Electronic Journal of Mathematical Analysis and Applications,
Vol. 2(1) Jan. 2014, pp. 149-163.
ISSN: 2090-792X (online)
http://fcag-egypt.com/Journals/EJMAA/

L^∞-SOLUTIONS FOR SOME DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS

A.C. CAVALHEIRO

Abstract. In this paper we are interested in the existence of solutions for Dirichlet problem associated to the degenerate quasilinear elliptic equations

$$-\sum_{j=1}^{n} D_j[\omega(x)\mathcal{A}_j(x,u,\nabla u)] + \omega(x)g(x,u(x),\nabla u(x)) + H(x,u,\nabla u)\,\omega(x) = f(x), \text{ on } \Omega$$

in the setting of the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega)$.

1. Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega)$ for the Dirichlet problem

$$(P)\begin{cases} Lu(x) &= f(x), \quad \text{on } \ \Omega \\ u(x) &= 0, \quad \text{on } \ \partial\Omega \end{cases}$$

where L is the partial differential operator

$$Lu(x) = -\mathrm{div}\big[\omega(x)\mathcal{A}(x,u,\nabla u)\big] + g(x,u,\nabla u)\,\omega(x) + H(x,u,\nabla u)\,\omega(x) \qquad (1.1)$$

where Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$, ω is a weight function, and the functions $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$, $g:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $H:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ are Carathéodory functions.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^N$ such that $\omega(x) > 0$ for a.e. $x\in\mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^N$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^N$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4], [5] and [7]).

2000 *Mathematics Subject Classification.* 37J70, 35J60.
Key words and phrases. degenerate quasilinear elliptic equations, weighted Sobolev spaces.
Submitted March 21, 2013.

A class of weights, which is particular well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [10]). These classes have found many usefull applications in harmonic analysis (see [12] and [13]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [8]). There are, in fact, many interesting examples of weights (see [7]) for p-admissible weights).

Equations like (1.1) have been studied by many authors in the non-degenerate case (i.e. with $\omega(x)\equiv 1$) (see e.g. [1] and the references therein). The degenerate case with different conditions haven been studied by many authors. In [4] Drabek, Kufner and Mustonen proved that under certain condition, the Dirichlet problem associated with the equation $-\mathrm{div}(a(x,u,\nabla u)) = h$, $h\in[W_0^{1,p}(\Omega,\omega)]^*$ has at least one solution $u\in W_0^{1,p}(\Omega,\omega)$, and in [3] the author proved the existence of solution when the nonlinear term $H(x,\eta,\xi)$ is equal to zero.

Firstly, we prove an L^∞ estimate for the bounded solutions of (P): we assume that $f/\omega\in L^q(\Omega,\omega)$, with $r/(r-1) < q < \infty$ (where $r>1$ as in Theorem 2.4) and we prove that any $u\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ that solves (P) satisfies $\|u\|_{L^\infty(\Omega)}\leq C$, where C depends only of the data, i.e., Ω, $N, p, q, \alpha_1, \alpha_2, C_0, C_1$ and $\|f/\omega\|_{L^q(\Omega,\omega)}$. After that, we prove the existence of solution for problem (P) if $f/\omega\in L^q(\Omega,\omega)$, with $p'r/(r-1) < q < \infty$.

Note that, in the proof of our main result, many ideas have been adapted from [1],[2] and [9].

The following theorem will be proved in section 3.

Theorem 1.1 *Let ω be an A_p-weight, $1<p<\infty$. Suppose that*
(H1) $x\mapsto\mathcal{A}(x,\eta,\xi)$ *is measurable in Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^N$*
$(\eta,\xi)\mapsto\mathcal{A}(x,\eta,\xi)$ *is continuous in $\mathbb{R}\times\mathbb{R}^N$ for almost all $x\in\Omega$.*
(H2) $[\mathcal{A}(x,\eta,\xi)-\mathcal{A}(x,\eta',\xi')].(\xi-\xi') > 0$, *whenever $\xi,\xi'\in\mathbb{R}^N$, $\xi\neq\xi'$;*
(H3) $\mathcal{A}(x,\eta,\xi).\xi\geq\alpha_1|\xi|^p$, *with $1<p<\infty$, where $\alpha_1>0$;*
(H4) $|\mathcal{A}(x,\eta,\xi)|\leq K_1(x)+h_1(x)|\eta|^{p/p'}+h_2(x)|\xi|^{p/p'}$, *where K_1, h_1 and h_2 are positive functions, with h_1 and $h_2\in L^\infty(\Omega)$, and $K_1\in L^{p'}(\Omega,\omega)$ $(1/p+1/p'=1)$.*
(H5) $x\mapsto g(x,\eta,\xi)$ *is measurable in Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^N$*
$(\eta,\xi)\mapsto g(x,\eta,\xi)$ *is continuous in $\mathbb{R}\times\mathbb{R}^N$ for almost all $x\in\Omega$.*
(H6) $|g(x,\eta,\xi)|\leq K_2(x)+h_3(x)|\eta|^{p/p'}+h_4(x)|\xi|^{p/p'}$, *where K_2, h_3 and h_4 are positive functions, with $h_3, h_4\in L^\infty(\Omega)$ and $K_2\in L^{p'}(\Omega,\omega)$.*
(H7) $g(x,\eta,\xi)\eta\geq\alpha_0|\eta|^p$, *for all $\eta\in\mathbb{R}$, where $\alpha_0>0$.*
(H8) $x\mapsto H(x,\eta,\xi)$ *is measurable in Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^N$*
$(\eta,\xi)\mapsto H(x,\eta,\xi)$ *is continuous in $\mathbb{R}\times\mathbb{R}^N$ for almost all $x\in\Omega$.*
(H9) $|H(x,\eta,\xi)|\leq C_0+C_1|\xi|^p$, *where C_0 and C_1 are positive constants.*
(H10) $f/\omega\in L^q(\Omega,\omega)$, *with $r/(r-1) < q < \infty$ (where $r>1$ as in Theorem 2.4).*
Let $u\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ be a solution of problem (P). Then there exists a constant $C>0$, which depends only on Ω, $n, p, \alpha_1, \alpha_0, C_0, C_1$ and $\|f/\omega\|_{L^q(\Omega,\omega)}$, such that $\|u\|_{L^\infty(\Omega)}\leq C$.

The main result of this article is given in the next theorem, which is proved in section 4.

Theorem 1.2 *Let us assume that* $(H1)-(H9)$ *hold true and suppose that*
(H11) $f/\omega \in L^q(\Omega,\omega)$, *with* $p'r/(r-1) < q < \infty$;
(H12) $H(x,\eta,\xi)\,\eta \geq 0$, *for all* $\eta \in \mathbb{R}$.
Then there exists at least one solution $u \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$ *of the problem* (P).

Theorem 1.2 will be proved by approximating problem (P) with the following problems

$$(P_m)\begin{cases} -\mathrm{div}\big[\omega\,\mathcal{A}(x,u,\nabla u)\big] + g(x,u,\nabla u)\,\omega + H_m(x,u,\nabla u)\,\omega = f(x), \text{ on } \Omega \\ u(x) = 0, \text{ on } \partial\Omega \end{cases}$$

where $H_m(x,\eta,\xi) = \dfrac{H(x,\eta,\xi)}{1 + \dfrac{1}{m}\,|H(x,\eta,\varepsilon)|}$, for $m \in \mathbb{N}$. Note that $|H_m| \leq |H|$ and that $|H_m| \leq m$.

2. Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^N$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \leq C_{p,\omega}$$

for all balls $B \subset \mathbb{R}^N$, where $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [6],[7],[13] or [14] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r))$$

for every ball $B = B(x,r) \subset \mathbb{R}^N$, where $\mu(B) = \int_B \omega(x)\,dx$.If $\omega \in A_p$, then ω is doubling (see Corollary 15.7 in [7]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see Corollary 4.4, Chapter IX in [13]). If $\varphi \in BMO(\mathbb{R}^N)$ then $\omega(x) = \mathrm{e}^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [12]).

Definition 2.1 Let ω be a weight, and let $\Omega \subset \mathbb{R}^N$ be open. For $0 < p < \infty$, we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,dx\right)^{1/p} < \infty.$$

Remark 2.2 If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [14]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$. □

Definition 2.3 Let $\Omega \subset \mathbb{R}^N$ be open, $1 < p < \infty$, and let ω be an A_p-weight, $1 < p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega,\omega)$ as the set of functions

$u \in L^p(\Omega,\omega)$ with weak derivatives $D_j u \in L^p(\Omega,\omega)$, for $j = 1, ..., N$. The norm of u in $W^{1,p}(\Omega,\omega)$ is given by

$$\|u\|_{W^{1,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx + \sum_{j=1}^{N} \int_\Omega |D_j u(x)|^p \omega(x)\, dx \right)^{1/p}. \qquad (2.1)$$

We also define $W_0^{1,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega,\omega)$, and

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} = \left(\sum_{j=1}^{N} \int_\Omega |D_j u(x)|^p \omega(x)\, dx \right)^{1/p}.$$

The dual space of $W_0^{1,p}(\Omega,\omega)$ is the space $[W_0^{1,p}(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$ (see [5]),

$$W^{-1,p'}(\Omega,\omega) = \{T = f_0 - \mathrm{div} f \,:\, f = (f_1, ..., f_N),\ f_j/\omega \in L^{p'}(\Omega,\omega),\ j = 0, ..., N\}.$$

It is evident that the weight ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (c_1 and c_2 positive constants), give nothing new (the space $\mathrm{W}_0^{1,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}_0^{1,p}(\Omega)$). Consequently, we shall interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

In this paper we use the following four results.

Theorem 2.4 (The Weighted Sobolev Inequality) *Let Ω be an open bounded set in $\mathbb{R}^N$ $(N \ge 2)$ and $\omega \in A_p$ $(1 < p < \infty)$. There exist constants C_Ω and δ positive such that for all $u \in C_0^\infty(\Omega)$ and all r satisfying $1 \le r \le \dfrac{N}{(N-1)} + \delta$,*

$$\|u\|_{L^{p^*}(\Omega,\omega)} \le C_\Omega \, \|\nabla u\|_{L^p(\Omega,\omega)}$$

where $p^* = p\, r$.

Proof. See Theorem 1.3 in [5]. □

The following lemma is due to Stampacchia (see [11], Lemme 4.1).

Lemma 2.5 *Let α, β, C, k_0 be real positive numbers, where $\beta > 1$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function such that*

$$\varphi(l) \le \frac{C}{(l-k)^\alpha} [\varphi(k)]^\beta$$

for all $l > k \ge k_0$. Then $\varphi(k_0 + d) = 0$, where $d^\alpha = C\, [\varphi(k_0)]^{\beta-1} 2^{\alpha\beta/(\beta-1)}$.

Lemma 2.6 *If $\omega \in A_p$, then $\left(\dfrac{|E|}{|B|}\right)^p \le C_{p,\omega} \dfrac{\mu(E)}{\mu(B)}$, whenever B is a ball in $\mathbb{R}^N$ and E is a measurable subset of B.*

Proof. See Theorem 15.5 *Strong doubling of A_p-weights* in [7]. □

By Lemma 2.6, if $\mu(E) = 0$ then $|E| = 0$.

Lemma 2.7 *Let $\omega \in A_p$, $1 < p < \infty$ and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega,\omega)$ satisfies* *(i)* $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega,\omega)$ *and* μ*-a.e. in* Ω*;*

(ii) $\int_\Omega \langle \mathcal{A}(x,u_n,\nabla u_n) - \mathcal{A}(x,u_n,\nabla u), \nabla(u_n-u)\rangle\,\omega\,dx \to 0$ *with* $n\to\infty$.

Then $u_n \to u$ *in* $W_0^{1,p}(\Omega,\omega)$.

Proof. The proof of this lemma follows the line of Lemma 5 in [2]. □

Definition 2.8 We say that $u\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ is a (weak) solution of problem (P) if

$$\int_\Omega \omega\,\mathcal{A}(x,u,\nabla u).\nabla\varphi\,dx + \int_\Omega g(x,u,\nabla u)\varphi\,\omega\,dx + \int_\Omega H(x,u,\nabla u)\,\varphi\,\omega\,dx = \int_\Omega f\,\varphi\,dx, \tag{2.2}$$

for all $\varphi\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$.

3. **Proof of Theorem 1.1**

Set $\lambda = \dfrac{C_1}{\alpha_1} + 1$ and define for $k>0$ the functions $\phi\in C^1(\mathbb{R})$ and $G_k\in W^{1,\infty}(\mathbb{R})$ by

$$\phi(s) = \begin{cases} \mathrm{e}^{\lambda s} - 1, & \text{if } s\ge 0,\\ -\mathrm{e}^{-\lambda s} + 1, & \text{if } s\le 0,\end{cases}$$

$$G_k(s) = \begin{cases} s-k, & \text{if } s\ge k,\\ 0, & \text{if } -k\le s\le k,\\ s+k, & \text{if } s\le -k.\end{cases}$$

If $u\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ is a solution of problem (P), define the set $A(k) = \{x\in\Omega : |u(x)| > k\}$.

We will use the test functions $v(x) = \phi(G_k(u(x)))$. We have

$$v(x) = \phi((|u|-k)^+)\,\chi_{A(k)}\mathrm{sign}(u),$$
$$\nabla v = \phi'((|u|-k)^+)\chi_{A(k)}\nabla u,$$

where $\chi_{A(k)}$ is the characteristic function of the set $A(k)$.
Since $u\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$, we have that $v\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$.

Using the function v in (2.2) we obtain

$$\int_\Omega \omega\,\mathcal{A}(x,u,\nabla u).\nabla v\,dx + \int_\Omega g(x,u,\nabla u)v\,\omega\,dx + \int_\Omega H(x,u,\nabla u)\,v\,\omega\,dx = \int_\Omega f\,v\,dx. \tag{3.1}$$

We have the following estimates.
(i) By (H3) we obtain

$$\begin{aligned}\int_\Omega \omega\, \mathcal{A}(x,u,\nabla u).\nabla v\, dx &= \int_{A(k)} \phi'((|u|-k)^+)\mathcal{A}(x,u,\nabla u).\nabla u\, \omega\, dx \\ &\geq \alpha_1 \int_{A(k)} |\nabla u|^p \phi'((|u|-k)^+)\, \omega\, dx.\end{aligned}$$

(ii) By (H7) we obtain

$$\begin{aligned}\int_\Omega g(x,u,\nabla u)\, v\, \omega\, dx &= \int_{A(k)} g(x,u,\nabla u)\, \phi((|u|-k)^+)\, \omega\, dx \\ &\geq \alpha_0 \int_{A(k)} |u|^{p-1}\phi((|u|-k)^+)\, \omega\, dx.\end{aligned}$$

(iii) Using (H9) we obtain

$$\begin{aligned}\left|\int_\Omega H(x,u,\nabla u)\, v\, \omega\, dx\right| &\leq \int_\Omega |H(x,u,\nabla u)||v|\, \omega\, dx \\ &\leq \int_{A(k)} (C_0 + C_1|\nabla u|^p)\phi((|u|-k)^+)\, \omega\, dx.\end{aligned}$$

And we also have

$$\left|\int_\Omega f\, v\, dx\right| \leq \int_{A(k)} |f|\, \phi((|u|-k)^+)\, dx.$$

Hence in (3.1) we obtain

$$\begin{aligned}&\alpha_1 \int_{A(k)} |\nabla u|^p \phi'((|u|-k)^+)\, \omega\, dx + \alpha_0 \int_{A(k)} |u|^{p-1}\phi((|u|-k)^+)\, \omega\, dx \\ &\leq \int_{A(k)} (C_0 + C_1|\nabla u|^p)\phi((|u|-k)^+)\omega\, dx + \int_{A(k)} |f|\phi((|u|-k)^+)\, dx. \quad (3.2)\end{aligned}$$

Since $\lambda = \dfrac{C_1}{\alpha_1} + 1$, we have for $s \geq 0$

$$\begin{aligned}&\alpha_1\phi'(s) - C_1\phi(s) = \alpha_1\lambda e^{\lambda s} - C_1(e^{\lambda s} - 1) \\ &= (\alpha_1\lambda - C_1)e^{\lambda s} + C_1 = \alpha_1 e^{\lambda s} + C_1 \\ &\geq \alpha_1 e^{\lambda s} = \frac{\alpha_1}{\lambda^p}[\lambda e^{\lambda s/p}]^p \\ &= \frac{\alpha_1}{\lambda^p}[\phi'(s/p)]^p. \qquad (3.3)\end{aligned}$$

Hence in (3.2) we obtain

$$\begin{aligned}&\int_{A(k)} \left[\alpha_1|\nabla u|^p\, \phi'((|u|-k)^+) - C_1|\nabla u|^p\, \phi((|u|-k)^+)\right]\omega\, dx \\ &+ \alpha_0 \int_{A(k)} |u|^{p-1}\phi((|u|-k)^+)\, \omega\, dx \\ &\leq \int_{A(k)} (|f| + C_0\, \omega)\phi((|u|-k)^+)\, dx. \qquad (3.4)\end{aligned}$$

Using (3.3) and $k < |u(x)|$ if $x \in A(k)$, we obtain

$$\frac{\alpha_1}{\lambda^p}\int_{A(k)}\left|\phi'\left(\frac{(|u|-k)^+}{p}\right)\nabla u\right|^p \omega\,dx + \alpha_0 k^{p-1}\int_{A(k)}\phi((|u|-k)^+)\,\omega\,dx$$
$$\leq \int_{A(k)}(|f|+C_0\,\omega)\,\phi((|u|-k)^+)\,dx. \tag{3.5}$$

Let us define the function ψ_k by $\psi_k(x) = \phi\left(\frac{(|u(x)|-k)^+}{p}\right)$. We have that $\psi_k \in W_0^{1,p}(\Omega,\omega)$ and

$$\nabla\psi_k = \frac{1}{p}\,\phi'\left(\frac{(|u|-k)^+}{p}\right)\chi_{A(k)}\mathrm{sign}(u)\,\nabla u. \tag{3.6}$$

We have that
(a) For all $s \geq 0$, $\mathrm{e}^{\lambda s} - 1 \geq (\mathrm{e}^{\lambda s/p} - 1)^p$;
(b) There exist a constant $C_2 > 0$ ($C_2 = C_2(\lambda, p)$) such that for all $s \geq 1$

$$\mathrm{e}^{\lambda s} - 1 \leq C_2(\mathrm{e}^{\lambda s/p} - 1)^p \text{ and } \lambda\mathrm{e}^{\lambda s} \leq C_2\,\lambda(\mathrm{e}^{\lambda s/p} - 1)^p.$$

This implies
(I1) $\phi((|u|-k)^+) = \mathrm{e}^{\lambda(|u|-k)^+} - 1 \geq (\mathrm{e}^{\lambda(|u|-k)^+/p} - 1)^p = |\,\psi_k|^p$ a.e. on Ω;
(I2) If $x \in \mathrm{A}(k+1)$ then
$\phi((|u|-k)^+) = \mathrm{e}^{\lambda(|u|-k)^+} - 1 \leq C_2(\mathrm{e}^{\lambda(|u|-k)^+/p} - 1)^p = C_2|\psi_k|^p$ and
$\phi'((|u|-k)^+) = \lambda\,\mathrm{e}^{\lambda(|u|-k)^+} \leq C_2\,\lambda\,(\mathrm{e}^{\lambda(|u|-k)^+/p} - 1)^p = C_2\,\lambda\,|\psi_k|^p$.
Combining (I1) and (I2) with (3.5) and (3.6) we obtain

$$\frac{\alpha_1 p^p}{\lambda^p}\int_\Omega |\nabla\psi_k|^p\omega\,dx + \alpha_0 k^{p-1}\int_\Omega |\psi_k|^p\,\omega\,dx$$
$$\leq \int_{A(k)}(|f|+C_0\,\omega)\,\phi((|u|-k)^+)\,dx$$
$$\leq \int_{A(k+1)}(|f|+C_0\,\omega)\,C_2\,|\psi_k|^p\,dx$$
$$+\int_{A(k)-A(k+1)}(|f|+C_0\,\omega)\phi((|u|-k)^+)\,dx. \tag{3.7}$$

Define the function $h = |f| + C_0\omega$. Since $f/\omega \in L^q(\Omega,\omega)$ and $\mu(\Omega) < \infty$, we have that $h/\omega \in L^q(\Omega,\omega)$. Hence

$$\begin{aligned}\int_\Omega h\,|\psi_k|^p\,dx &= \int_\Omega \frac{h}{\omega}\,|\psi_k|^p\,\omega^{1/q}\,\omega^{1/q'}\,dx\\ &\leq \|h/\omega\|_{L^q(\Omega,\omega)}\|\psi_k\|^p_{L^{p\,q'}(\Omega,\omega)}.\end{aligned} \tag{3.8}$$

If $x \in A(k) - A(k+1)$, we have $k < |u| < k+1$. Hence

$$\phi((|u|-k)^+) = \mathrm{e}^{\lambda(|u|-k)^+} - 1 \leq \mathrm{e}^\lambda - 1$$

and we obtain

$$\int_{A(k)-A(k+1)} (|f| + C_0\,\omega)\,\phi((|u|-k)^+)\,dx \;\leq\; \int_{A(k)-A(k+1)} (\mathrm{e}^{\lambda} - 1)\,h\,dx \;\leq\; \mathrm{e}^{\lambda} \int_{A(k)} h\,dx. \tag{3.9}$$

By Theorem 2.4, (3.7) and (3.9) we have

$$\frac{\alpha_1\,p^p}{\lambda^p}\,\frac{1}{C_\Omega^p}\bigg(\int_\Omega |\psi_k|^{p^*}\omega\,dx\bigg)^{p/p^*} + \;\alpha_0\,k_0^{p-1}\int_\Omega |\psi_k|^p\,\omega\,dx$$
$$\leq C_2 \int_\Omega h\,|\psi_k|^p\,dx + \mathrm{e}^{\lambda}\int_{A(k)} h\,dx$$

Therefore, there exist positive constants C_3 and C_4 (depending only on Ω, α_1, p,λ and C_2) such that

$$C_3\bigg(\int_\Omega |\psi_k|^{p^*}\omega\,dx\bigg)^{p/p^*} + C_4\,\alpha_0\,k_0^{p-1}\int_\Omega |\psi_k|^p\,\omega\,dx$$
$$\leq \int_\Omega h\,|\psi_k|^p\,dx + \int_{A(k)} h\,dx \tag{3.10}$$

Since $r/(r-1) < q$ then $q' < r$ and $p < p\,q' < p^*$. For $0 < \theta < 1$ such that $\dfrac{1}{p\,q'} = \dfrac{\theta}{p} + \dfrac{1-\theta}{p^*}$, using an interpolation inequality, Young's inequality (with $0 < \gamma < \infty$) and Hölder's inequality with exponents q and q' we thus obtain

$$\int_\Omega h\,|\psi_k|^p\,dx \leq \|h/\omega\|_{L^q(\Omega,\omega)}\|\psi_k\|^p_{L^{p\,q'}(\Omega,\omega)}$$
$$\leq \|h/\omega\|_{L^q(\Omega,\omega)}\|\psi_k\|^{\theta\,p}_{L^p(\Omega,\omega)}\|\psi_k\|^{(1-\theta)p}_{L^{p^*}(\Omega,\omega)}$$
$$\leq (1-\theta)\gamma^{1/(1-\theta)}\|\psi_k\|^p_{L^{p^*}(\Omega,\omega)} + \theta\,\gamma^{-1/\theta}\|h/\omega\|^{1/\theta}_{L^q(\Omega,\omega)}\|\psi_k\|^p_{L^p(\Omega,\omega)}. \tag{3.11}$$

Hence in (3.10) we obtain

$$C_3\bigg(\int_\Omega |\psi_k|^{p^*}\omega\,dx\bigg)^{p/p^*} + C_4\,\alpha_0\,k_0^{p-1}\int_\Omega |\psi_k|^p\,\omega\,dx$$
$$\leq (1-\theta)\gamma^{1/(1-\theta)}\|\psi_k\|^p_{L^{p^*}(\Omega,\omega)} + \theta\,\gamma^{-1/\theta}\|h/\omega\|^{1/\theta}_{L^q(\Omega,\omega)}\|\psi_k\|^p_{L^p(\Omega,\omega)}$$
$$+ \int_{A(k)} h\,dx. \tag{3.12}$$

Now, we can choose γ in order to have $(1-\theta)\,\gamma^{1/(1-\theta)} = C_3/2$ and k_0 such that

$$C_4\alpha_0 k_0^{p-1} = \theta\,\gamma^{-1/\theta}\|h/\omega\|^{1/\theta}_{L^q(\Omega,\omega)}.$$

We obtain, from (3.12), that for every $k \geq k_0$ it results

$$\frac{C_3}{2}\left(\int_\Omega |\psi_k|^{p^*}\,\omega\,dx\right)^{p/p^*} \le \int_{A(k)} h\,dx \le \|h/\omega\|_{L^q(\Omega,\omega)}\,[\mu(A_k)]^{1/q'}.$$

Hence for all $k \ge k_0$ we have

$$\begin{aligned}\int_\Omega |\psi_k|^{p^*}\,\omega\,dx &\le \left(\frac{2}{C_3}\|h/\omega\|_{L^q(\Omega,\omega)}[\mu(A(k))]^{1/q'}\right)^{p^*/p}\\ &= \left(\frac{2}{C_3}\right)^{p^*/p}\|h/\omega\|^{p^*/p}_{L^q(\Omega,\omega)}[\mu(A(k))]^{p^*/p\,q'}\\ &= C_5[\mu(A(k))]^{p^*/p\,q'}.\end{aligned}$$

Let us now take $l > k \ge k_0$ we have

$$\begin{aligned}\mu(A(l))\left[\lambda\left(\frac{l-k}{p}\right)\right]^{p^*} &\le \mu(A(k))|\phi((l-k)/p)|^{p^*}\\ &\le \int_{A(k)} |\psi_k|^{p^*}\omega\,dx\\ &\le \int_\Omega |\psi_k|^{p^*}\omega\,dx.\end{aligned}$$

Therefore foll all $l > k \ge k_0$ we obtain

$$\begin{aligned}(l-k)^{p^*}\mu(A(l)) &\le \frac{p^{p^*}}{\lambda^{p^*}}C_5[\mu(A(k))]^{p^*/p\,q'}\\ &= C_6[\mu(A(k))]^{p^*/p\,q'}.\end{aligned}$$

that is, $\mu(A(l)) \le \dfrac{C_6}{(l-k)^{p^*}}[\mu(A(k))]^{p^*/p\,q'}$.

Let $\varphi(k) = \mu(A(k))$. Since $\beta = p^*/p\,q' > 1$, by Lemma 2.5 there exists a constant $C_7 > 0$ such that

$$\mu(A(k)) = 0,\ \forall\, k \ge C_7.$$

Using Lemma 2.6 we have $|A(k)| = 0$ for all $k \ge C_7$. Therefore any solution u of problem (P) satisfies the estimate $\|u\|_{L^\infty(\Omega)} \le C_7$.

□

4. **Proof of Theorem 1.2**

Step 1. Let us define for $m \in \mathbb{N}$ the approximation

$$H_m(x,\eta,\xi) = \frac{H(x,\eta,\xi)}{1 + \dfrac{1}{m}\,|H(x,\eta,\xi)|}.$$

We have that $|H_m(x,\eta,\xi)| \le |H(x,\eta,\xi)|$, $|H_m(x,\eta,\xi)| < m$ and $H_m(x,\eta,\xi)$ satisfies the conditions (H9) and (H12). We consider the approximate problem

$$(P_m)\begin{cases} -\operatorname{div}\big[\omega\,\mathcal{A}(x,u,\nabla u)\big]+g(x,u,\nabla u)\,\omega+H_m(x,u,\nabla u)\,\omega=f(x)\ ,\ \text{on }\Omega\\ u(x)=0,\ \text{on }\partial\Omega\end{cases}$$

We say that $u\in W_0^{1,p}(\Omega,\omega)$ is (weak) solution of problem (P_m) if

$$\begin{aligned}\int_\Omega \mathcal{A}(x,u,\nabla u).\nabla\varphi\,\omega\,dx+\int_\Omega g(x,u,\nabla u)\,\varphi\,\omega\,dx\\ +\int_\Omega H_m(x,u,\nabla u)\,\varphi\,\omega\,dx=\int_\Omega f\,\varphi\,dx,\end{aligned}\tag{4.1}$$

for all $\varphi\in W_0^{1,p}(\Omega,\omega)$. We will prove that there exists at least one solution u_m of the problem (P_m).

For $u,v,\varphi\in W_0^{1,p}(\Omega,\omega)$ we define

$$\begin{aligned} B(u,v,\varphi) &= \int_\Omega \omega\,\mathcal{A}(x,u,\nabla v).\nabla\varphi\,dx,\\ B_m(u,\varphi) &= \int_\Omega g(x,u,\nabla u)\,\varphi\,\omega\,dx+\int_\Omega H_m(x,u,\nabla u)\,\varphi\,\omega\,dx,\\ T(\varphi) &= \int_\Omega f\,\varphi\,dx.\end{aligned}$$

Then $u\in W_0^{1,p}(\Omega,\omega)$ is a (weak) solution of problem (P_m) if

$$B(u,u,\varphi)+B_m(u,\varphi)=T(\varphi),\quad \text{for all } \varphi\in W_0^{1,p}(\Omega,\omega).$$

Let $a(u,v,\varphi)=B(u,v,\varphi)+B_m(u,\varphi)$.
(i) Using (H4) we obtain

$$\begin{aligned}|B(u,v,\varphi)| \leq\ & \Big(\|K_1\|_{L^{p'}(\Omega,\omega)}+\|h_1\|_{L^\infty(\Omega)}\,\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\\ & +\|h_2\|_{L^\infty(\Omega)}\|v\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\Big)\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

(ii) Using (H6) and $|H_m(x,\eta,\xi)|\leq m$, we obtain

$$\begin{aligned}|B_m(u,\varphi)| \leq\ & \Big(\|K_2\|_{L^{p'}(\Omega,\omega)}+\|h_3\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}+\|h_4\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\\ & +m\,[\mu(\Omega)]^{1/p'}\Big)\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}\end{aligned}$$

Hence,

$$\begin{aligned}&|\,a(u,v,\varphi)|\\ &\leq\Big(\|K_1\|_{L^{p'}(\Omega,\omega)}+(\|h_1\|_{L^\infty(\Omega)}+\|h_3\|_{L^\infty(\Omega)}+\|h_4\|_{L^\infty(\Omega)})\,\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\\ &\quad+\|K_2\|_{L^{p'}(\Omega,\omega)}+\|h_2\|_{L^\infty(\Omega)}\|v\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}+m\,[\mu(\Omega)]^{1/p'}\Big)\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

Since $a(u,v,.)$ is linear and continuous for each $(u,v)\in W_0^{1,p}(\Omega,\omega)\times W_0^{1,p}(\Omega,\omega)$, there exists a linear and continuous operator $A(u,v):W_0^{1,p}(\Omega,\omega)\to[W_0^{1,p}(\Omega,\omega)]^*$ such that $\langle A(u,v),\varphi\rangle = a(u,v,\varphi)$. We set $\tilde{A}(u)=A(u,u)$ for all $u\in W_0^{1,p}(\Omega,\omega)$. The operator $\tilde{A}:W_0^{1,p}(\Omega,\omega)\to[W_0^{1,p}(\Omega,\omega)]^*$ is semimonotone, that is, by similar arguments as in the proof of Theorem 2 in [9] we have

(i) $\langle A(u,u)-A(u,v),u-v\rangle\geq 0$ for all $u,v\in W_0^{1,p}(\Omega,\omega)$;

(ii) For each $u\in W_0^{1,p}(\Omega,\omega)$, the operator

$$v\mapsto A(u,v)$$

is hemicontinuous and bounded from $W_0^{1,p}(\Omega,\omega)$ to $[W_0^{1,p}(\Omega,\omega)]^*$ and for each $v\in W_0^{1,p}(\Omega,\omega)$ the operator

$$u\mapsto A(u,v)$$

is hemicontinuous and bounded from $W_0^{1,p}(\Omega,\omega)$ to $[W_0^{1,p}(\Omega,\omega)]^*$;

(iii) If $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega)$ and $\langle A(u_n,u_n)-A(u_n,u),u_n-u\rangle\to 0$, then $A(u_n,u)\rightharpoonup A(u,v)$ in $[W_0^{1,p}(\Omega,\omega)]^*$ as $n\to\infty$ for all $v\in W_0^{1,p}(\Omega,\omega)$;

(iv) If $v\in W_0^{1,p}(\Omega,\omega)$, $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega)$ and $A(u_n,v)\rightharpoonup\tilde{v}$ in $[W_0^{1,p}(\Omega,\omega)]^*$ then $\langle A(u_n,v),u_n\rangle\to\langle\tilde{v},u\rangle$ as $n\to\infty$;

(v) The operator $\tilde{A}:W_0^{1,p}(\Omega,\omega)\to[W_0^{1,p}(\Omega,\omega)]^*$ is bounded.

Hence the operator $\tilde{A}:W_0^{1,p}(\Omega,\omega)\to[W_0^{1,p}(\Omega,\omega)]^*$ is pseudomonotone (see [15]).

(vi) By (H3), (H7) and (H12) we have

$$\langle\tilde{A}(u),u\rangle\geq\alpha_1\int_\Omega|\nabla u|^p\omega\,dx+\alpha_0\int_\Omega|u|^p\omega\,dx\geq\alpha_1\|u\|^p_{W_0^{1,p}(\Omega,\omega)}.$$

Since $p>1$, we have

$$\frac{\langle\tilde{A}(u),u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega)}}\to\infty\ \text{ as }\ \|u\|_{W_0^{1,p}(\Omega,\omega)}\to\infty,$$

that is, the operator $\tilde{A}$ is coercive. Then, by Theorem 27.B in [15], for each $T\in[W_0^{1,p}(\Omega,\omega)]^*$, the equation

$$\tilde{A}u=T,\quad u\in W_0^{1,p}(\Omega,\omega)$$

has a solution. Therefore, the problem (P_m) has a solution $u_m\in W_0^{1,p}(\Omega,\omega)$.

Step 2. We will show that $u_m\in L^\infty(\Omega)$ and $\|u_m\|_{L^\infty(\Omega)}\leq C$, where C is independent of m. If $u\in W_0^{1,p}(\Omega,\omega)$ is a solution of problem (P_m) we define

$$u_n(x)=\begin{cases} u(x), & \text{if } |u(x)|\leq n,\\ n, & \text{if } u(x)>n,\\ -n, & \text{if } u(x)<-n.\end{cases}$$

We have $D_iu_n=D_iu$ if $|u(x)|\leq n$. For $k>0$, let us define the function $\psi_n(x)=\operatorname{sign}(u_n(x))\max\{|u_n(x)|-k,0\}$. We have $\psi_n\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$. Now consider the function

$$\Phi(t)=\begin{cases} t+k, & \text{if } t\leq -k,\\ 0, & \text{if } |t|\leq k,\\ t-k, & \text{if } t\geq k.\end{cases}$$

We have that Φ is a Lipschitz function and $\Phi(0) = 0$. Then $\Phi(\psi_n) \in W_0^{1,p}(\Omega, \omega)$. Moreover $D_i\Phi(\psi_n) = \Phi'(\psi_n)D_i\psi_n$ and

$$\Phi'(u_n)\nabla u_n \to \Phi'(u)\nabla u, \quad \mu - a.e. \text{ in } \Omega.$$

We also have, for all measurable subset $E \subset \Omega$

$$\int_E |\Phi'(u_n)\nabla u_n|^p \,\omega\, dx \leq \int_E |\nabla u_n|^p \,\omega\, dx.$$

By applying the Vitali's Convergence Theorem, with $\psi = \Phi(u)$, we obtain

$$\nabla\psi_n \to \nabla\psi \quad \text{in} \quad L^p(\Omega, \omega). \tag{4.2}$$

Since $u \in W_0^{1,p}(\Omega, \omega)$ is a solution of problem (P_m) and $\psi_n \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned} &\int_\Omega \mathcal{A}(x, u, \nabla u).\nabla\psi_n \,\omega\, dx + \int_\Omega g(x, u, \nabla u)\, \psi_n \,\omega\, dx \\ &+ \int_\Omega H_m(x, u, \nabla u)\, \psi_n \,\omega\, dx = \int_\Omega f\, \psi_n\, dx. \end{aligned} \tag{4.3}$$

Using (H4), (H6), $|H_m(x, \eta, \xi)| \leq m$ and (4.2), we obtain in (4.3) as $n \to \infty$

$$\begin{aligned} &\int_\Omega \mathcal{A}(x, u, \nabla u).\nabla\psi \,\omega\, dx + \int_\Omega g(x, u, \nabla u)\, \psi \,\omega\, dx \\ &+ \int_\Omega H_m(x, u, \nabla u)\, \psi \,\omega\, dx = \int_\Omega f\, \psi\, dx. \end{aligned}$$

Using $\varphi = \psi\, \chi_{A(k)}$ in (4.1) (where $A(k) = \{x \in \Omega : |u(x)| > k\}$) we obtain

$$\begin{aligned} &\int_{A(k)} \mathcal{A}(x, u, \nabla u).\nabla\psi \,\omega\, dx + \int_{A(k)} g(x, u, \nabla u)\, \psi \,\omega\, dx \\ &+ \int_{A(k)} H_m(x, u, \nabla u)\, \psi \,\omega\, dx = \int_{A(k)} f\, \psi\, dx. \end{aligned} \tag{4.4}$$

Since

$$\psi = \Phi(u) = \begin{cases} u + k, & \text{if } u \leq -k \\ 0, & \text{if } |u| \leq k \\ u - k, & \text{if } u \geq k, \end{cases}$$

we obtain:

(i) By (H7) we have $g(x, \eta, \xi)\, \eta \geq 0$ for all $\eta \in \mathbb{R}$, and

$$\begin{aligned} \int_{A(k)} g(x, u, \nabla u)\, \psi \,\omega\, dx &= \int_{\{u \leq -k\}} g(x, u, \nabla u)(u + k)\, \omega\, dx \\ &+ \int_{\{u \geq k\}} g(x, u, \nabla u)(u - k)\, \omega\, dx \geq 0; \end{aligned}$$

(ii) Using (H12) we have $H_m(x, \eta, \xi)\, \eta \geq 0$ for all $\eta \in \mathbb{R}$, and

$$\begin{aligned}\int_{A(k)} H_m(x,u,\nabla u)\,\psi\,\omega\,dx &= \int_{\{u\le -k\}} H_m(x,u,\nabla u)(u+k)\,\omega\,dx\\ &+ \int_{\{u\ge k\}} H_m(x,u,\nabla u)(u-k)\,\omega\,dx \ge 0.\end{aligned}$$

We have $\nabla\psi = \nabla u$ in $A(k)$. Using (H3), (i) and (ii) we obtain in (4.4)

$$\alpha_1 \int_{A(k)} |\nabla u|^p\,\omega\,dx \le \int_{A(k)} f\,\psi\,dx. \tag{4.5}$$

By Theorem 2.1.14 in [14] there is a positive constant C such that

$$\int_\Omega |\psi|^p \omega\,dx \le C\int_\Omega |\nabla\psi|^p\omega\,dx.$$

Then we obtain

$$\begin{aligned}\int_{A(k)} f\,\psi\,dx &\le \left(\int_{A(k)} \left|\frac{f}{\omega}\right|^{p'}\omega\right)^{1/p'}\left(\int_{A(k)} |\,\psi|^p\omega\,dx\right)^{1/p}\\ &\le C\left(\int_{A(k)} \left|\frac{f}{\omega}\right|^{p'}\omega\right)^{1/p'}\left(\int_{A(k)} |\,\nabla\psi|^p\omega\,dx\right)^{1/p}\\ &= C\left(\int_{A(k)} \left|\frac{f}{\omega}\right|^{p'}\omega\right)^{1/p'}\left(\int_{A(k)} |\,\nabla u|^p\omega\,dx\right)^{1/p}.\end{aligned} \tag{4.6}$$

Using (4.6) and Young's inequality we obtain in (4.5) (for all $\varepsilon > 0$)

$$\begin{aligned}\alpha_1 \int_{A(k)} |\,\nabla u|^p\omega\,dx &\le C\left(\int_{A(k)} \left|\frac{f}{\omega}\right|^{p'}\omega\right)^{1/p'}\left(\int_{A(k)} |\,\nabla u|^p\omega\,dx\right)^{1/p}\\ &\le C\left[\varepsilon\int_{A(k)} |\,\nabla u|^p\,\omega\,dx + C(\varepsilon)\int_{A(k)} \left|\frac{f}{\omega}\right|^{p'}\omega\,dx\right],\end{aligned}$$

where $C(\varepsilon) = (\varepsilon p)^{-p'/p}/p'$. We can choose $\varepsilon > 0$ so that $C\,\varepsilon = \alpha_1/2$, and there exists a constant C_8 such that

$$\int_{A(k)} |\nabla u|^p\omega\,dx \le C_8\int_{A(k)} \left|\frac{f}{\omega}\right|^{p'}\omega\,dx. \tag{4.7}$$

Using the Sobolev's inequality (Theorem 2.4) and Hölder's inequality with exponents q and q' we obtain (since $q > p'r/(r-1) > p'$)

$$\begin{aligned}&\left(\int_{A(k)} (|\,u| - k)^{p^*}\,\omega\,dx\right)^{p/p^*} = \left(\int_{A(k)} |\psi|^{p^*}\,\omega\,dx\right)^{p/p^*}\\ &\le C\int_{A(k)} |\nabla\psi|^p\omega\,dx = C\int_{A(k)} |\nabla u|^p\omega\,dx\\ &\le C\,C_8\int_{A(k)} \left|\frac{f}{\omega}\right|^{p'}\omega\,dx \le C_9\left(\int_\Omega \left|\frac{f}{\omega}\right|^q\omega\,dx\right)^{p'/q}[\mu(\Omega)]^{1-\frac{p'}{q}}.\end{aligned}$$

Let us now take $l > k > 0$, and observe that $A(l) \subset A(k)$. Then, from the previous inequality, it follows that

$$\begin{aligned}
\mu(A(l))\,(l-k)^{p^*} &= \int_{A(l)} (l-k)^{p^*}\,\omega\,dx \\
&\le \int_{A(l)} (|\,u| - k)^{p^*}\,\omega\,dx \\
&\le \int_{A(k)} (|\,u| - k)^{p^*}\,\omega\,dx \\
&\le C_9 \left(\int_\Omega \left|\frac{f}{\omega}\right|^q \omega\,dx \right)^{\frac{p'\,p^*}{q\,p}} [\mu(A(k))]^{(1-\frac{p'}{q})\frac{p^*}{p}}.
\end{aligned}$$

Hence we obtain

$$\mu(A(l)) \le \frac{C_9 \left(\int_\Omega \left|\frac{f}{\omega}\right|^q \omega\,dx \right)^{\frac{p'\,p^*}{q\,p}}}{(l-k)^{p^*}}\,[\mu(A(k))]^{\left(1-\frac{p'}{q}\right)\frac{p^*}{p}}.$$

Since $\left(1-\frac{p'}{q}\right)\frac{p^*}{p} > 1$, by Lemma 2.5 there exists a constant $C_{10} > 0$ such that $\mu(A(k)) = 0$ for all $k \ge C_{10}$, and using Lemma 2.6 we obtain $|A(k)| = 0$. Therefore if u_m is a solution of problem (P_m) we have $\|u_m\|_{L^\infty(\Omega)} \le C_{10}$ and C_{10} is independent of m.

Step 3. Since $u_m \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$ and $\|u_m\|_{L^\infty(\Omega)} \le C_{10}$, then the sequence $\{u_m\}$ is relative compact in the strong topology of $W_0^{1,p}(\Omega,\omega)$ (by apply the analogous results of [2] and Lemma 2.7). Then, by extracting a subsequence $\{u_{m_k}\}$ which strongly converges in $W_0^{1,p}(\Omega,\omega)$ (there exists $u \in W_0^{1,p}(\Omega,\omega)$ such that $u_{m_k} \to u$ in $W_0^{1,p}(\Omega,\omega)$), we have for all $\varphi \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,u_{m_k},\nabla u_{m_k}).\nabla\varphi\,\omega\,dx + \int_\Omega g(x,u_{m_k},\nabla u_{m_k})\,\varphi\,\omega\,dx \\
&+ \int_\Omega H_{m_k}(x,u_{m_k},\nabla u_{m_k})\,\varphi\,\omega\,dx \\
&\to \int_\Omega \mathcal{A}(x,u,\nabla u).\nabla\varphi\,\omega\,dx + \int_\Omega g(x,u,\nabla u)\,\varphi\,\omega\,dx + \int_\Omega H(x,u,\nabla u)\,\varphi\,\omega\,dx.
\end{aligned}$$

Therefore $u \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$ is the solution of problem (P).

Example. Let $\Omega = \{(x,y) \in \mathbb{R}^2 \;: x^2 + y^2 < 1\}$, and consider the weight function $\omega(x,y) = (x^2+y^2)^{-1/2}$ ($\omega \in A_2$), the functions $\mathcal{A} : \Omega\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$, $g : \Omega\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ and $H : \Omega\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned}
&\mathcal{A}((x,y),\eta,\xi) = h_2(x,y)\,\xi, \\
&g((x,y),\eta,\xi) = \eta\,(\cos^2(xy) + 1), \\
&H((x,y),\eta,\xi) = |\xi|^2 \sin^2(xy).\arctan(\eta),
\end{aligned}$$

where $h_2(x,y) = 2e^{x^2+y^2}$. Let us consider the partial differential operator

$$\begin{aligned} Lu(x,y) &= -\text{div}\Big[\omega(x,y)\mathcal{A}((x,y),u,\nabla u)\Big] + \omega(x,y)g((x,y),u,\nabla u) \\ &+ \omega(x,y)\,H((x,y),\eta,\xi), \end{aligned}$$

and $f(x,y) = (x^2+y^2)^{-1/3q}\cos(1/(x^2+y^2))$, with $q > 2r/(r-1) > 2$. Therefore, by Theorem 1.2, the problem

$$(P)\begin{cases} Lu(x,y) &= f(x,y) \text{ on } \Omega \\ u(x,y) &= 0, \text{ on } \partial\Omega \end{cases}$$

has a solution $u \in W_0^{1,2}(\Omega,\omega) \cap L^\infty(\Omega)$.

References

[1] L. Boccardo, F. Murat and J.P. Puel, *L^∞- Estimate for some nonlinear elliptic partial differential equations and application to an existence result*, SIAM J.Math. Anal., vol. 23, No.2, march 1992, 326-333.

[2] L. Boccardo. F. Murat and J.P.Puel, *Existence of bounded solutions for non linear elliptic unilateral problems*, Ann. Mat. Pura Appl., 152 (1988), 183-196.

[3] A.C.Cavalheiro, *Existence of solutions for some degenerate quasilinear elliptic equations*, Le Matematiche, Vol. LXIII (2008) - Fasc. II, pp. 101-112.

[4] P. Drábek, A. Kufner and V. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators*, Bull. Austral. Math. Soc., vol.58 (1998), 213-221.

[5] E. Fabes, C. Kenig, R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. PDEs 7 (1982), 77-116.

[6] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies 116, (1985).

[7] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monographs, Clarendon Press, (1993).

[8] A. Kufner, *Weighted Sobolev Spaces*, John Wiley & Sons, (1985).

[9] J.Leray and J.L.Lions, *Quelques reséultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France, 93 (1965), 97-107.

[10] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Am. Math. Soc. 165 (1972), 207-226.

[11] G. Stampachhia, *Équations elliptiques du second ordre à coefficients discontinus*, Séminaires de Mathématiques Supérieures, 16, Presses de l'Université de Montréal, Montréal, 1966.

[12] E. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[13] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, San Diego, (1986).

[14] B.O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Lecture Notes in Mathematics, vol. 1736, Springer-Verlag, (2000).

[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol.II/B, Springer-Verlag, (1990).

Albo Carlos Cavalheiro
Department of Mathematics
State University of Londrina, 86057-970 Londrina - PR, Brazil
E-mail address: accava@gmail.com