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DYNAMIC PROPERTIES FOR A GENERAL SEIV EPIDEMIC MODEL

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Abstract. An SEIV epidemic model with vaccination, treatment and a general nonlinear incidence rate is considered. A formula to determine the reproduction number R_0 for the formulated model is deduced and it plays an important role in studying the equilibria. It is shown that the disease-free equilibrium is asymptotic stable if $R_0 < 1$(hence the disease will dies out) and unstable if $R_0 > 1$. Sufficient conditions are established to grantee the asymptotic stability of the endemic equilibria for $R_0 > 1$ using Gerschgorin disks and Routh-Hurwitz theorems. Some numerical simulations are given for some special cases of incidence rate to illustrate the idea of the obtained results.

1. Introduction

Mathematical Models help us to improve controlling the spread of diseases by studying the effect of different methods that affect. Vaccination participates by an essential part of this controlling, but it is very difficult to vaccinate all susceptible individuals. Sometimes adequate attention must be paid to exceed a threshold level of vaccinate to reduce the spread of disease. The rate of new infections is called incidence rate, and it depends on susceptible and infectious individuals. This incidence rate is really nonlinear in many models due to the complicated process of transmission of disease. So Stability of the epidemic models with nonlinear incidence rate and vaccination have received great interest. Recently the authors in [1, 4, 8, 9, 13] discussed global stability of different epidemic models with nonlinear incidence rates and the authors in [11-13, 15] discussed the role of vaccination and treatment in controlling disease.

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In this paper we investigate an SEIV epidemic model for childhood diseases with non-permanent. We consider a general form of the incidence rate as follows:

$$\begin{aligned} \dot{S} &= (1-p)A - \mu\, S + \omega V - I\, F(S,I), \\ \dot{V} &= pA - \mu\, V - \omega\, V + \delta\, I, \\ \dot{E} &= I\, F(S,I) - \mu\, E - \sigma\, E, \\ \dot{I} &= \sigma\, E - \mu\, I - \delta\, I, \end{aligned} \tag{1}$$

where S,V, E and I represent respectively the number of susceptible, vaccinated-treated, exposed but not yet infectious and infectious individual. All the parameters are positive, p is the fraction of vaccinated recruited individuals, A is the recruitment rate of children (either by birth or by immigration) into the population (assumed susceptible), μ is the natural mortality rate, ω is the rate at which vaccine wanes. The incidence rate is given by a general nonlinear function $I\, F(S,I)$, δ is the rate of treatment of infected individuals, and σ is the rate of entering the exposed individuals to be infectious.

Systems similar to model (1) were discussed by [1,3, 12, 16] for some special cases of the incidence rate. Bilinear incidence rate was considered by Li et al in [12] in an SEIVR epidemic model with latent stage and vaccination. They used Lyapunov function to prove the global stability of the disease-free equilibrium if $R_0 \leq 1$ and for the unique endemic equilibria if $R_0 > 1$. Cai and Li [3] considered Eq. (1) when $F(S,I) = \beta\, S(t)/\phi(I)$. They show that if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable, while for $R_0 > 1$ they deduced conditions for global asymptotic stability of the unique endemic equilibrium. In [16] Zhoa and Cui studied the special case when $F(S,I) = \beta\, S(t)\,(1 + \alpha\, I(t))$. They discussed the backward bifurcation phenomenon and the global stability of the endemic equilibrium for their model using the center manifold theory and a generalized Poincare Bendixson criterion. Finally Abdulrazak et al [1] discussed system (1) when $F(S,I) = \beta\, S(t)\,/\phi(\,I)$ and used Bellman and Cooke's theorem to analyze the stability behavior of their model.

The aim of this paper is to discuss the stability behavior of the equilibria of system (1) related with the reproduction number R_0 and $F(S,I)$ using the methods of Gerschgorin disks and Routh-Hurwitz theorem. This paper is organized as follows: Section 2 represents the equilibria and evaluation of the reproduction number R_0 of system (1). Also sufficient conditions for asymptotic stability of the equilibria are given using Gerschgorin disks technique. In section metricconverterProductID3, a3, a reduced three dimensional system corresponding to system (1) is considered and sufficient conditions are given that guarantee the asymptotic stability of the equilibria. Moreover an excepectation of chaotic behavior solutions is illustrated. In section 4, some simulation examples using Matlab are solved. And finally we give our conclusions in section 5.

2. Equilibria and Reproduction Number

It is clear that the equilibrium points of (1) are obtained by solving the nonlinear algebraic equations:

$$\dot{S} = \dot{V} = \dot{E} = \dot{I} = 0 \tag{2}$$

The physical subset region of R^4 associated with system (1) can be determined by considering it, we have,

$$(\dot{S + V + E + I}) = \mu\,[A/\mu - (S + V + E + I)\,] \tag{3}$$

So, we can write

$$\lim_{t\to\infty} \sup\,(S + V + E + I) = A/\mu. \tag{4}$$

Hence the considerable region of system (1) is

$$D^* = \{\,(S, V, E, I\,):\ S + V + E + I \le A/\mu\,,\quad S > 0,\ V \ge 0,\ E \ge 0\,, I \ge 0\,\}$$

From Eq. (3), it is clear that the region $D^*$is a positively invariant set with respect to system (1). Now solving Eq. (2) we have $E = \frac{\mu+\delta}{\sigma} I$, $V = \frac{pA+\delta\,I}{\mu+\omega}$, $S = \frac{1}{\mu}\,[A(1 - P + \frac{p\omega}{\mu+\omega}) - (\frac{(\mu+\delta)(\mu+\sigma)}{\sigma} - \frac{\delta\omega}{\mu+\omega})I\,]$, $I[F(S, I) - \frac{(\mu+\delta)(\mu+\sigma)}{\sigma}] = 0$ So if $I = 0$, then we have the disease-free equilibrium point $P_0^*(\,\frac{A}{\mu}\,[1 - P + \frac{p\omega}{\mu+\omega}\,]\,,\ \frac{pA}{\mu+\omega}\ , 0\,, 0\,)$, while we have infected equilibria $P_1^*(S_1, V_1, E_1, I_1)$, if

$$F(S_1, I_1) = \frac{(\mu+\delta)(\mu+\sigma)}{\sigma} \tag{5}$$

where

$$S_1 = \frac{1}{\mu}\,[A(1-P+\frac{p\omega}{\mu+\omega})-(\frac{(\mu+\delta)(\mu+\sigma)}{\sigma}-\frac{\delta\omega}{\mu+\omega})I_1\,], V_1 = \frac{pA+\delta\,I_1}{\mu+\omega}\ ,\ E_1 = \frac{\mu+\delta}{\sigma}I_1 \tag{6}$$

To determine the reproduction number R_0, we can rewrite system (1) in the following form:
$\dot{X} = H(X) - G(X)$, $X = (E, I, S, V)^T$, where

$$H(X) = \begin{pmatrix} I\,F(S, I) \\ 0 \\ 0 \\ 0 \end{pmatrix}\ ,\quad and\ \ G(X) = \begin{pmatrix} (\mu+\sigma)\,E \\ (\mu+\delta)\,I - \sigma\,E \\ IF(S,\,I) + \mu\,S - \omega V\ - (1-p)A \\ (\mu+\omega)\,V - \delta\,I - pA \end{pmatrix}$$

Using the Jacobian matrix of $H(X)$ *and* $G(X)$ *at* P_0^* and Theorem metricconverterProductID2 in2 in [5], we can conclude that the formula of the reproduction number of (1) is

$$R_0 = \frac{\sigma\,(I\frac{\partial F}{\partial I} + F(S, I)\,)_{P_0^*}}{(\mu+\sigma)(\mu+\delta)}. \tag{7}$$

Now we study the stability behavior of system (1). It always has the disease-free equilibrium point P_0^* while it has an infected equilibrium point p_1^* if and only if

$F(S,I)$ satisfies condition (5). The following theorem discusses the stability of P_0^*.

Theorem 2.1
Suppose that the function $F(S,I)$ in system (1) satisfies that

$$(I\frac{\partial F}{\partial S})_{P_0^*} = 0 \tag{8}$$

Then the free-disease equilibrium point P_0^* is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.
Proof:
The characteristic equation of system (1) at P_0^* and when condition (8) holds, is:

$$(\lambda+\mu)(\lambda+\mu+\omega)[(\lambda+\mu+\sigma)(\lambda+\mu+\delta)-\sigma(I\frac{\partial F}{\partial I}+F(S,I))_{P_0^*}]=0 \tag{9}$$

The eigenvalues of Eq. (9) are $\lambda_1 = -\mu$, $\lambda_2 = -(\mu+\omega)$ and the two roots of the simplified quadratic equation $\lambda^2 + \lambda(2\mu+\sigma+\delta) + (\mu+\sigma)(\mu+\delta)(1-R_0) = 0$. Hence:

$$\lambda_{3,4} = \frac{1}{2}[-(2\mu+\sigma+\delta) \pm \sqrt{(2\mu+\sigma+\delta)^2 - 4(\mu+\sigma)(\mu+\delta)(1-R_0)}]. \tag{10}$$

Thus it is clear that $\lambda_{3,4}$ have negative real parts if $R_0 < 1$ which guarantee that P_0^* is locally asymptotically stable, while if $R_0 > 1$, P_0^* is unstable where there is an eigenvalues with positive real part.
Note that in the case P_0^* is the only critical point of (1), if (8) is satisfied, we have the following result.
Theorem 2.2
Let P_0^* be the only critical point for (1) and condition (8) is satisfied, then P_0^* is globally asymptotically stable if $R_0 < 1$.

The following result deals with the case when system (1) has an infected equilibrium point $P_1^* \in D^*$,
Theorem 2.3
Suppose that there is an infected equilibrium point $P_1^* \in D^*$ of system (1) and the function $F(S,I)$ satisfies that:

$$(I\frac{\partial F}{\partial S})_{P_1^*} > -\mu/2 \tag{11}$$

$$\left|(I\frac{\partial F}{\partial I}+F(S,I))_{P_1^*}\right| < \mu/2 \tag{12}$$

Then P_1^* is asymptotically stable
Proof:

Since the Jacobin matrix of the system (1) in terms of P_1^* is

$$J(P_1^*) = \begin{pmatrix} -\mu - (I\frac{\partial F}{\partial S})_{P_1^*} & \omega & 0 & -(I\frac{\partial F}{\partial I} + F(S,I))_{P_1^*} \\ 0 & -(\mu+\omega) & 0 & \delta \\ (I\frac{\partial F}{\partial S})_{P_1^*} & 0 & -(\mu+\sigma) & (I\frac{\partial F}{\partial I} + F(S,I))_{P_1^*} \\ 0 & 0 & \sigma & -(\mu+\delta) \end{pmatrix}$$

Then the Gerschgorin column disks C_i in the complex plane which defined by $C_i = the\ set\ of\ all\ z\ where\ \ |z - a_{ii}| \le c_i\ \ and\ c_i = \sum_{k=1,\ k\neq i}^{4} |a_{ki}|\ ,\ i = 1:4$

Then we have:

$$C_1 : \left| z + (\mu + (I\frac{\partial F}{\partial S})_{P_1^*}) \right| \le \left| (I\frac{\partial F}{\partial S})_{P_1^*} \right| , \quad C_2 : |z + (\mu + \omega)| \le \omega,$$

$C_3 : |z + (\mu + \sigma)| \le \sigma$ and $C_4 : |z + (\mu + \delta)| \le \delta + 2\left|(I\frac{\partial F}{\partial I} + F(S,I))_{P_1^*}\right|$.
It is clear that C_2 and C_3 lie in the left side of the imaginary axis. Now for the first disk C_1, it lies in the left side of the imaginary axis if $\mu + (I\frac{\partial F}{\partial S})_{P_1^*} > \left|(I\frac{\partial F}{\partial S})_{P_1^*}\right|$, which is satisfied by condition (11). Similarly for the last disk C_4. Since the required condition to lie in the left side is $\mu + \delta > \delta + 2\left|(I\frac{\partial F}{\partial I} + F(S,I))_{P_1^*}\right|$, then condition (12) with Theorem 7.9.9 in [2] guarantee this fact. Thus it follows that all the eigenvalues of $J(P_1^*)$ lie in the left part of the complex plane. Consequently P_1^* is asymptotically stable.

3. Stability of Equilibria in The Three Dimensional Reduced System

According to Eq. (4), we can write a reduction limit system to system (1) in the form:

$$\begin{aligned} \dot{S} &= (1-p)A - \mu S + \omega V - I\,F(S,I), \\ \dot{V} &= pA - \mu V - \omega V + \delta I, \\ \dot{I} &= \sigma\,[A/\mu - (S+V+I)] - \mu I - \delta I. \end{aligned} \tag{13}$$

The equilibria of system (13) are $P_0(\frac{A}{\mu}[1 - P + \frac{p\omega}{\mu+\omega}], \frac{pA}{\mu+\omega}, 0)$ in addition to $P_1(S_1, V_1, I_1)$ when (5) holds, where $P_0, P_1 \in D \equiv \{(S,V,I) : S + V + I \le A/\mu,\ S > 0,\ V \ge 0\ and\ I \ge 0\}$ and P_1 is defined in (6).
The Jacobian matrix of (13) in term of P_i is

$$J_{P_i} = \begin{pmatrix} -\mu - (I\frac{\partial F}{\partial S})_{P_i} & \omega & -(I\frac{\partial F}{\partial I} + F(S,I))_{P_i} \\ 0 & -(\mu+\omega) & \delta \\ -\sigma & -\sigma & -(\mu+\delta+\sigma) \end{pmatrix}$$

And the characteristic equation of J_{P_i} is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{14}$$

$$\begin{aligned} a_1 &= a + b + c + (I\frac{\partial F}{\partial S})_{P_i}, \\ a_2 &= ab + a\delta + \delta\sigma + (a+b+\delta)(\mu + (I\frac{\partial F}{\partial S})_{P_i}) - \sigma(I\frac{\partial F}{\partial I} + F(S,I))_{P_i}, \\ a_3 &= [a(b+\delta) + \delta\sigma]\,[\mu + (I\frac{\partial F}{\partial S})_{P_i}] + \omega\delta\sigma - a\sigma(I\frac{\partial F}{\partial I} + F(S,I))_{P_i}, \end{aligned} \tag{15}$$

and $a = \mu + \omega$, $b = \mu + \sigma$, $c = \mu + \delta$.
The coefficients of the characteristic equation (14) around disease-free equilibrium P_0 takes the following forms when considering substitution by the value of the reproduction number R_0:

$$\begin{aligned} a_1 &= a + b + c + (I\tfrac{\partial F}{\partial S})_{P_0}, \\ a_2 &= a(b+c) + bc\,(1 - R_0) + (a+b+\delta)\,(I\tfrac{\partial F}{\partial S})_{P_0}, \\ a_3 &= abc\,(1-R_0) + (ab + a\,\delta + \delta\,\sigma)\,(I\tfrac{\partial F}{\partial S})_{P_0}. \end{aligned} \tag{16}$$

Also we can write

$$\begin{aligned} a_1\,a_2 - a_3 = {} & a(b+c)(a+b+c) + bc(b+c)(1-R_0) + (a+b+\delta)\,((I\tfrac{\partial F}{\partial S})_{P_0})^2 + \\ & (I\tfrac{\partial F}{\partial S})_{P_0}\,[(a+\mu+c)(a+b+\delta) + b(a+\sigma) + bc\,(1-R_0)\,] \end{aligned} \tag{17}$$

The following theorem classifies the stability conditions of P_0.
Theorem 3.1
Assume in system (13), that the function $F(S, I)$satisfies

$$(I\frac{\partial F}{\partial S})_{P_0} > M = \max\{\,-(a+b+c)\,,\, -\frac{ab+ac+bc(1-R_0)}{a+b+\delta}\,,\, -\frac{abc\,(1-R_0)}{ab+a\delta+\delta\sigma}\,\} \tag{18}$$

$$\begin{aligned} & (I\tfrac{\partial F}{\partial S})_{P_0}\,[(a+\mu+c)(a+b+\delta) + b(a+\sigma) + bc\,(1-R_0)\,] \\ & \quad > -\,[a(b+c)(a+b+c) + bc(b+c)(1-R_0) + (a+b+\delta)\,((I\tfrac{\partial F}{\partial S})_{P_0})^2] \end{aligned} \tag{19}$$

Then the disease-free equilibrium P_0 of system (13) is asymptotically stable.
Proof:
Applying the Routh Hurwitz theorem [15] on the characteristic equation (14) with the coefficients (16), it is clear that (18) guarantees that $a_i > 0$, $i = 1, 2, 3$. To complete the proof, we should have $a_1\,a_2 - a_3 > 0$, which is directly obtained using (19) in Eq. (17).
In the special case of Theorem 3.1 when $(I\frac{\partial F}{\partial S})_{P_0} = 0$, we have the following result,
Theorem 3.2
Assume that $(I\frac{\partial F}{\partial S})_{P_0} = 0$. Then the disease-free equilibria P_0 is
(i)asymptotically stable when $R_0 < 1$ (ii) stable when $R_0 = 1$ (iii)unstable when $R_0 > 1$.
Proof:

(1) If $R_0 < 1$, then according to Theorem 3.1 P_0 is asymptotically stable.
(2) When $R_0 = 1$, we have $a_1 > 0$, $a_2 > 0$ *and* $a_3 = 0$. Then the eigenvalues are $\lambda_1 = 0$, *Real* $\lambda_{2,3} < 0$ which grantee that P_0 is stable.
(3) If $R_0 > 1$, we have $a_1 > 0$, $a_3 < 0$. Then applying Descart's principle we have a positive eigenvalues and so P_0 is unstable.

Now we discuss the stability of infection equilibria when $F(S, I)$ satisfies (5).
Theorem 3.3
Assume that system (13) has an infected equilibria P_1 , the values of a, b and c are defined in Eq

(14) and the function $F(S,I)$ satisfies that:

$$(I\frac{\partial F}{\partial S})_{P_1}+(a+b+c)>0, \tag{20}$$

$$\sigma\,(I\frac{\partial F}{\partial I})_{P_1}-m<0\ \ ,\ \ m=\min\{\,a(b+c)+(a+b+\delta)\,(I\frac{\partial F}{\partial S})_{P_1}\,,\,(b+\delta+\frac{\delta\,\sigma}{a})\,(I\frac{\partial F}{\partial S})_{P_1}\,\}, \tag{21}$$

$$a(b+c)(a+b+c)+(a+b+\delta)\,[(I\tfrac{\partial F}{\partial S})_{P_1}]^2+[(a+b)\,(a+b+c)+ac+\delta(\mu+c)]\,(I\tfrac{\partial F}{\partial S})_{P_1}>\sigma(I\tfrac{\partial F}{\partial I})_{P_1}\,[b+c+\,(I\tfrac{\partial F}{\partial S})_{P_1}\,] \tag{22}$$

Then P_1 is asymptotically stable.

Proof:

The coefficients of the characteristic equation (14)-(15) using Eq. (5) are

$$\begin{aligned} a_1 &= a+b+c+(I\tfrac{\partial F}{\partial S})_{P_1}\\ a_2 &= ab+ac+(a+b+\delta)(I\tfrac{\partial F}{\partial S})_{P_1}-\sigma(I\tfrac{\partial F}{\partial I})_{P_1}\\ a_3 &= [a(b+\ \delta)+\delta\,\sigma\,]\,(I\tfrac{\partial F}{\partial S})_{P_1}-a\sigma(I\tfrac{\partial F}{\partial I})_{P_1} \end{aligned} \tag{23}$$

It is clear that condition (20) insure that $a_1>0$ and condition (21) guarantees that $a_2>0$ *and* $a_3>0$. Now since
$a_1\,a_2-a_3=[a+b+c+(I\frac{\partial F}{\partial S})_{P_1}]\,[ab+ac+(a+b+\delta)(I\frac{\partial F}{\partial S})_{P_1}-\sigma(I\frac{\partial F}{\partial I})_{P_1}]-[a(b+\ \delta)+\delta\,\sigma\,]\,(I\frac{\partial F}{\partial S})_{P_1}+a\sigma(I\frac{\partial F}{\partial I})_{P_1}$
then by simplification, we have

$$a_1\,a_2-a_3=a(b+c)(a+b+c)+(a+b+\delta)\,((I\tfrac{\partial F}{\partial S})_{P_1})^2+(I\tfrac{\partial F}{\partial S})_{P_1}\,[(a+b)\,(a+b+c)+ac+\delta(\mu+c)\,]-\sigma\,(I\tfrac{\partial F}{\partial S}\,\tfrac{\partial F}{\partial I})_{P_1}\,-\sigma(b+c)\,(I\tfrac{\partial F}{\partial S})_{P_1} \tag{24}$$

Applying condition (22), we see that $a_1\,a_2-a_3>0$. Then by the Routh-Hurwitz theorem [15], it follows that the real parts of the eigenvalues of Eq. (14) with the coefficients (23) are negative.

Now we try to answer the important question, Can system (13) have periodic solutions?
It is easy to prove that the linear part of system (13) has a family of periodic solutions around P_1 if the following conditions are satisfied:
(i) $a(b+c)+(a+b+\delta)\,(I\frac{\partial F}{\partial S})_{P_1}-\sigma\,(I\frac{\partial F}{\partial I})_{P_1}>0$
(ii) $a(b+c)(a+b+c)+(a+b+\delta)\,((I\frac{\partial F}{\partial S})_{P_1})^2+(I\frac{\partial F}{\partial S})_{P_1}\,[(a+b)\,(a+b+c)+ac+\delta(\mu+c)\,]=\sigma\,(I\frac{\partial F}{\partial I})_{P_1}\,[b+c+\,(I\frac{\partial F}{\partial S})_{P_1}\,]$

These periodic solutions have period $T=2\pi\,/\,\varpi$, where $\varpi=[a(b+c)+(a+b+\delta)\,(I\frac{\partial F}{\partial S})_{P_1}-\sigma\,(I\frac{\partial F}{\partial I})_{P_1}]^{0.5}$
Hence we can expect chaotic behavior of system (13) around the infection point P_1 for values of system parameters near that satisfy conditions (i) and (ii).

4. Discussion

Now we consider special cases of the function $F(S,I)$ as examples to illustrate the results.

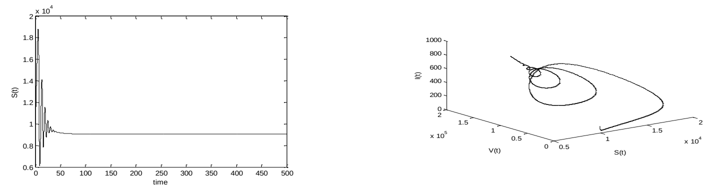

Figure 1. (a) time response of S(t), (b) phase portrait in (S,I,V) phase space of system (1)

Example1: Let $F(S,I) = \beta_1 S(t)(1+\beta_2 I(t))$. Hence $P_0^* = (\frac{A}{\mu}[1-p+\frac{p\,\omega}{\mu+\omega}], \frac{p\,A}{\mu+\omega}, 0, 0)$, and the
reproduction number is $R_0 = \frac{A\sigma\,\beta_1}{bc\,\mu}(1-p+\frac{p\,\omega}{a})$. To get P_1^*, we have $F(S_1, I_1) = bc/\sigma$. Substituting from (5) we get the quadratic equation $d_1 I_1^2 + d_2 I_1 + d_3 = 0$, where $d_1 = \beta_1 \beta_2(\delta\sigma\omega - abc) < 0$, $d_2 = abc(\mu\beta_2 R_0 - \beta_1) + \beta_1 \delta\sigma\omega$ *and* $d_3 = abc\mu(R_0 - 1)$. When $R_0 < 1$, we have $d_1 < 0$, $d_3 < 0$ *then if* $d_2 < 0$ there is no positive value for I_1, while if $d_2 > 0$, the required condition for existence of two positive values for I_1 is $d_2^2 - 4\,d_1\,d_3 > 0$, otherwise there is no positive values. Now consider the case $R_0 > 1$. Then$d_1 < 0$, $d_3 > 0$. Thus there is a unique positive value for $I_1 = (1/2\,d_1)[-d_2 - \sqrt{d_2^2 - 4\,d_1\,d_3}]$. Consider the set values of the parameters $A = 10000$,$\beta_1 = 0.0009$, $\beta_2 0.0005$,$\omega = 0.05$, $\mu = 0.06$, $\delta = 11$, $\sigma = 2$, $p = 0.8$, we have $R_0 = 7.4216$. Then by applying Theorem 3.3, we find that the conditions (20) –(22) are satisfied, hence the unique infected equilibria P_1 is asymptotically stable . A numerical solution of system (13) is given in figures (1.a) and (1.b).

It is clear that the above result agree with the obtained results by Zhou et al in [16] which is a special case of our results.
Example2: Another choices of the coefficients, we consider$A = 11000$,$\beta_1 = 0.00048$, $\beta_2 0.0005$, $\omega = 0.05$, $\mu = 0.06$, $\delta = 20$, $\sigma = 2.070701$ and $p = 0.6$, we have $R_0 = 2.868$. Simple calculation show that conditions (i) and (ii) are satisfied, so P_1 of the linearized system of (13) has periodic solutions around P_1. Figure (2.a, 2.b) shows a numerical solution for the linearized system while figure (3.a, 3.b) gives the solutions for the same parameters for system(13).
Moreover, for the same values in example 2 except the value of σ is increased by a small amount to be $\sigma = 2.5$, chaos appears around P_1. Figure (4.a, 4.b) illustrate this result.

If we consider $F(S,I) = \beta\,S(t)$ (the bilinear case), hence condition (8) is satisfied

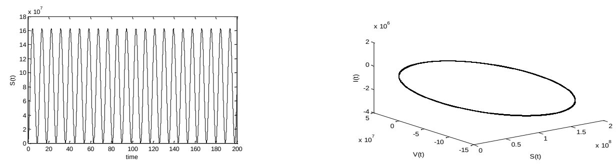

FIGURE 2. (a) time response of S(t) for linear part , (b) phase portrait in (S, V,I) phase space of linear part of system (1)

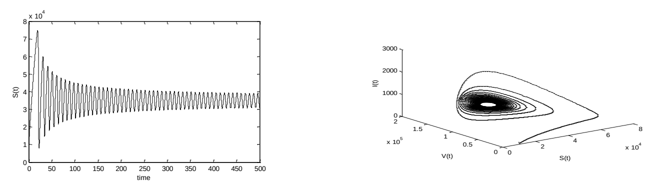

FIGURE 3. (a) time response of S(t), (b) phase portrait in (S, V,I) phase space of system (1)

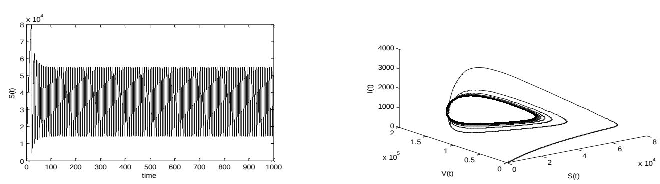

FIGURE 4. (a) time response of S(t), (b) phase portrait in (S, V, I) space of system (1)

where $(I\frac{\partial F}{\partial S})_{P_0^*} = 0$. By applying Theorem 2.1 or 2.3, the disease-free equilibrium point P_0^* is aymptotically stable when $R_0 < 1$ and unstable for $R_0 > 1$, see examples 3 and 4.

Example3: Consider the following values for the parameters of system (1) $A = 0.2$,$\beta = 0.8$, $\omega = 0.1$, $\mu = 0.2$, $\delta = 0.5$, $\sigma = 0.3$and $p = 0.1$, we have $R_0 = 0.64$. Hence applying theorem 2.2 , the disease-free equilibrium point P_0^* is asymptotically stable. The obtained result agrees with that obtained by A. J. Abdulrazak et al [1].

Example4: For the same values of the parameters considered in example 3 except $\beta = 1.0$, $\sigma = 1.0$, we have $R_0 = 1.1111$. And conditions of Theorem

3.3 are satisfied, henceP_1 (and consequently $P_1^*(0.84, 0.1129, 0.194, 0.0277)$is asymptotically stable. A numerical solution is shown in figure 5(a, b).

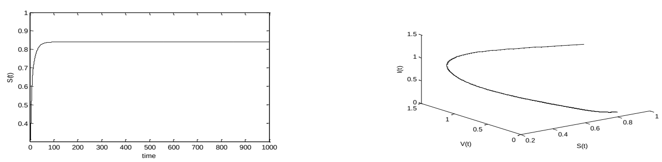

FIGURE 5. (a) time response of S(t), (b) phase portrait in (S, V, I) space of system (1)

5. Conclusion

In this paper we considered an SEIV epidemic model (1) with general non-linear incidence rate. We investigate a general formula to determine the reproduction number R_0 in Eq.(7). Our results show that the value of R_0 completely determine the stability behavior of the disease-free equilibrium point P_0^* when $(I\frac{\partial F}{\partial S})_{P_0^*} = 0$. To discuss the stability conditions for the infected equilibrium point P_1 when it is exist, we use Gerschgorin theorem to give simple conditions that guarantee the asymptotic stability of system (1). Then we investigate sufficient conditions insure the asymptotic stability of the equilibria for an SIV reduction system (13), using Routh Hurwitz theorem. Moreover required conditions are given to guarantee the existence of Hopf bifurcation of the linearized system of (13) around the infected equilibria P_1. And excepectation of existence of chaos is discussed. To illustrate our results we considered a special case of $F(S,I) = \beta_1\,S(t)\,(1+\beta_2\,I(t)\,)$. We note that our obtained results agree with those obtained by Zhou et al [16]. Moreover Theorem 3.3 can be considered as a general form for Theorem metricconverterProductID3.4 in3.4 in [16]. Also we consider $F(S,I) = \beta\,S(t)$which gives the bilinear case of the incidence rate. We find that when $R_0 < 1$, the disease-free equilibrium point P_0^* is asymptotically stable. For $R_0 > 1$, P_0^* is unstable and there is only infected equilibria P_1^* which is asymptotically stable where the conditions of Theorem 3 are satisfied.

References

1. A. J. Abdulrazak, M. O. Ibrahim, I. O. Usman, A Seiv Vaccination Model with General Non-Linear Incidence Rate and Waning Preventive Vaccine, IOSR Journal of Mathematics, Volume 4, Issue 2(Nov.– Dec. 2012) 44-51.

2. Robert L. Borrelli, Courtney S. Coleman, Differential Equations: A Modeling Perspective. John Wiley & Sons, Inc (1998).
3. Li-Ming Cai, Xue-Zhi Li, Analysis Of A SEIV Epidemic Model With A Nonlinear Incidence Rate, Applied Mathematical Modeling 33, 2919-2926 (2009).
4. Li- Ming Cai, Xue-Zhi Li, Mini Ghosh, Global Stability Of Stage-Structured Epidemic Model With A Nonlinear Incidence, Applied Mathematics And Computation 214, 73-82 (2009).
5. L. Cai, X. Li, M. Ghosh And B. Guo, Stability Analysis Of An HIV/AIDS Epidemic Model With Treatment, Journal Of Computational And Applied Mathematics, 229 (2009) 313-323.
6. Van Den Driessche, P. Watmough, J., Reproduction Numbers And Sub-Threshold Endemic Equilibria For Compartmental Models Of Disease Transmission. Math. Biosci. 180, 24-48 (2002).
7. L. H. Erbe, S. V. H. Rao, H. Freedman, Three-Species Food Chain Models With Mutual Interference And Time Delays, Math. Biosci. 80, 57-80 (1986).
8. Qiang Hou, Zhen Jin, Shigui Ruan, Dynamics of Rabies Epidemics and the Impact of Control Efforts in Guangdong Province, China, Journal of Theoritical Biology, 300, 39-47 (2012).
9. Andrei Korobeinikov, Global Properties Of Infectious Disease Models With Nonlinear Incidence, Bulletin Of Mathematical Biology 69, 1871-1886 (2007).
10. A. Lahrouz, L. Omari, D. Kiouach, Global Analysis Of A Deterministic And Stochastic Non Linear SIRS Epidemic Model, Nonlinear Analysis: Modeling And Control, 16, 59-76 (2011).
11. Xue-Zhi Li, Jing Wang, Mini Ghosh, Stability And Bifurcation Of An SIVS Epidemic Model With Treatment And Age Of Vaccination, Applied Mathematical Modeling, 34, 437-450 (2010).
12. Jianquan Li, Yali Yang, Yicang Zhou, Global Stability Of An Epidemic Model With Latent Stage And Vaccination, Nonlinear Analysis: Real World Applications 12, 2163-2173 (2011).
13. Xue-Zhi Li, Lin-Lin Zhou, Global Stability Of An SEIR Epidemic Model With Vertical Transmission And Saturating Contact Rate, Chaos Solutions And Fractals 40,873-884 (2009).
14. Dan Long, Zhongyi Xiang, On The Study Of An SEIV Epidemic Model Concerning Vaccination And Vertical Transmission, Journal Of Applied Mathematics, Bioinformatics 1, 21-30 (2011).
15. M. R. M. Rao, Ordinary Differential Equations, Theory And Applications. Edward Arnold, London (1981).
16. Xueyon Zhou, Jingan Cui, Analysis Of Stability And Bifurcation For An SEIV Epidemic Model With Vaccination And Nonlinear Incidence Rate, Nonlinear Dyn, Springer Science ,Business Media B. V. (2010).

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