Electronic Journal of Mathematical Analysis and Applications,
Vol. 2(1) Jan. 2014, pp. 91-98.
ISSN: 2090-792X (online)
http://fcag-egypt.com/Journals/EJMAA/

EXISTENCE OF NONTRIVIAL SOLUTIONS FOR NEUMANN BOUNDARY (p, q)-LAPLACIAN SYSTEMS

EL. M. HSSINI, M. MASSAR, M. TALBI, N. TSOULI

Abstract. In this paper, existence results of nontrivial solutions to a Neumann boundary (p, q)-Laplacian systems is established. Our technical is based on Bonanno's general critical points theorem.

1. Introduction

In this paper we are concerned with the existence of nontrivial weak solutions for the following systems of (p, q)-Laplacian type

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda F_u(x, u, v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda F_v(x, u, v) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where $q \geq p > N$, Ω is a nonempty bounded open set of $\mathbb{R}^N$, with a boundary of class C^1 and ν is the outer unit normal to $\partial\Omega$. Here, λ is a real positive parameter, $a, b \in L^\infty(\Omega)$, with $\operatorname{ess\,inf}_\Omega a \geq 0$, $\operatorname{ess\,inf}_\Omega b \geq 0$ and $a \not\equiv 0$, $b \not\equiv 0$. $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(., s, t)$ is continuous in Ω, for all $(s, t) \in \mathbb{R}^2$ and $F(x, ., .)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, and F_u, F_v denote the partial derivatives of F, with respect to u, v respectively. Moreover, $F(x, s, t)$ satisfies the following condition

(F)

$$\sup_{|s| \leq \sigma, |t| \leq \sigma} (|F_u(., s, t)| + |F_v(., s, t)|) \in L^1(\Omega), \text{ for all } \sigma > 0.$$

The investigation of existence and multiplicity of solutions for systems involving p-Laplacian operators has been the subject of numerous studies, see [2, 3, 7, 11, 14] and references therein. As well as the multiplicity results for perturbed Neumann problems has been extensively obtained by several authors, see, for instance [1, 6, 8, 9, 10, 13].

In this paper an interval of real parameters λ, for which the problem (1) admits at least one nontrivial solution, is established. Our approach is variational and main tool is a local minimum theorem due to Bonanno [4].

2000 *Mathematics Subject Classification.* 35J40, 35J60.
Key words and phrases. Neumann problem; p-Laplacian; critical points.
Submitted May 02, 2013.

In the sequel, X will denote the space $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$, which is a reflexive Banach space endowed with the norm

$$||(u,v)|| = ||u||_p + ||v||_q,$$

where

$$||u||_p = \left(\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx\right)^{1/p} \quad \text{and} \quad ||v||_q = \left(\int_\Omega (|\nabla v|^q + b(x)|v|^q)dx\right)^{1/q}.$$

Let

$$k := \max\left\{ \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u(x)|^p}{||u||_p^p}, \sup_{v\in W^{1,q}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|v(x)|^q}{||v||_q^q} \right\}. \tag{2}$$

Since $p > N$ and $q > N$, the Rellich Kondrachov theorem assures that $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $W^{1,q}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ are compact, and hence $k < \infty$. Moreover

$$\min(k||a||_1, k||b||_1) \geq 1, \tag{3}$$

where

$$||a||_1 = \int_\Omega |a(x)|dx \ \text{ and } \ ||b||_1 = \int_\Omega |b(x)|dx.$$

In addition, if Ω is convex, it is known that

$$\sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u(x)|}{||u||_p} \leq 2^{\frac{p-1}{p}} \max\left\{ \left(\frac{1}{||a||_1}\right)^{1/p}; \frac{d}{N^{1/p}}\left(\frac{p-1}{p-N}m(\Omega)\right)^{\frac{p-1}{p}} \frac{||a||_\infty}{||a||_1} \right\}, \tag{4}$$

where $d = \text{diam}(\Omega)$ and $m(\Omega)$ is the Lebesgue measure of the set Ω (see [5]), and equality occurs when Ω is a ball.

We recall that $(u,v)\in X$ is weak solution of problem (1) if (u,v) satisfies the following condition

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_\Omega a(x)|u|^{p-2}u\varphi dx + \int_\Omega |\nabla v|^{q-2}\nabla v\nabla\psi dx$$
$$+ \int_\Omega b(x)|v|^{p-2}v\psi dx - \lambda\int_\Omega F_u(x,u,v)\varphi dx - \lambda\int_\Omega F_v(x,u,v)\psi dx = 0,$$

for all $(\varphi,\psi)\in X$.

We see that weak solutions of system (1) are critical points of the functional $I_\lambda : X \to \mathbb{R}$, given by

$$I_\lambda(u,v) = \Phi(u,v) - \lambda\Psi(u,v) \ \text{ for all } (u,v)\in X,$$

where

$$\Phi(u,v) = \frac{1}{p}||u||_p^p + \frac{1}{q}||v||_q^q \quad \text{and} \quad \Psi(u,v) = \int_\Omega F(x,u,v)dx.$$

Since X is compactly embedded in $C^0(\overline{\Omega})\times C^0(\overline{\Omega})$, it is well known that Φ and Ψ are well defined Gâteaux differentiable functionals whose Gâteaux derivatives at $(u,v)\in X$ are given by

$$\begin{aligned} \langle \Phi'(u,v),(\varphi,\psi)\rangle &= \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_\Omega a(x)|u|^{p-2}u\varphi dx \\ &+ \int_\Omega |\nabla v|^{q-2}\nabla v\nabla\psi dx + \int_\Omega b(x)|v|^{p-2}v\psi dx, \end{aligned}$$

$$\langle \Psi'(u,v),(\varphi,\psi)\rangle = \int_\Omega F_u(x,u,v)\varphi dx + \int_\Omega F_v(x,u,v)\psi dx,$$

for all $(\varphi,\psi) \in X$. Moreover, by the weakly lower semicontinuity of norm, we see that Φ is sequentially weakly lower semi continuous and whose Gateaux derivative admits a continuous inverse on X^* (see Proposition 1 in [7]). Thanks to $p, q > N$ and (F), Ψ has compact derivative, it follows that Ψ is sequentially weakly continuous.

Our main tools are two consequences of a local minimum theorem [4], which is a more general version of the Ricceri Variational Principle (see [12]). Given a set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, put

$$\beta(r_1,r_2) := \inf_{u\in\Phi^{-1}(]r_1,r_2[)} \frac{\left(\sup_{v\in\Phi^{-1}(]r_1,r_2[)} \Psi(v)\right) - \Psi(u)}{r_2 - \Phi(u)} \tag{5}$$

$$\rho_2(r_1,r_2) := \sup_{u\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(u) - \left(\sup_{v\in\Phi^{-1}(]-\infty,r_1[)} \Psi(v)\right)}{\Phi(u) - r_1} \tag{6}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and

$$\rho(r) := \sup_{u\in\Phi^{-1}(]r,+\infty[)} \frac{\Psi(u) - \left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right)}{\Phi(u) - r} \tag{7}$$

for all $r \in \mathbb{R}$.

Theorem 1.[[4], Theorem 5.1] Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gateaux differentiable function whose Gateaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that

$$\beta(r_1,r_2) < \rho_2(r_1,r_2), \tag{8}$$

where β, ρ are given by (5) and (6).
Then, for each $\lambda \in \left]\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$, there is $u_{0,\lambda} \in \Phi^{-1}(]r_1,r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1,r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.[[4],Theorem 5.3] Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative admits a continuous inverse on X^*.$\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that

$$\rho(r) > 0 \tag{9}$$

where ρ is given by (7), for each $\lambda > \frac{1}{\rho(r)}$, the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Then, for $\lambda > \frac{1}{\rho(r)}$, there is $u_{0,\lambda} \in \Phi^{-1}(]r,+\infty[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r,+\infty[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

2. Main results

Given two nonnegative constants c, d such that

$$\frac{c^p}{q\left(k^{1/p}+k^{1/q}\right)^p} \neq \frac{d^q}{p}\left(||a||_1+||b||_1\right),$$

where k is given by (2). Put

$$A_d(c) := \frac{\int_\Omega \max_{|s|+|t|\leq c} F(x,s,t)dx - \int_\Omega F(x,d,d)dx}{\frac{c^p}{q\left(k^{1/p}+k^{1/q}\right)^p} - \frac{d^q}{p}\left(||a||_1+||b||_1\right)}$$

Now we are ready to state our main results.

Theorem 3. Assume that there exist three constants c_1, c_2, d with

$$(k^{1/p}+k^{1/q}) \max\left(q^{1/p}, (||a||_1+||b||_1)^{1/p}\right) \leq c_1 < c_2, \tag{10}$$

and

$$\frac{c_1}{\left(k^{1/p}+k^{1/q}\right)\left(||a||_1+||b||_1\right)^{1/p}} < d < \left(\frac{pc_2^p}{q\left(k^{1/p}+k^{1/q}\right)^p\left(||a||_1+||b||_1\right)}\right)^{1/q} \tag{11}$$

such that

$$A_d(c_2) < A_d(c_1)$$

Then, for each

$$\lambda \in \left]\frac{1}{A_d(c_1)}, \frac{1}{A_d(c_2)}\right[,$$

problem (1) admits at least one nontrivial weak solution $(\widetilde{u}, \widetilde{v})$ such that

$$\frac{c_1^p}{q\left(k^{1/p}+k^{1/q}\right)^p} < \Phi(\widetilde{u}, \widetilde{v}) < \frac{c_2^p}{q\left(k^{1/p}+k^{1/q}\right)^p}.$$

Proof. Let Φ, Ψ be the functionals defined in Section 1, it is well known that they satisfy all regularity assumptions requested in Theorem 1. So, our aim is to verify condition (8), put

$$u_0(x) = d \quad \text{for all } x \in \Omega.$$

Clearly, $(u_0, u_0) \in X$, and

$$\Psi(u_0, u_0) = \int_\Omega F(x, u_0, u_0)dx = \int_\Omega F(x, d, d)dx. \tag{12}$$

From the definition of Φ and the above conditions, we have

$$\frac{d^p}{q}\left(||a||_1+||b||_1\right) \leq \Phi(u_0, u_0) \leq \frac{d^q}{p}\left(||a||_1+||b||_1\right). \tag{13}$$

Fix c_1, c_2, d satisfying conditions (10) and (11), put

$$r_1 = \frac{c_1^p}{q\left(k^{1/p}+k^{1/q}\right)^p} \text{ and } r_2 = \frac{c_2^p}{q\left(k^{1/p}+k^{1/q}\right)^p}.$$

Then, we obtain $r_1 < \Phi(u_0, u_0) < r_2$.

Moreover, for all $(u, v) \in X$ such that $(u, v) \in \Phi^{-1}(]-\infty, r_2[)$, we get, from (2) and (10), that

$$|u| + |v| \leq c_2,$$

therefore

$$\Psi(u,v)=\int_\Omega F(x,u,v)dx\le\int_\Omega\max_{|s|+|t|\le c_2}F(x,s,t)dx\quad\text{for all }(u,v)\in\Phi^{-1}(]-\infty,r_2[).$$

Hence

$$\sup_{(u,v)\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u,v)\le\int_\Omega\max_{|s|+|t|\le c_2}F(x,s,t)dx \tag{14}$$

by the same argument, we obtain

$$\sup_{(u,v)\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u,v)\le\int_\Omega\max_{|s|+|t|\le c_1}F(x,s,t)dx. \tag{15}$$

Combining (12), (13), (14) and (15), we get

$$\begin{aligned}\beta(r_1,r_2) &\le \frac{\left(\sup_{(u,v)\in\Phi^{-1}(]r_1,r_2[)}\Psi(u,v)\right)-\Psi(u_0,u_0)}{r_2-\Phi(u_0,u_0)}\\ &\le \frac{\int_\Omega\max_{|s|+|t|\le c_2}F(x,s,t)dx-\int_\Omega F(x,d,d)dx}{\frac{c_2^p}{q\left(k^{1/p}+k^{1/q}\right)^p}-\frac{d^q}{p}\left(||a||_1+||b||_1\right)}=A_d(c_2)\end{aligned}$$

and

$$\begin{aligned}\rho_2(r_1,r_2) &\ge \frac{\Psi(u_0,u_0)-\left(\sup_{(u,v)\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u,v)\right)}{\Phi(u_0,u_0)-r_1}\\ &\ge \frac{\int_\Omega F(x,d,d)dx-\int_\Omega\max_{|s|+|t|\le c_1}F(x,s,t)dx}{\frac{d^q}{p}\left(||a||_1+||b||_1\right)-\frac{c_1^p}{q\left(k^{1/p}+k^{1/q}\right)^p}}=A_d(c_1).\end{aligned}$$

so, from our assumption it follows that

$$\beta(r_1,r_2)<\rho_2(r_1,r_2).$$

Hence, by Theorem 1 for each $\lambda\in\left]\frac{1}{A_d(c_1)},\frac{1}{A_d(c_2)}\right[$, the functional I_λ admit at least one critical point $(\widetilde{u},\widetilde{v})$ such that

$$\frac{c_1^p}{q\left(k^{1/p}+k^{1/q}\right)^p}<\Phi(\widetilde{u},\widetilde{v})<\frac{c_2^p}{q\left(k^{1/p}+k^{1/q}\right)^p}.$$

Now, we give an application of Theorem 2.

Theorem 4. Assume that

(i) there exist two constants $\overline{c},\overline{d}$ with

$$\max\left(1,\left(\frac{q}{||a||_1+||b||_1}\right)^{1/p}\right)\le\frac{\overline{c}}{\left(k^{1/p}+k^{1/q}\right)\left(||a||_1+||b||_1\right)^{1/p}}<\overline{d} \tag{16}$$

such that

$$\int_\Omega\max_{|s|+|t|\le\overline{c}}F(x,s,t)dx<\int_\Omega F(x,\overline{d},\overline{d})dx \tag{17}$$

(ii)

$$\limsup_{|s|+|t|\to+\infty}\frac{F(x,s,t)}{|s|^p+|t|^q}\le 0\ \text{ uniformly in } X. \tag{18}$$

Then, for each $\lambda > \overline{\lambda}$, where

$$\overline{\lambda} = \frac{\frac{\overline{c}^p}{q\left(k^{1/p}+k^{1/q}\right)^p} - \frac{\overline{d}^q}{p}\left(||a||_1 + ||b||_1\right)}{\int_\Omega \max_{|s|+|t|\leq \overline{c}} F(x,s,t)dx - \int_\Omega F(x,\overline{d},\overline{d})dx},$$

problem (1) admits at least one nontrivial weak solution $(\widetilde{u},\widetilde{v})$ such that

$$\Phi(\widetilde{u},\widetilde{v}) > \frac{\overline{c}^p}{q\left(k^{1/p}+k^{1/q}\right)^p}.$$

Proof. First, note that $\min_X \Phi = \Phi(0,0) = 0$. Moreover, the condition (18) and (F) implies that, for every $\varepsilon > 0$ there exists $l_\varepsilon \in L^1(\Omega)$ such that

$$F(x,s,t) \leq \varepsilon(|s|^p + |t|^q) + l_\varepsilon(x) \quad \text{for all } (x,s,t) \in \Omega \times \mathbb{R}^2,$$

thus

$$\int_\Omega F(x,u,v)dx \leq \varepsilon\left(C_p||u||_p^p + C_q||v||_q^q\right) + \int_\Omega l_\varepsilon(x)dx \quad \text{for all } (u,v) \in X,$$

where C_p, C_q are constants of Sobolev. Therefore

$$I_\lambda(u,v) \quad \geq \quad (\frac{1}{p} - \varepsilon C_p)||u||_p^p + (\frac{1}{q} - \varepsilon C_q)||v||_q^q - \int_\Omega l_\varepsilon(x)dx.$$

So, choosing ε small enough we deduce that I_λ is coercive. Our aim is to verify condition (9) of Theorem 2. Indeed, let

$$u_0 = \overline{d} \quad \text{for all } x \in \Omega.$$

Working as in the proof of Theorem 3, put

$$r = \frac{\overline{c}^p}{q\left(k^{1/p}+k^{1/q}\right)^p}.$$

For all $(u,v) \in X$ such that $(u,v) \in \Phi^{-1}(]-\infty,r[)$, one has

$$|u| + |v| \leq \overline{c},$$

and we have

$$\begin{aligned} \rho(r) \quad &\geq \quad \frac{\Psi(u_0,u_0) - \left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right)}{\Phi(u_0,u_0) - r} \\ &\geq \quad \frac{\int_\Omega F(x,\overline{d},\overline{d})dx - \int_\Omega \max_{|s|+|t|\leq\overline{c}} F(x,s,t)dx}{\frac{\overline{d}^q}{p}\left(||a||_1 + ||b||_1\right) - \frac{\overline{c}^p}{q\left(k^{1/p}+k^{1/q}\right)^p}} > 0. \end{aligned}$$

Hence, Theorem 2 ensures the existence of nontrivial solution $(\widetilde{u},\widetilde{v})$ of (1), such that

$$\Phi(\widetilde{u},\widetilde{v}) > \frac{\overline{c}^p}{q\left(k^{1/p}+k^{1/q}\right)^p}.$$

A further consequence of Theorem 4 is the following result.

Theorem 5. Assume that F_u and F_v are nonnegative, in addition

(i) there exist two constants $\overline{c}, \overline{d}$ with

$$\max\left(1, \left(\frac{q}{||a||_1 + ||b||_1}\right)^{1/p}\right) \leq \frac{\overline{c}}{\left(k^{1/p} + k^{1/q}\right)\left(||a||_1 + ||b||_1\right)\right)^{1/p}} < \overline{d} \tag{19}$$

such that

$$\frac{F(\overline{c}, \overline{c})}{\overline{c}^p} < \frac{F(\overline{d}, \overline{d})}{\left(k^{1/p} + k^{1/q}\right)^p \left(||a||_1 + ||b||_1\right) \overline{d}^p} \tag{20}$$

(ii)

$$\limsup_{|s|+|t| \to +\infty} \frac{F(x, s, t)}{|s|^p + |t|^q} = 0 \text{ uniformly in } X. \tag{21}$$

Then, for each

$$\lambda > \frac{\frac{\overline{c}^p}{q\left(k^{1/p}+k^{1/q}\right)^p} - \frac{\overline{d}^q}{p}\left(||a||_1 + ||b||_1\right)}{F(\overline{c}, \overline{c}) - F(\overline{d}, \overline{d})},$$

problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda F_u(u, v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda F_v(u, v) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{in } \partial\Omega, \end{cases} \tag{22}$$

admits at least one nontrivial weak solution.

Proof. Clearly, (21) implies (18), and by a simple computations show that (20) implies (17). Hence Theorem 4 ensures the conclusion.

Example. The problem

$$\begin{cases} -u'' + u = \lambda\left(\frac{3}{2}v\sqrt{u} + v\sqrt{v}\right) & \text{in } (0,1), \\ -v'' + v = \lambda\left(\frac{3}{2}u\sqrt{v} + u\sqrt{u}\right) & \text{in } (0,1), \\ u'(0) = v'(0) = u'(1) = v'(1) = 0, \end{cases} \tag{23}$$

admits at least one nontrivial solution for every

$$\lambda > \frac{\sqrt{2} - 2^{20}}{2^6 - 2^{26}}.$$

In fact, if we choose, for example $\overline{c} = 4$, $\overline{d} = 2^{10}$ and $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a function defined by

$$F(s, t) = st\left(\sqrt{s} + \sqrt{t}\right),$$

Taking into account that (3) and the estimate (4) implies $1 \leq k \leq \sqrt{2}$, we deduce that

$$\frac{\frac{\overline{c}^p}{q\left(k^{1/p}+k^{1/q}\right)^p} - \frac{\overline{d}^q}{p}\left(||a||_1 + ||b||_1\right)}{F(\overline{c}, \overline{c}) - F(\overline{d}, \overline{d})} \leq \frac{\sqrt{2} - 2^{20}}{2^6 - 2^{26}},$$

and all hypotheses of Theorem 5 are satisfied.

References

[1] D. Averna, G. Bonanno, Three solutions for a Neumann boundary value problem involving the p-Laplacian, *Le Matematiche*, **60** (Fasc. I) (2005), 81-91.

[2] G. A. Afrouzi, S. Heidarkhani, A. Hadjian and S. Shakeri, On a Class of Neumann Boundary Value Equations Driven by a $(p_1, .., p_n)$-Laplacian Operator, *J. Part. Diff. Eq.* Vol. **25**, No. 1, 2012, pp. 21-31.

[3] G. A. Afrouzi, S. Heidarkhani, D. O'Regan, Three solutions to a class of Neumann doubly eigenvalue elliptic systems driven by a $(p_1, .., p_n)$-Laplacian, *Bull. Korean Math. Soc.*, **47**(6) (2010), 1235-1250.

[4] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Analysis,* **75** (2012), 2992-3007.
[5] G. Bonanno, P. Candito, Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian, Arch. Math. (Basel) **80** (2003) 424-429.
[6] G. Bonanno, G. D'Aguì, Multiplicity results for a perturbed elliptic Neumann problem, Abstr. Appl. Anal., **2010** (2010), 1-10.
[7] G. Bonanno, S. Heidarkhani, D. O'Rengan, Multiple solutions for class of Drichlet quasilinear elliptic systems driven by a (p,q)-Laplacian operator, Dynamic Systems and Applications **20** (2011) 89-100.
[8] G. Bonanno, G. Molica Bisci, A remark on perturbed elliptic Neumann problems, *Studia Univ. Babeş-Bolyai, Mathematica*, Volume **LV**, Number 4, December 2010.
[9] G. Bonanno, A. Sciammetta, Existence and multiplicity results to Neumann problems for elliptic equations involving the p-Laplacian, *J. Math. Anal. Appl.* **390** (2012) 59-67.
[10] F. Faraci, Multiplicity results for a Neumann problem involving the p-Laplacian, *J. Math. Anal. Appl.*, **277** (2003), 180-189.
[11] C. Li, L. C. Tang , Three solutions for a class of quasilinear elliptic systems involving the (p,q)-Laplacian, *Nonlinear Anal.*, **69** (2008), 3322-3329.
[12] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl.Math.*, **133** (2000),401-410.
[13] B. Ricceri, A note on the Neumann problem, *Complex Var. Elliptic Equ.*, **55** (2010), 593-599.
[14] D. M. Shoorabi, G. A. Afrouzi, Infinitely many solutions for class of Neumann quasilinear elliptic systems, *Boundary Value Problems,* 2012, 2012:54.

EL Miloud Hssini
University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco.
E-mail address: hssini1975@yahoo.fr

Mohammed Massar
University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco.
E-mail address: massarmed@hotmail.com

Mohamed Talbi
Centre Régional de Métiers de l'Éducation et de Formation (CRMEF), Oujda, Morocco.
E-mail address: talbi_md@yahoo.fr

Najib Tsouli
University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco.
E-mail address: tsouli@hotmail.com