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UNIFIED PRESENTAION FOR MULTIVALENT HARMONIC FUNCTIONS

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Abstract. In this paper we introduced a class defined by certain combination of starlike and convex multivalent harmonic functions and obtained growth and distortion theorems. Also convolution properties for functions in the class are obtained.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic in a complex domain D if both u and v are harmonic in D. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small [3]).

Denote by H the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

For $m \in \mathbb{N} = \{1, 2, ...\}$, h and g analytic in U, denote by $H(m)$ the set of all multivalent harmonic functions $f = h + \overline{g}$ defined in U, where h and g defined by

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1},\ g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1},\ |b_m| < 1. \qquad (1)$$

The class $H(m)$ was studied by Ahuja and Jahangiri [1] and for $m = 1$ was studied by Jahangiri et al. [5].

For, $m \geq 1, 0 \leq \alpha < 1$, Ahuja and Jahangiri [1, 2] defined the class of $m-$valent harmonic starlike functions of order $\alpha, SH(m, \alpha)$ which consisting of functions $f = h + \overline{g} \in H(m)$ and satisfy the condition

$$\frac{\partial}{\partial \theta} \left(\arg(f(re^{i\theta})) \right) \geq m\alpha, \qquad (2)$$

where $z = .re^{i\theta}, 0 \leq \theta < 2\pi$ and $0 \leq r < 1$. For $\alpha = 0$ this class was studied by Sheil- Small [7].

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Denote by $TH(m,\alpha)$ the class of functions $f = h + \overline{g} \in SH(m,\alpha)$ of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1}, \qquad g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \quad |b_m| < 1. \tag{3}$$

The class of the form (3) was defined by Ahuja and Jahangiri [1] and for $m = 1$ was studied by Silverman [8] (see also Sheil Small [7] and Silverman and Silvia [9]).

Ananolgous to $TH(m,\alpha)$ is the class $KH(m,\alpha)$ of $m-$valent harmonic convex functions of order $\alpha (0 \leq \alpha < 1)$ consisting of functions $f = h + \overline{g}$ of the form (3) which satisfy

$$\frac{\partial}{\partial\theta}\left(\arg(\frac{\partial}{\partial\theta} f(re^{i\theta}))\right) \geq m\alpha,$$

where $z = re^{i\theta}, 0 \leq \theta < 2\pi$ and $0 \leq r < 1$.

It is clear that

$$f(z) \in KH(m,\alpha) \text{ if and only if } \frac{1}{m} z f'(z) \in TH(m,\alpha).$$

For functions $f = h + \overline{g}$ of the form (3) Ahuja and Jahangiri [1, 2] proved the following lemmas (see also [4]).

Lemma 1. *Let $f = h + \overline{g}$ be given by* (3). *Then $f \in TH(m,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}| + \frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|\right) \leq 2, \tag{it4}$$

where $a_m = 1$ and $m \geq 1$.

Lemma 2. *Let $f = h + \overline{g}$ be given by* (3). *Then $f \in KH(m,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{n+m-1}{m}\left(\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}| + \frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|\right) \leq 2, \tag{it5}$$

where $a_m = 1$ and $m \geq 1$.

In view of Lemma 1 and Lemma 2 and for $\beta \geq 0$, we define the new class $T^*S_H(m,\alpha,\beta)$ consisting of functions $f = h + \overline{g}$, where h and g are of the form (3) and satisfy:

$$\sum_{n=1}^{\infty} \frac{m+\beta(n-1)}{m}\left(\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}| + \frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|\right) \leq 2. \tag{6}$$

We clearly see that

$$T^*S_H(m,\alpha,0) = TH(m,\alpha) \text{ and } T^*S_H(m,\alpha,1) = KH(m,\alpha),$$

that is, that $T^*S_H(m,\alpha,\beta)$ can be written in the form:

$$T^*S_H(m,\alpha,\beta) = (1-\beta)TH(m,\alpha) + \beta KH(m,\alpha).$$

In this paper we obtain growth and distortion theorems and also convolution properties for functions in the class $T^*S_H(m,\alpha,\beta)$.

2. Main results

Unless otherwise mentioned, we assume that $0 \leq \alpha < 1, \beta \geq 0,\ a_m = 1$ and $m \in \mathbb{N}$.

The following theorem gives a distortion property for functions belonging to the class $T^*S_H(m,\alpha,\beta)$.

Theorem 1. *Let $f = h + \overline{g}$, where h and g are of the form* (3) *and belonging to the class $T^*S_H(m,\alpha,\beta)$, then for $\frac{1+\alpha}{1-\alpha}|b_m| < 1$ and $0 \leq |z| = r < 1$, we have*

$$|f(z)| \leq (1+|b_m|)r^m + \frac{m^2(1-\alpha)}{(m+\beta)(1+m-m\alpha)}\left(1 - \frac{1+\alpha}{1-\alpha}|b_m|\right)r^{m+1} \qquad \text{(it7)}$$

and

$$|f(z)| \geq (1-|b_m|)r^m - \frac{m^2(1-\alpha)}{(m+\beta)(1+m-m\alpha)}\left(1 - \frac{1+\alpha}{1-\alpha}|b_m|\right)r^{m+1}. \qquad \text{(it8)}$$

The bounds in (7) *and* (8) *are sharp for the functions*

$$f(z) = (1+b_m)\overline{z}^m + \left(\tfrac{m^2(1-\alpha)}{(m+\beta)(1+m-m\alpha)} - \tfrac{m^2(1+\alpha)}{(m+\beta)(1+m-m\alpha)}b_m\right)\overline{z}^{m+1}$$

and

$$f(z) = (1-|b_m|)\overline{z}^m - \left(\tfrac{m^2(1-\alpha)}{(m+\beta)(1+m-m\alpha)} - \tfrac{m^2(1+\alpha)}{(m+\beta)(1+m-m\alpha)}b_m\right)\overline{z}^{m+1},$$

respectively .

Proof. Since, $f \in T^*S_H(m,\alpha,\beta)$, we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_m|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}| + |b_{n+m-1}|)r^{n+m-1} \\
&\leq (1+|b_m|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}| + |b_{n+m-1}|)r^{m+1} \\
&\leq (1+|b_m|)r^m + \tfrac{m^2(1-\alpha)}{(m+\beta)(1+m-m\alpha)}\sum_{n=2}^{\infty}\tfrac{(m+\beta)(1+m-m\alpha)}{m^2(1-\alpha)}(|a_{n+m-1}| + |b_{n+m-1}|)r^{m+1} \\
&\leq (1+|b_m|)r^m + \tfrac{m^2(1-\alpha)}{(m+\beta)(1+m-m\alpha)}\sum_{n=2}^{\infty}\left(\tfrac{n-1+m(1-\alpha)}{m^2(1-\alpha)}|a_{n+m-1}| + \tfrac{n-1+m(1+\alpha)}{m^2(1-\alpha)}|b_{n+m-1}|\right)r^{m+1} \\
&\leq (1+|b_m|)r^m + \frac{m^2(1-\alpha)}{(m+\beta)(1+m-m\alpha)}\left(1 - \frac{1+\alpha}{1-\alpha}|b_m|\right)r^{m+1}\ (|z| = r < 1),
\end{aligned}$$

this proves (7). The proof of (8) is similarly and so we omit it. This complets the proof of Theorem 1.

Theorem 2. *Let $f = h + \overline{g}$, where h and g are of the form* (3) *and belonging to the class $T^*S_H(m,\alpha,\beta)$, then for $\frac{1+\alpha}{1-\alpha}|b_m| < 1$ and $0 \leq |z| = r < 1$,*

$$|f'(z)| \leq m(1+|b_m|)r^{m-1} + \tfrac{m^2(m+1)(1-\alpha)}{(m+\beta)(1+m-m\alpha)}\left(1 - \frac{1+\alpha}{1-\alpha}|b_m|\right)r^m \qquad \text{(it9)}$$

and

$$|f'(z)| \geq m(1-|b_m|)r^{m-1} - \tfrac{m^2(m+1)(1-\alpha)}{(m+\beta)(1+m-m\alpha)}\left(1 - \frac{1+\alpha}{1-\alpha}|b_m|\right)r^m. \qquad \text{(it10)}$$

The bounds in (9) *and* (10) *are sharp.*

Proof. The proof is similar to that of Theorem 1 and hence we omit it.

Putting $\beta = 1$ in Theorems 1 and 2, we have the following result.

Corollary 1. *Let $f = h + \overline{g}$, where h and g are of the form (3) and belonging to the class $KH(m,\alpha)$, then for $\frac{1+\alpha}{1-\alpha}|b_m| < 1$ and $0 \le |z| = r < 1$, we have*

$$\begin{aligned} &(1-|b_m|)r^m - \tfrac{m^2(1-\alpha)}{(m+1)(1+m-m\alpha)}\left(1-\tfrac{1+\alpha}{1-\alpha}|b_m|\right)r^{m+1} \\ \le\ & |f(z)| \le (1+|b_m|)r^m + \tfrac{m^2(1-\alpha)}{(m+1)(1+m-m\alpha)}\left(1-\tfrac{1+\alpha}{1-\alpha}|b_m|\right)r^{m+1} \end{aligned}$$

and

$$\begin{aligned} &m(1-|b_m|)r^{m-1} - \tfrac{m^2(m+1)(1-\alpha)}{(m+1)(1+m-m\alpha)}\left(1-\frac{1+\alpha}{1-\alpha}|b_m|\right)r^m \\ \le\ & |f'(z)| \le m(1+|b_m|)r^{m-1} + \tfrac{m^2(m+1)(1-\alpha)}{(m+1)(1+m-m\alpha)}\left(1-\frac{1+\alpha}{1-\alpha}|b_m|\right)r^m. \end{aligned}$$

The bounds are sharp.

Let the functions f_j $(j = 1, 2)$ be defined by

$$f_j(z) = z^m - \sum_{n=2}^{\infty}|a_{n+m-1,j}|\,z^n + \sum_{n=1}^{\infty}|b_{n+m-1,j}|\,\overline{z}^n \ (z \in U), \tag{11}$$

then the Hadamard prodct (or convolution) is defined by

$$(f_1 * f_2)(z) = z^m - \sum_{n=2}^{\infty}|a_{n+m-1,1}|\,|a_{n+m-1,2}|\,z^n + \sum_{n=1}^{\infty}|b_{n+m-1,1}|\,|b_{n+m-1,2}|\,\overline{z}^n. \tag{12}$$

The next theorem shows that the class $T^*S_H(m,\alpha,\beta)$ is closed under convolution.

Theorem 3. *For $0 \le \gamma \le \alpha < 1$, let $f_1 \in T^*S_H(m,\alpha,\beta)$ and $f_2 \in T^*S_H(m,\gamma,\beta)$. Then*

$$(f_1 * f_2)(z) \in T^*S_H(m,\alpha,\beta) \subset T^*S_H(m,\gamma,\beta).$$

Proof. In order to prove the theorem we must show that the coefficients in (12) must satisfy the condition (6). Now, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\tfrac{m+\beta(n-1)}{m}\left(\tfrac{n-1+m(1-\gamma)}{m(1-\gamma)}|a_{n+m-1,1}|\,|a_{n+m-1,2}| + \tfrac{n-1+m(1+\gamma)}{m(1-\gamma)}|b_{n+m-1,1}|\,|b_{n+m-1,2}|\right) \\ \le\ & \sum_{n=1}^{\infty}\tfrac{m+\beta(n-1)}{m}\left(\tfrac{n-1+m(1-\gamma)}{m(1-\gamma)}|a_{n+m-1,1}| + \tfrac{n-1+m(1+\gamma)}{m(1-\gamma)}|b_{n+m-1,1}|\right) \\ \le\ & \sum_{n=1}^{\infty}\tfrac{m+\beta(n-1)}{m}\left(\tfrac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1,1}| + \tfrac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1,1}|\right) \le 2. \end{aligned}$$

This completes the proof of Theorem 3.

The next theorem shows that the class $T^*S_H(m,\alpha,\beta)$ is closed under convex combination.

Theorem 4. *The class $T^*S_H(m,\alpha,\beta)$ is closed under convex combination.*

Proof. Let f_j $(j = 1, 2, ...)$ be defined by (11) belongs to the class $T^*S_H(m,\alpha,\beta)$, then

$$\sum_{n=1}^{\infty} \tfrac{m+\beta(n-1)}{m} \left(\tfrac{n-1+m(1-\alpha)}{m(1-\alpha)} |a_{n+m-1,j}| + \tfrac{n-1+m(1+\alpha)}{m(1-\alpha)} |b_{n+m-1,j}| \right) \leq 2. \tag{13}$$

Let, for $0 \leq t_j \leq 1$, $\sum_{j=1}^{\infty} t_j = 1$, the convex combination of f_j be in the form

$$\sum_{j=1}^{\infty} t_j f_j = z^m - \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} t_j |a_{n+m-1,j}| z^n + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} t_j |b_{n+m-1,j}| \overline{z}^n. \tag{14}$$

Using (13), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \tfrac{m+\beta(n-1)}{m} \left(\tfrac{n-1+m(1-\alpha)}{m(1-\alpha)} \sum_{j=1}^{\infty} t_j |a_{n,j}| + \tfrac{n-1+m(1+\alpha)}{m(1-\alpha)} \sum_{j=1}^{\infty} t_j |b_{n,j}| \right) \\ = \;& \sum_{j=1}^{\infty} t_j \left(\sum_{n=1}^{\infty} \tfrac{m+\beta(n-1)}{m} \left(\tfrac{n-1+m(1-\alpha)}{m(1-\alpha)} |a_{n+m-1,j}| + \tfrac{n-1+m(1+\alpha)}{m(1-\alpha)} |b_{n+m-1,j}| \right) \right) \\ \leq \;& 2 \sum_{j=1}^{\infty} t_j = 2. \end{aligned}$$

This leads to $\sum_{j=1}^{\infty} t_j f_j \in T^* S_H(m, \alpha, \beta)$. This completes the proof of Theorem 4.

Remarks. (i) *Taking $\beta = 0$ in the above results, we obtain the results corresponding to the class $TH(m, \alpha)$* (see [1,2]);

(ii) *Taking $\beta = 1$ in Theorems 3 and 4, we obtain the results corresponding to the class $KH(m, \alpha)$*;

(iii) *Taking $m = 1$ in the above results, we obtain the results of Joshi and Darus [6].*

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