t_1}} dx +$$

$$+ Z(r, t) e^{-\lambda(V_0)(t_1 - t)} \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\frac{\ln\frac{V_0}{K_1} + \left(\mu-b-\frac{s_V^2}{2}\right)t_1}{s_V}} f\left(V_0 e^{(\mu-b-\frac{s_V^2}{2})t_1 + s_V x}\right) e^{-\frac{x^2}{2t_1}} dx$$

$$= I_1 + Z(r, t) e^{-\lambda(V_0)(t_1 - t)} I_2. \tag{25}$$

Here

$$I_1 = R_u Z(r, t) \left[1 - e^{-\lambda(V_0)(t_1 - t)}\right] \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + \left(\mu-b-\frac{s_V^2}{2}\right)t_1\right]} e^{-\frac{x^2}{2t_1}} dx$$

$$= R_u Z(r, t) \left[1 - e^{-\lambda(V_0)(t_1 - t)}\right] N\left[d_-\left(\frac{V_0}{K_1}, \mu, t_1\right)\right], \tag{26}$$

$$I_2 = \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + \left(\mu-b-\frac{s_V^2}{2}\right)t_1\right]} f\left(V_0 e^{(\mu-b-\frac{s_V^2}{2})t_1 + s_V x}\right) e^{-\frac{x^2}{2t_1}} dx$$

$$= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + \left(\mu-b-\frac{s_V^2}{2}\right)t_1\right]} [g_{21}(x) + g_{22}(x) + g_{23}(x) + g_{24}(x)] e^{-\frac{x^2}{2t_1}} dx$$

$$= I_{21} + I_{22} + I_{23} + I_{24}. \tag{27}$$

Here we used the fact that $g_1(r, t, V_0)$ does not depend on x and $f = g_{21} + g_{22} + g_{23} + g_{24}$. Now we calculate I_{2i}. For simplicity of symbol, let denote

$$\alpha_1 = \frac{1}{s_V}\left[\ln\frac{V_0}{K_1} + \left(\mu - b - \frac{s_V^2}{2}\right)t_1\right] = d_-\left(\frac{V_0}{K_1}, \mu, t_1\right). \tag{28}$$

Then from (21), we have

$$I_{21} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_1} g_{21}(x\sqrt{t_1})e^{-\frac{x^2}{2}}dx = \frac{R_u}{2\pi} \int_{-\infty}^{\alpha_1} dx \int_{-\infty}^{\alpha_2} e^{-\frac{x^2}{2}-\frac{1}{2}\left(y+\frac{x\sqrt{t_1}}{\sqrt{t_2-t_1}}\right)^2} dy.$$
(29)

The exponent of the integrand of (29) can be written as a bivariate quadratic form $-\frac{1}{2}\xi^\perp A\xi$, where

$$A = \begin{pmatrix} \frac{t_2}{t_2-t_1} & \sqrt{\frac{t_1}{t_2-t_1}} \\ \sqrt{\frac{t_1}{t_2-t_1}} & 1 \end{pmatrix}, \quad \det A = 1, \xi^\perp = (x,y).$$
(30)

Thus I_{21} is represented by the cumulated distribution function N_2 of the bivariate normal distribution as follows:

$$I_{21} = R_u \frac{\sqrt{\det A}}{2\pi} \int_{-\infty}^{\alpha_1} dx \int_{-\infty}^{\alpha_2} e^{-\frac{1}{2}\xi^\perp A\xi}dy = R_u N_2(\alpha_1, \alpha_2; A).$$
(31)

Similarly, from (23), we have the representation of I_{23} by the cumulated distribution function N_2 of the bivariate normal distribution:

$$I_{23} = R_u R_e \frac{\sqrt{\det \tilde{A}}}{2\pi} \int_{-\infty}^{\alpha_1} dx \int_{-\infty}^{-\alpha_2} e^{-\frac{1}{2}\xi^\perp \tilde{A}\xi}dy = R_u N_2(\alpha_1, -\alpha_2; \tilde{A}).$$
(32)

Here

$$\tilde{A} = \begin{pmatrix} \frac{t_2}{t_2-t_1} & -\sqrt{\frac{t_1}{t_2-t_1}} \\ -\sqrt{\frac{t_1}{t_2-t_1}} & 1 \end{pmatrix}, \quad \det \tilde{A} = 1.$$
(33)

From (27) and (22) we directly get

$$I_{22} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_1} g_{22}(x\sqrt{t_1})e^{-\frac{x^2}{2}}dx =$$
$$= \frac{1-R_u}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_1} F(x)N\left(\alpha_2 + x\sqrt{\frac{t_1}{t_2-t_1}}\right) e^{-\frac{x^2}{2}}dx.$$
(34)

Here

$$F(x) = \exp\left\{-(t_2-t_1)\lambda\left(V_0 e^{\left(\mu-b-\frac{s_V^2}{2}\right)t_1+s_V x\sqrt{t_1}}\right)\right\}.$$

From (27) and (24) we directly get

$$I_{24} = \frac{R_e(1-R_u)}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_1} F(x)N\left(-\alpha_2 - x\sqrt{\frac{t_1}{t_2-t_1}}\right) e^{-\frac{x^2}{2}}dx.$$
(35)

Substitute (31), (32),(34) and (35) into (27) to get I_2. Then substitute I_2 and I_1 into (25) to get $E(C_0(r,t;V_1 > K_1|V_0))$ of (17). Then our bond price is given by

$$C(r,t;V_0) = E(C_0(r,t;V_1 > K_1|V_0)) + E(C_0(r,t;V_1 \le K_1|V_0))$$

which gives the above formula (6).

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