Electronic Journal of Mathematical Analysis and Applications
Vol. 2(1) Jan. 2014, pp. 53-63.
ISSN: 2090-792X (online)
http://fcag-egypt.com/Journals/EJMAA/

EXISTENCE, UNIQUENESS AND REGULARITY PROPERTY OF SOLUTIONS TO FOKKER-PLANCK TYPE EQUATIONS

RONGRONG TIAN

ABSTRACT. In this paper, we study the existence, uniqueness and H^k-regularity property of solutions to a class of Fokker-Planck type equations with Sobolev coefficients and L^2 initial conditions. Meantime, as a slight extension of Le Bris and Lions' work ([1]), we obtain the existence and uniqueness of weak L^p-solutions for Fokker-Planck type equations with L^p initial values.

1. **Introduction**

Our purpose in this article is to prove the existence, uniqueness and regularity property of solutions for Fokker-Planck type equations with coefficients in Sobolev spaces. In the present article, we consider the Fokker-Planck type equation of 'divergence' form below:

$$\begin{cases} \partial_t f - b_i \partial_i f - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f) = 0, \text{ in } (0,T] \times \mathbb{R}^N, \\ f(t=0,\cdot) = f_0, \text{ in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

Firstly, let us recall some prototypical results for the case of $\sigma_{ik} = \delta_{ik}$ (where δ_{ik} is the standard Kroneck-delta signa) and $b = 0$, i.e.

$$\begin{cases} \partial_t f - \frac{1}{2}\Delta f = 0, \text{ in } (0,T] \times \mathbb{R}^N, \\ f(t=0,\cdot) = f_0, \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1$'$}$$

in classical parabolic theory.

It is well known in classical parabolic theory that, once the existence and uniqueness of a weak solution to $(1.1)'$ in $L^2([0,T]; H^1) \cap H^1([0,T]; H^{-1})$ are proved, then under the additional assumptions that the first kth derivatives of f_0 with the spatial variable exist and belong to L^2, one may prove that the solution $f(t,x)$ to $(1.1)'$ belongs to $L^2([0,T]; H^{k+1}) \cap H^1([0,T]; H^{k-1})$ as well. It is easy to see that if we presume further that f_0 is smooth, then Sobolev embedding theorems also imply the smoothness of f.

On the other hand, formally, we can trade spatial-regularity property against temporal-regularity property at a cost of two spatial derivatives for one temporal

---

2000 *Mathematics Subject Classification.* 35A05; 35B65.
Key words and phrases. Existence, Uniqueness, Regularity.
Submitted June 14, 2013.

derivative. It inspires us to get higher spatial regularity property for solutions by multiplying a factor t^α when the initial value does not have good properties. Based on this fact, let us consider our present work to (1.1).

We mention that, in [1], Le Bris and Lions have proved if $f_0 \in L^2 \cap L^\infty$, there exists a unique weak solution f to (1.1) as well as the following properties:

$$f \in L^\infty([0,T]; L^2 \cap L^\infty), \quad \sigma^t \nabla f \in L^2([0,T]; L^2).$$

Besides, if $\sigma\sigma^t$ is uniformly positive, they also get

$$f \in L^2([0,T]; H^1).$$

Do the solutions have better properties, such as the H^k regularity property ? They do not answer it and it is our concerning. It consists the main part of our present article Section 2.

It is remarked that our motivation to study (1.1) stems from SDEs directly (for more details for SDE, one can refer to [2]). In fact, if we consider autonomous SDE

$$dX(t,x) = -b(X(t,x))dt + \sigma(X(t,x))dW_t, \tag{1.2}$$

and let f be the law of X_t (here $X_t = X(t,x)$) , then by the Itô rule, f satisfies (1.1) starting from the law of $X_t|_{t=0}$. Correspondingly, the existence (or uniqueness) of solutions to (1.1) may transfer the existence (or uniqueness) of solutions to (1.2).

It is noted that, different from [1], there are many other research works on Fokker-Planck equations, such as [3-5] and the references cited up there.

This paper is organized as follows: in Section 2, we prove the H^2 regularity property of the solutions with spatial variable, Section 3 is devoted to extend our existence, uniqueness and regularity property results to Fokker-Planck type equation (1.1) to L^p initial conditions.

Before our argument, we introduce some requisite notations for future use.

The summation convention is enforced throughout this article, wherein summation is understood with respect to repeated indices. The letter c will mean a positive constant, whose values may change in different places. The differential operator of gradient and divergence we used by ∇ and div and the differential operator with the subscript i, i.e. ∂_i denotes $\frac{\partial}{\partial x_i}$, with $1 \le i \le N$, for $x = (x_1, \dots, x_N)$.

2. Fokker-Planck type equations with L^2 initial conditions

This section is intended to state and prove the existence, uniqueness and regularity property for solutions of (1.1). To reach this aim, firstly let us make precise the mathematical setting and look back some notions.

Consider the following Fokker-Planck equation of divergence form

$$\partial_t f - b_i \partial_i f - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f) = 0, \tag{2.1}$$

with initial value $f_0 \in L^2(\mathbb{R}^N)$. We call

$$f \in L^\infty([0,T]; L^2(\mathbb{R}^N)), \quad \sigma^t \nabla f \in L^2([0,T]; L^2(\mathbb{R}^N))$$

a weak solution to (2.1) if for any $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$, the following identity

$$\int_0^T\int f\partial_t\varphi + \int f_0\varphi(0,\cdot) - \int_0^T\int fb_i\partial_i\varphi + \mathrm{div}b\varphi f - \frac{1}{2}\int_0^T\int \sigma_{ik}\sigma_{jk}\partial_j f\partial_i\varphi = 0 \tag{2.2}$$

holds, where $\mathcal{D}([0,T)\times\mathbb{R}^N)$ denotes the set of all smooth functions on $[0,T)\times\mathbb{R}^N$ with compact supports.

2.1 Sobolev drift coefficients and constant diffusion coefficients

We begin with the Sobolev drift coefficient and constant diffusion coefficient, that is

$$\partial_t f - b_i\partial_i f - \frac{1}{2}\Delta f = 0, \tag{2.3}$$

with the initial data $f(t=0,\cdot) = f_0 \in L^2\cap L^\infty(\mathbb{R}^N)$.

By standard steps that are: an a priori estimate and a regularization procedure, it is easy to get: there exists a unique

$$f \in L^\infty([0,T];L^2\cap L^\infty(\mathbb{R}^N))\cap L^2([0,T];H^1(\mathbb{R}^N))$$

solving (2.3) supplied with initial value f_0, if the vector field b (for simplicity we assume b is independent of time) satisfies

$$\begin{cases} (Q_1) \quad b\in W^{1,1}_{loc}(\mathbb{R}^N); \\ (Q_2) \quad \mathrm{div}b \in L^\infty(\mathbb{R}^N); \\ (Q_3) \quad \frac{b}{1+|x|}\in L^\infty(\mathbb{R}^N). \end{cases}$$

The details we will give in Section 3 and now we omit it.

Remark 2.1. Of course above conclusion is also true if b is time dependent by requiring L^1 integrability in the temporal variable.

A natural question will be raised: How to get the high order regularity property property?

From classical partial differential equations theory [see [6] chapter 7] we have the following fact: if $u_0\in H^k$, the unique weak solution of heat equation

$$\begin{cases} u_t - \Delta u = 0, \text{ in } (0,T]\times\mathbb{R}^N, \\ u(t=0) = u_0, \text{ in } \mathbb{R}^N, \end{cases}$$

meets:

$$u \in L^\infty([0,T];H^k(\mathbb{R}^N))\cap L^2([0,T];H^{k+1}).$$

But on the other hand, as the statement in introduction, formally we can 'trade' spatial-regularity property against temporal-regularity property at a cost of two spatial derivatives for one temporal derivative. It enlightens us to get H^k spatial regularity property for $t^\alpha f$ associated with bad initial conditions for f_0.

Theorem 2.2. Assume vector field b satisfies $(Q_1)-(Q_3)$, along with the additional presumption $\frac{\partial_i b_j\,+\,\partial_j b_i}{2} \le cId$ in the sense of symmetric matrices for some constant

c independent of x. Then for each initial datum $f_0 \in L^2 \cap L^\infty(\mathbb{R}^N)$, the unique solution of (2.3) satisfies

$$t^{\frac{1}{2}} f \in L^\infty([0,T]; H^1(\mathbb{R}^N)) \cap L^2([0,T]; H^2(\mathbb{R}^N)). \tag{2.4}$$

Proof. To begin with, we differentiate (2.3) with respect to the mth spatial variable x_m

$$\partial_t \partial_m f - \partial_m b_i \partial_i f - b_i \partial_{im}^2 f - \frac{1}{2}\Delta \partial_m f = 0. \tag{2.5}$$

If we denote by $g_m = t^{\frac{1}{2}} \partial_m f$, then by differentiating t with g_m, we get

$$\begin{aligned}
\partial_t g_m &= \frac{1}{2} t^{-\frac{1}{2}} \partial_m f + t^{\frac{1}{2}} \partial_t \partial_m f \\
&= \frac{1}{2} t^{-\frac{1}{2}} \partial_m f + t^{\frac{1}{2}} (\partial_m b_i \partial_i f + b_i \partial_{im}^2 f + \frac{1}{2}\Delta \partial_m f) \\
&= \frac{1}{2} t^{-\frac{1}{2}} \partial_m f + \partial_m b_i \partial_i (t^{\frac{1}{2}} f) + b_i \partial_i g_m + \frac{1}{2}\Delta g_m.
\end{aligned} \tag{2.6}$$

Multiplying by $g_m \phi_r$, where ϕ_r is a smooth cut-off function i.e.

$$\phi \in \mathcal{D}(\mathbb{R}^N), \quad \phi(x) = \begin{cases} 1 & \text{on } B(0;1), \\ \\ 0 & \text{on } B(0;2)^c, \end{cases} \quad ,$$

and

$$0 \le \phi(x) \le 1, \quad \phi_r(x) = \phi(\frac{x}{r}),$$

summing over m and integrating in $\mathbb{R}^N$, this leads to

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int |\nabla (t^{\frac{1}{2}} f)|^2 + \frac{1}{2}\int \mathrm{div} b |\nabla (t^{\frac{1}{2}} f)|^2 + \frac{1}{2}\int |\nabla^2 (t^{\frac{1}{2}} f)|^2 \\
= \; & \frac{1}{2}\int |\nabla f|^2 + \int \partial_m b_i \partial_i (t^{\frac{1}{2}} f) \partial_m (t^{\frac{1}{2}} f) \\
\le \; & \frac{1}{2}\int |\nabla f|^2 + c\int |\nabla (t^{\frac{1}{2}} f)|^2,
\end{aligned}$$

by virtue of integration by parts formula and tending r to infinity, where in the last inequality we used $\frac{\partial_i b_j + \partial_j b_i}{2} \le cId$.

Notice that $\mathrm{div} b \in L^\infty$, the Grönwall inequality applies, we derive

$$t^{\frac{1}{2}} f \in L^\infty([0,T]; H^1(\mathbb{R}^N)) \cap L^2([0,T]; H^2(\mathbb{R}^N)).$$

It is time for us to conclude H^3 regularity property.

Theorem 2.3. Let b be as in Theorem 2.2, in addition that $b \in \dot{W}^{2,\infty}$, where the dot on $W^{2,\infty}$ denotes the corresponding homogeneous space. Then the unique solution satisfies:

$$tf \in L^\infty([0,T]; H^2(\mathbb{R}^N)) \cap L^2([0,T]; H^3(\mathbb{R}^N)).$$

Proof. Firstly we differentiate (2.3) with respect to ith and jth spatial variables x_i, x_j:

$$\partial_t(\partial_{ij}^2 f) - \partial_{ij}^2 b_k \partial_k f - \partial_i b_k \partial_j (\partial_k f) - \partial_j b_k \partial_i (\partial_k f) - b_k \partial_k (\partial_{ij}^2 f) - \frac{1}{2}\Delta(\partial_{ij}^2 f) = 0. \tag{2.7}$$

Denote by $f_{ij} = \partial^2_{ij} f$ and differentiate tf_{ij} with t, combining above identity we obtain

$$\partial_t(tf_{ij}) = f_{ij} + t\partial^2_{ij} b_k f_k + t(\partial_i b_k f_{kj} + \partial_j b_k f_{ki}) + \frac{1}{2}\Delta(tf_{ij}) + tb_k f_{kij}. \tag{2.8}$$

Multiply (2.8) by $tf_{ij}\phi_r$, sum over i, j and integrate over $\mathbb{R}^N$, one gains

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |tf_{ij}|^2 + \frac{1}{2}\int \mathrm{div} b|tf_{ij}|^2 + \frac{1}{2}\int |\nabla(tf_{ij})|^2 \\ = \quad & \int |t^{\frac{1}{2}} f_{ij}|^2 + \int (\partial_i b_k (tf_{jk})(tf_{ji}) + \partial_j b_k (tf_{ik})(tf_{ij})) + \int \partial^2_{ij} b_k (tf_k)(tf_{ij}),\end{aligned} \tag{2.9}$$

by applying integration by parts formula and taking r to infinity.

Note that $\frac{\partial_i b_j + \partial_j b_i}{2} \leq cId$ and $\partial^2_{ij} b_k \in L^\infty$, on using Cauchy-Schwarz's inequality, one asserts

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |tf_{ij}|^2 + \frac{1}{2}\int \mathrm{div} b|t^{\frac{1}{2}} f_{ij}|^2 + \frac{1}{2}\int |\nabla(tf_{ij})|^2 \\ \leq \quad & c\int |tf_{ij}|^2 + \int |t^{\frac{1}{2}} f_{ij}|^2 + \int |t\nabla f|^2.\end{aligned} \tag{2.10}$$

The Grönwall inequality and the condition $\mathrm{div} b \in L^\infty$ also yield that

$$tf \in L^\infty([0,T]; H^2(\mathbb{R}^N)) \cap L^2([0,T]; H^3(\mathbb{R}^N))$$

and this completes our proof.

Remark 2.4. By virtue of the Lions-Aubin lemma, we also have

$$tf \in \mathcal{C}([0,T]; H^2(\mathbb{R}^N)),$$

and this implies

$$f \in \mathcal{C}((0,T]; H^2(\mathbb{R}^N)).$$

2.2 L^2-initial condition

For L^2 initial condition we can also verify that

Theorem 2.5 [Lipschitz regular coefficients, L^2 initial condition]. Let b be as in Theorem 2.3, that σ is time independent and fulfills

$$\sigma \in (\mathrm{Lip}(\mathbb{R}^N))^{N\times K}.$$

Then for each initial datum $f_0 \in L^2 \cap L^\infty$, there exists a unique weak solution f to (2.1). Moreover,

$$t^{\frac{1}{2}} f \in L^\infty([0,T]; H^1(\mathbb{R}^N)) \cap L^2([0,T]; H^2(\mathbb{R}^N)),$$

if we assume in addition that $\sigma\sigma^t$ is uniformly positive definite in $-\Delta$, i.e for any $f \in H^1(\mathbb{R}^N)$

$$\langle -\Delta f,\ \sigma\sigma^t f\rangle_{(H^{-1}(\mathbb{R}^N),\ H^1(\mathbb{R}^N))} \geq \alpha\langle -\Delta f,\ f\rangle_{(H^{-1}(\mathbb{R}^N),\ H^1(\mathbb{R}^N))}, \tag{Q_4}$$

for some real number $\alpha > 0$.

Proof. The existence and uniqueness of solutions to (2.1) have been proved in [1] and the solution meets

$$f \in L^\infty([0,T]; L^2(\mathbb{R}^N)) \cap L^\infty([0,T] \times \mathbb{R}^N).$$

Now it suffices to check :

$$t^{\frac{1}{2}}f \in L^{\infty}([0,T];H^1(\mathbb{R}^N)) \cap L^2([0,T];H^2(\mathbb{R}^N)).$$

In fact, if one denotes f_m by $\frac{\partial}{\partial x_m}f$, then

$$\partial_t(t^{\frac{1}{2}}f_m) = \frac{1}{2}t^{-\frac{1}{2}}f_m + t^{\frac{1}{2}}(b_i\partial_i f)_m + \frac{1}{2}t^{\frac{1}{2}}(\sigma_{jk}\sigma_{ik}\partial_j f)_{im}. \tag{2.11}$$

Multiplying $t^{\frac{1}{2}}f_m$ to the both hand sides, summing over with m and integrating over $\mathbb{R}^N$, one gains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla(t^{\frac{1}{2}}f)|^2 &= \frac{1}{2}\int|\nabla f|^2 + \int t^{\frac{1}{2}}(b_i\partial_i f)_m(t^{\frac{1}{2}}f_m)\\ &\quad +\frac{1}{2}\int t^{\frac{1}{2}}(\sigma_{jk}\sigma_{ik}f_j)_{im}(t^{\frac{1}{2}}f_m).\end{aligned} \tag{2.12}$$

By applying integration by parts formula, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla(t^{\frac{1}{2}}f)|^2 &= \frac{1}{2}\int|\nabla f|^2 + \int \partial_m b_i(t^{\frac{1}{2}}f_i)(t^{\frac{1}{2}}f_m)\\ &\quad -\frac{1}{2}\int \mathrm{div}b|\nabla(t^{\frac{1}{2}}f)|^2\\ &\quad -\frac{1}{2}\int \sigma_{jk}\sigma_{ik}(t^{\frac{1}{2}}f_j)(t^{\frac{1}{2}}f_{mmi}).\end{aligned} \tag{2.13}$$

By (Q_4) and Cauchy-Schwarz's inequality, one concludes with the estimate

$$\frac{1}{2}\frac{d}{dt}\int|\nabla(t^{\frac{1}{2}}f)|^2 + \frac{\alpha}{2}\int|\nabla^2(t^{\frac{1}{2}}f)|^2 \le \frac{1}{2}\int|\nabla f|^2 + c\int|\nabla(t^{\frac{1}{2}}f)|^2, \tag{2.14}$$

which implies

$$t^{\frac{1}{2}}f \in L^{\infty}([0,T];H^1(\mathbb{R}^N)) \cap L^2([0,T];H^2(\mathbb{R}^N)),$$

with the aid of the Grönwall lemma.

Remark 2.6. (i) If we presume further that

$$b \in \dot{W}^{2,\infty}(\mathbb{R}^N) \tag{Q_5}$$

and $\sigma\sigma^t$ is uniformly positive definite in $(-\Delta)^2$, i.e for any $f \in H^2(\mathbb{R}^N)$

$$\langle(-\Delta)^2 f,\ \sigma\sigma^t f\rangle_{(H^{-2}(\mathbb{R}^N),\ H^2(\mathbb{R}^N))} \ \ge\ \alpha\langle(-\Delta)^2 f,\ f\rangle_{(H^{-2}(\mathbb{R}^N),\ H^2(\mathbb{R}^N))}, \tag{Q_6}$$

where $\alpha > 0$ is a constant. Then the unique solution f satisfies

$$tf \in L^{\infty}([0,T];H^2(\mathbb{R}^N)) \cap L^2([0,T];H^3(\mathbb{R}^N)).$$

In fact, if one denotes f_{mn} by $\frac{\partial^2}{\partial x_m \partial x_n}f$, then

$$\partial_t(tf_{mn}) = f_{mn} + t(b_i\partial_i f)_{mn} + \frac{1}{2}t(\sigma_{jk}\sigma_{ik}\partial_i f)_{imn}. \tag{2.15}$$

Multiplying tf_{mn} to the both hand sides, summing over with m,n and integrating over $\mathbb{R}^N$, one ends up with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla(\nabla(tf))|^2 &= \int|\nabla(\nabla(t^{\frac{1}{2}}f))|^2 + \int\partial_{mn}b_i(tf_i)(tf_{mn}) + \int\partial_m b_i(tf_{ni})(tf_{mn})\\ &\quad + \int\partial_n b_i(tf_{mi})(tf_{mn}) - \frac{1}{2}\int\sigma_{jk}\sigma_{ik}(tf_j)(tf_{mnmni}),\end{aligned} \tag{2.16}$$

by applying integration by parts.

On using assumptions (Q_5) and (Q_6), we gain

$$\frac{1}{2}\frac{d}{dt}\int |\nabla(\nabla(tf))|^2 + \frac{\alpha}{2}\int |\nabla^3(tf)|^2 \leq \int |\nabla(\nabla(t^{\frac{1}{2}}f))|^2 + c\int |\nabla(\nabla(tf))|^2, \quad (2.17)$$

which hints

$$tf \in L^\infty([0,T]; H^2(\mathbb{R}^N)) \cap L^2([0,T]; H^3(\mathbb{R}^N)).$$

3. Fokker-Planck equations of divergence form: $L^p -$ theory $(1 < p < \infty)$

In this section, we consider the following Fokker-Planck equations of divergence form with initial data $f_0 \in L^p$,

$$\begin{cases} \partial_t f - b_i\partial_i f - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f) = 0, & \text{in } (0,T]\times\mathbb{R}^N, \\ f(t=0,\cdot) = f_0, \text{ in } \mathbb{R}^N. \end{cases} \quad (3.1)$$

To be simple, we grant $f_0 \in L^p \cap L^\infty(\mathbb{R}^N)$ and b and σ are time independent and meet at least

$$b \in L^1_{loc}(\mathbb{R}^N;\mathbb{R}^N), \quad \sigma \in L^2_{loc}(\mathbb{R}^N; R^{N\times K}).$$

We present two notions as follows:

Definition 3.1. We call $f \in L^\infty([0,T]; L^p \cap L^\infty(\mathbb{R}^N))$ a weak solution of (3.1) corresponding to initial condition $f_0 \in L^p \cap L^\infty(\mathbb{R}^N)$ if for any $\varphi \in \mathcal{D}([0,\text{ T})\times\mathbb{R}^N)$, the below identity holds:

$$\int_0^T\int f\partial_t\varphi + \int f_0\varphi(t=0,\cdot) - \int_0^T\int [b_i\partial_i\varphi + \text{div}b\varphi]f - \frac{1}{2}\int_0^T\int \sigma_{ik}\sigma_{jk}\partial_j f\partial_i\varphi = 0. \quad (3.2)$$

Definition 3.2. $f \in L^\infty([0,T]; L^p \cap L^\infty(\mathbb{R}^N))$ is said to be a strong solution to (3.1) associated with initial data $f_0 \in L^p \cap L^\infty(\mathbb{R}^N)$, if it is a weak solution and $(3.1)_1$ holds for almost all $(t,x) \in (0,T]\times\mathbb{R}^N$.

It time for us to state our first result in this section.

Theorem 3.3 [**L^p-weak solution**]. Let b and σ be time-independent and satisfy :

$$b \in (W^{1,1}_{loc}(\mathbb{R}^N))^N, \quad \frac{b}{1+|x|} \in (L^1 + L^\infty)(\mathbb{R}^N), \quad \text{div}b \in L^\infty(\mathbb{R}^N),$$
$$\sigma \in (W^{1,2}_{loc}(\mathbb{R}^N))^{N\times K}, \quad \frac{\sigma}{1+|x|} \in ((L^2 + L^\infty)(\mathbb{R}^N))^{N\times K}.$$

Then for each initial value $f_0 \in L^p \cap L^\infty(\mathbb{R}^N)$, there is a unique weak solution of equation (3.1) in the sense of Definition 3.1.

Our techniques of proof throughout this result come from [7] which is introduced by DiPerna and Lions in order to establish the existence and uniqueness of solutions to linear transport equations $\partial_t f - b_i\partial_i f = 0$ with Sobolev coefficients and for more in this topic, one can read [8-9] and the references.

The first step consists in establishing formal **an a priori estimate** on the tentative solutions f. This is performed by multiplying the equation by some function $\beta'(f)$ (where β is some convenient good function) and then integrate over the spatial variable.

Using the first step, in view of compactness method and regularization procedure, it is easy to get the existence.

The second major step is a regularization procedure which is based on the so-called Di Perna-Lions lemma. Concretely speaking, by regularizing to get the uniqueness.

Proof. • an a priori estimate for tentative solution f.

Multiplying $|f|^{p-2}f$ to the both hand sides of (3.1) and integrating the spatial variable over $\mathbb{R}^N$, it follows that

$$\frac{1}{p}\frac{d}{dt}\int|f|^p+\frac{1}{p}\int \mathrm{div} b|f|^p+\frac{2(p-1)}{p^2}\int|\sigma^t\nabla(|f|^{\frac{p-2}{2}}f)|^2=0, \tag{3.3}$$

where we used the integration by parts formula. By applying Grönwall's lemma we deduce that

$$f\in L^\infty([0,T];L^p(\mathbb{R}^N)),\quad \sigma^t\nabla|f|^{\frac{p}{2}}\in L^2([0,T];L^2(\mathbb{R}^N)).$$

In addition, since $f_0\in L^\infty$, the maximum principle suggests $f\in L^\infty([0,T];L^\infty(\mathbb{R}^N))$ and if σ is uniformly positive define we also have $|f|^{\frac{p}{2}}\in L^2([0,T];H^1(\mathbb{R}^N))$.

• regularization and the existence of L^p weak solutions.

Approximate coefficients b and σ by b_ε and σ_ε respectively, where

$$b_\varepsilon\in\mathcal{C}_b^1,\ \sigma_\varepsilon\in\mathcal{C}_b^1,\ \text{ and }\ b_\varepsilon\to b\ \text{ in }\ (W^{1,1}_{loc}(\mathbb{R}^N))^N,\ \ \sigma_\varepsilon\to\sigma\ \text{ in }\ (W^{1,2}_{loc}(\mathbb{R}^N))^{N\times K}.$$

Using the first step, through the compactness method, it is easy to get the existence and we skip it.

• regularization and the uniqueness.

Let $f_\varepsilon=f*\rho_\varepsilon$, where ρ_ε is the standard modifier, then f_ε yields:

$$\begin{aligned}\partial_t f_\varepsilon=\rho_\varepsilon*\partial_t f &= \rho_\varepsilon*(b_i\partial_i f+\frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f))\\ &= (\rho_\varepsilon*(b_i\partial_i f)-b_i\partial_i f_\varepsilon)+b_i\partial_i f_\varepsilon+\frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f_\varepsilon)+I_\varepsilon.\end{aligned}$$

Equivalent well,

$$\partial_t f_\varepsilon-b_i\partial_i f_\varepsilon-\frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f_\varepsilon)=T_\varepsilon+I_\varepsilon, \tag{3.4}$$

where

$$T_\varepsilon=\rho_\varepsilon*(b_i\partial_i f)-b_i\partial_i f_\varepsilon,$$
$$I_\varepsilon=\frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f)*\rho_\varepsilon-\frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f_\varepsilon)$$

and

$$T_\varepsilon\to 0\ \text{ in } L^\infty([0,T];L^1_{loc}),\ I_\varepsilon\to\ 0 \text{ in } L^\infty([0,T];\dot W^{-1,1}_{loc}),$$

where the dot on $W^{-1,1}_{loc}$ denotes the corresponding homogeneous Sobolev space.

Multiplying $|f_\varepsilon|^{p-2}f_\varepsilon\phi_r$, where ϕ_r is a smooth cut-off function described as in Theorem 2.1 to both hand sides of identity (3.4) and integrating over $\mathbb{R}^N$, we achieve

$$\frac{1}{p}\frac{d}{dt}\int|f_\varepsilon|^p\phi_r+\frac{1}{p}\int \mathrm{div} b|f_\varepsilon|^p\phi_r+\frac{2(p-1)}{p^2}\int|\sigma^t\nabla(|f_\varepsilon|^{\frac{p-2}{2}}f_\varepsilon)|^2\phi_r$$

$$\begin{aligned}
&= -\frac{1}{p}\int |f_\varepsilon|^p b\cdot\nabla\phi_r - \int \sigma_{ik}\sigma_{jk}\partial_j f_\varepsilon |f_\varepsilon|^{p-2} f_\varepsilon \partial_i\phi_r \\
&\quad + \int |f_\varepsilon|^{p-2} f_\varepsilon \phi_r T_\varepsilon + \frac{1}{p}\int |f_\varepsilon|^{p-2} f_\varepsilon \phi_r I_\varepsilon \\
&\le c\int_{r\le|x|\le 2r} |f_\varepsilon|^p \frac{|b|}{1+|x|} + |\sigma^t\nabla(|f_\varepsilon|^{\frac{p-2}{2}} f_\varepsilon)||f_\varepsilon|^{\frac{p}{2}}\frac{|\sigma|}{1+|x|} \\
&\quad + \int [I_\varepsilon + T_\varepsilon]|f_\varepsilon|^{p-2} f_\varepsilon\phi_r. \qquad (3.5)
\end{aligned}$$

On the other hand $|f_\varepsilon| \le c$, since $f \in L^\infty([0,T]; L^\infty(\mathbb{R}^N))$. It follows that if we let ε go to zero first and r approach to infinity next, we obtain

$$\frac{1}{p}\frac{d}{dt}\int |f|^p + \frac{1}{p}\int \mathrm{div} b|f|^p + \frac{2(p-1)}{p^2}\int |\sigma^t\nabla(|f|^{\frac{p-2}{2}} f)|^2 \le 0, \qquad (3.6)$$

for

$$\frac{b}{1+|x|} \in ((L^1 + L^\infty)(\mathbb{R}^N))^N, \quad \frac{\sigma}{1+|x|} \in ((L^2 + L^\infty)(\mathbb{R}^N))^{N\times K},$$

$$f \in L^\infty([0,T]; L^p \cap L^\infty(\mathbb{R}^N)).$$

This inequality and the condition $\mathrm{div} b \in L^\infty$ leads to $f = 0$ if $f_0 = 0$, and one fulfills the proof.

Remark 3.4. (i) Theorem 3.3 yields the existence and uniqueness results for general non-homogeneous equations, namely

$$\partial_t f - b_i\partial_i f - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j f) = h, \qquad (3.7)$$

where $h \in L^p([0,T]\times\mathbb{R}^N)$ and b, σ are time dependent, i.e.

$$b \in L^1([0,T]; (W^{1,1}_{loc}(\mathbb{R}^N))^N), \quad \sigma \in L^2([0,T]; (W^{1,2}_{loc}(\mathbb{R}^N))^{N\times K}).$$

(ii) It is possible to prove that the weak solution is indeed a strong solution if we presume in addition that $\alpha I \le \sigma\sigma^t \le \beta I$, where $0 < \alpha < \beta$.

Precisely speaking, we have

Theorem 3.5 [**L^p-strong solution**] Let b and σ be as in Theorem 3.3, that there is a positive constant α such $\langle\sigma\sigma^t f,\ f\rangle \ge \alpha\|f\|^2$, then the unique weak solution in Theorem 3.3 is indeed a strong solution.

Proof. Denote by $g(t,x) = tf(t,x)$, then g is the unique weak solution of the following non-homogeneous equation

$$\begin{cases} \partial_t g - b_i\partial_i g - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j g) = f, \text{ in } (0,T]\times\mathbb{R}^N, \\ g(t=0,\ \cdot) = 0, \text{ in } \mathbb{R}^N. \end{cases} \qquad (3.8)$$

Firstly, we consider a deformation form of above equation

$$\begin{cases} \partial_t u - b_i\partial_i u - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j u) = \tilde{f}, \text{ in } \ Q_{T,2r} = (0,T]\times B_{2r}, \\ u(t=0,\ \cdot) = 0, \text{ in } B_{2r}, \\ u(t,x) = 0, \text{ on } \ (0,T]\times\partial B_{2r}, \end{cases} \qquad (3.9)$$

where B_{2r} denotes the open ball with center zero and diameter $4r$,

$$\tilde{f} = \begin{cases} f, \text{ when } (t,x) \in (0,T] \times B_r, \\ \\ 0, \text{ when } (t,x) \in (0,T] \times (B_{2r} \backslash B_{\frac{3r}{2}}), \end{cases}$$

and $\tilde{f} \in L^p$.

By standard parabolic (equation) theory (such as see [10]), there is a unique strong solution $u \in W_p^{2,1}$ to (3.9) where

$$W_p^{1,2} = \{\ f(t,x);\ f \in L^p,\ \nabla f \in L^p,\ (\nabla)^2 f \in L^p,\ \partial_t f \in L^p\ \}.$$

Clearly it is also a weak solution. Notice that $\tilde{f}|_{Q_{T,r}} = f$, by the uniqueness and the arbitrariness of r, we remark that

$$g \in W_{p,loc}^{2,1},\ i.e.\ tf \in W_{p,loc}^{2,1},$$

which implies tf yields (3.9) almost surely, it follows that f satisfies (3.2) almost surely and we finish our proof.

Remark 3.6. (i) For the sake of $\alpha I \leq \sigma\sigma^t \leq \beta I$, we have more precise result, that is

$$tf \in L_{loc}^p(\mathbb{R}^N; W^{1,p}([0,T])) \cap L^p([0,T]; W_{loc}^{2,p}(\mathbb{R}^N)).$$

By using Sobolev embedding theorems, one obtains

$$tf \in L_{loc}^p(\mathbb{R}^N; \mathcal{C}^{1-\frac{1}{p}}([0,T])) \cap L^p([0,T]; \mathcal{C}^{2-\frac{N}{p}}(\mathbb{R}^N)),\ \text{if } \frac{N}{p} < 2,$$

which implies for almost $x \in \mathbb{R}^N$, $tf \in \mathcal{C}^{1-\frac{1}{p}}([0,T])$ and for almost all $t \in [0,T]$, $tf \in \mathcal{C}(\mathbb{R}^N)$, thus $f \in \mathcal{C}(\mathbb{R}^N)$.

(ii) It is possible to get the high order regularity property by multiplying a factor t^α where $\alpha > 0$ is a positive real number and this procedure is machinery and tedious, as we have discussed it in Section 2.2 now we omit it .

(iii) If we assume $b \in (W^{1,1}(\mathbb{R}^N))^N$ and $\sigma \in (W^{1,2}(\mathbb{R}^N))^{N\times K}$, then we do not need

$$\frac{b}{1+|x|} \in ((L^1 +\ L^\infty)(\mathbb{R}^N))^N, \qquad \frac{\sigma}{1+|x|} \in ((L^2 + L^\infty)(\mathbb{R}^N))^{N\times K}.$$

References

[1] C. Le Bris and P.L. Lions, Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients, Rapport de Recherche du CERMICS, 349 (2007), 1-46.
[2] L. Arnod, Stochastic Differential Equations: Theory and Applications, John Wiley NY, 1974.
[3] R.J. Di Perna and P.L. Lions, On the Fokker-Planck-Boltzmann equation, Communications in Mathematical Physics, 120 (1988), 1-23.
[4] A. Figalli, Existence and uniqueness of the martingale solutions for SDEs with rough or degenerate coefficients, Journal of Functional Analysis, 254 (2008), 109-153.
[5] M. Röchner and X. Zhang, Weak uniqueness of Fokker-Planck equations with degenerate and bounded coefficients, Comptes Rendus Mathematique, 348 (2010): 435-438.
[6] L.C. Evans, Partial differential equations, American Mathematical Society, Rhode Island, 1998.
[7] R.J. Di Perna and P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Inventiones Mathematicae, 98 (1989) 511-547.
[8] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Inventiones mathematicae, 158 (2004) 227-260.

[9] L. Ambrosio, The flow associated to weakly differentiable vector fields: recent results and open problems, Nonlinear Conservation Law and Applications, Springer, 2011.
[10] A. Friedmann, Partial differential equations of parabolic type, Englewood Cliffs, NJ, 1964.

Rongrong Tian
School of Shanxi Normal University, Linfen 041000, Shanxi, P. R. China
E-mail address: tianrr06@sina.com