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MEAN SQUARE CONVERGENT THREE AND FIVE POINTS FINITE DIFFERENCE SCHEME FOR STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we focus on the use of two finite difference schemes in order to approximate the solution of stochastic parabolic partial differential equations problem. The conditions of the mean square convergence of the numerical solution are studied.

1. INTRODUCTION

Stochastic partial differential equations (SPDEs) are defined as partial differential equations involving random inputs. In recent years, some of the main numerical methods for solving stochastic partial differential equations (SPDEs), like finite difference and finite element schemes. This paper is organized as follows. In Section 2, some important preliminaries are discussed. In section 3, the stochastic parabolic partial differential equation is discussed.

2. PRELIMINARIES

[15]. Let us consider the properties of a class of real r.v.'s
X_{11}, X_{12}, ... , X_{21}, X_{22}, ... , X_{nk}, ..., and BbbE(X_{11}^2), $\mathbb{E}(X_{12}^2)$, ... , $\mathbb{E}(X_{21}^2)$, $\mathbb{E}(X_{22}^2)$, ... , $\mathbb{E}(X_{nk}^2)$, ... are finite. In this case, they are called "second order random variables", (2.r.v's).

[15]. A sequence of r.v's $\{X_{nk}, n, k > 0\}$converges in mean square (m.s) to a random variable X if

$$\lim_{n,k\to\infty} \|X_{nk} - X\| = 0 \qquad \text{i.e.} \quad X_{nk} \overset{m.s}{\to} X$$

3. STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATION (SPPDE)

In this section the stochastic finite difference method is used for solving the SPPDE. Consider the following stochastic parabolic partial differential equation in the form:

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$$\begin{aligned} u_t(x,t) &= \beta u_{xx}(x,t) + \sigma u(x,t) d\,W(t) \quad , t \in [0,T], x \in [0,X] && (1) \\ u(x,0) &= u_0(x) \quad , \ x \in [0,X] && (2) \\ u(0,T) &= u(X,T) = 0 && (3) \end{aligned}$$

Where $dW(t)$ is the white noise stochastic process[[**?**]] and β, σ are constants.

3.1. **Stochastic difference scheme(with three points).** For difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis. Notational, u_k^n will be approximate of $u(x,t)$ at point $(k\Delta x, n\Delta t)$ Hence for 1 the difference scheme is:

$$\begin{aligned} u_k^{n+1} &= u_k^n + r\beta(u_{k+1}^n - 2u_k^n + u_{k-1}^n) + \sigma u_k^n (W_k^{n+1} - W_k^n) && (4) \\ u_k^0 &= u_0(x_k) && (5) \\ u_0^n &= u_X^n = 0 && (6) \end{aligned}$$

Where $r = \frac{\Delta t}{\Delta x^2}$, and it can be written in the form:

$$u_k^{n+1} = (1 - 2r\beta)u_k^n + r\beta(u_{k+1}^n - u_{k-1}^n) + \sigma u_k^n (W_k^{n+1} - W_k^n) \quad (7)$$

3.1.1. ***Consistency Of (SFDS)***. [[6],[9],[11]]. A random difference scheme $L_k^n u_k^n = G_k^n$ approximating RPDE $Lv = G$ is consistent in mean square at time $t = (n+1)\Delta t$, if for any continuously differentiable function $\Phi = \Phi(x,t)$, we have in mean square:

$$\mathbb{E}\left|(L\Phi - G)_k^n - (L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n)\right|^2 \to 0$$

as $k \to \infty, n \to \infty, \Delta t \to 0$, $\Delta x \to 0$ and $(k\Delta x, n\Delta t) \to (x,t)$

Theorem 1. *The stochastic difference scheme (7) is consistent in mean square sense*

Proof. Assume that $\Phi(x,t)$ be a smooth function then:

$$L(\Phi)_k^n = \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \beta_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s) ds - \sigma_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s) dW(s)$$

And:

$$\begin{aligned} L_k^n \Phi &= \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - r\beta(\Phi((k+1)\Delta x, n\Delta t) - 2\Phi(k\Delta x, n\Delta t) \\ &\quad + \Phi((k-1)\Delta x, n\Delta t)) - \sigma\Phi(k\Delta x, n\Delta t)(W((n+1)\Delta t) - W(n\Delta t)) \end{aligned}$$

Then we have:

$$\begin{aligned} \mathbb{E}\left|(L\Phi)_k^n - L_k^n \Phi\right|^2 &= \mathbb{E}\Big| -\beta_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s) ds + \frac{\Delta t}{\Delta x^2} \beta(\Phi((k+1)\Delta x, n\Delta t) \\ &\quad -2\Phi(k\Delta x, n\Delta t) + \Phi((k-1)\Delta x, n\Delta t)) - \sigma_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s) dW(s) \\ &\quad + \sigma\Phi(k\Delta x, n\Delta t)(W((n+1)\Delta t) - W(n\Delta t))^2\Big| \\ &= \mathbb{E}\Big| -\beta[_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s) ds - \frac{\Delta t}{\Delta x^2}(\Phi((k+1)\Delta x, n\Delta t) \\ &\quad -2\Phi(k\Delta x, n\Delta t)) + \Phi((k-1)\Delta x, n\Delta t))] - \sigma[_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s) dW(s) \\ &\quad + \Phi(k\Delta x, n\Delta t)(W((n+1)\Delta t) - W(n\Delta t))]^2\Big|. \end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\left|(L\Phi)_k^n - L_k^n\Phi\right|^2 \quad\leq\quad & 2\beta^2\mathbb{E}\left|_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds - \frac{1}{\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t)\right.\\ & -2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t))ds|^2\\ & +2\sigma^2\mathbb{E}\left|_{n\Delta t}^{(n+1)\Delta t}[\Phi(k\Delta x,s)-\Phi(k\Delta x,n\Delta t)]dW(s)\right|^2\end{aligned}$$

And from the inequality[[8]]:

$$\mathbb{E}\left|_{t_0}^{t}[f(s,w)dW_s\right|^{2n} \leq (t-t_0)^{n-1}[n(2n-1)]^n_{t_0}{}^t\mathbb{E}[|f(s,w)|^{2n}]ds$$

And $\Phi(x,t)$ is deterministic function we have:

$$\begin{aligned}\mathbb{E}\left|(L\Phi)_k^n - L_k^n\Phi\right|^2 \quad\leq\quad & 2\beta^2\mathbb{E}\left|_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds - \frac{1}{\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t)\right.\\ & -2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t))ds|^2\\ & +2\sigma^2_{n\Delta t}{}^{(n+1)\Delta t}\left|\Phi(k\Delta x,s)-\Phi(k\Delta x,n\Delta t)\right|^2 ds\end{aligned}$$

at time $t=(n+1)\Delta t, \Delta t\to 0$, $\Delta x\to 0$, and $(k\Delta x,n\Delta t)\to(x,t)$ then:

$$\mathbb{E}\left|(L\Phi)_k^n - (L_k^n\Phi(k\Delta x,n\Delta t))\right|^2\to 0$$

Hence the stochastic difference scheme (7)is consistent in mean square sense. □

3.1.2. ***Stability Of (SFDS)***. [[6],[9],[11]]. A random difference scheme is stable in mean square if there exist some positive constants ε,δ and constants k,b such that:

$$\mathbb{E}\left|u_k^{n+1}\right|^2 \leq k\mathbb{E}^{bt}\sup_k\mathbb{E}\left|u^0\right|^2$$

For all $0\leq t=(n+1)\Delta t$, $0\leq\Delta x\leq\varepsilon$ and $0\leq\Delta t\leq\delta$

Theorem 2. *The stochastic difference scheme (7) is stable in mean square sense.*

Proof. Since

$$u_k^{n+1}=(1-2r\beta)u_k^n+r\beta(u_{k+1}^n-u_{k-1}^n)+\sigma u_k^n(W_k^{n+1}-W_k^n),$$

then

$$\mathbb{E}\left|u_k^{n+1}\right|^2=\mathbb{E}\left|(1-2r\beta)u_k^n+r\beta(u_{k+1}^n-u_{k-1}^n)+\sigma u_k^n(W_k^{n+1}-W_k^n)\right|^2$$

Since $\{W(..,t)-W(..,s)\}$ is normally distributed with mean zero and variance $t-s$, increments of the Wiener process are independent u_k^n and

$$(W_k^{n+1}-W_k^n)=(W(k\Delta x,(n+1)\Delta t)-W(k\Delta x,n\Delta t))$$

we will have:

$$\begin{aligned}\mathbb{E}\left|u_k^{n+1}\right|^2 \quad=\quad & (1-2r\beta)^2\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta\right|\left|1-2r\beta\right|\mathbb{E}\left|u_k^n(u_{k+1}^n+u_{k-1}^n)\right|\\ & +(r\beta)^2\mathbb{E}\left|u_{k+1}^n+u_{k-1}^n\right|^2+\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2\end{aligned}$$

Since

$$\mathbb{E}\left|X+Y\right|^2\leq k(\mathbb{E}\left|X\right|^2+\mathbb{E}\left|Y\right|^2)\text{ , }k=1\text{ at }s\leq 1\text{ and }k=2^{s-1}\text{ at }s\geq 1$$

then we have:

$$\begin{aligned}\mathbb{E}\left|u_k^{n+1}\right|^2 &\leq (1-2r\beta)^2\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta\right|\left|1-2r\beta\right|(\mathbb{E}\left|u_k^nu_{k+1}^n\right|+\mathbb{E}\left|u_k^nu_{k-1}^n\right|)\\ &\quad +2(r\beta)^2(\mathbb{E}\left|u_{k+1}^n\right|^2+\mathbb{E}\left|u_{k-1}^n\right|^2)+\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2\\ &\leq (1-2r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+4\left|r\beta\right|\left|1-2r\beta\right|\sup_k\mathbb{E}\left|u_k^n\right|^2\\ &\quad +4(r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2\end{aligned}$$

Since

$$\left|r\beta\right|\left|1-2r\beta\right|=\left|(r\beta)(1-2r\beta)\right|,$$

then

$$\mathbb{E}\left|u_k^{n+1}\right|^2\leq[\left|(1-2r\beta)\right|+2\left|(r\beta)\right|]^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2$$

Now with

$$0\leq r\beta=\frac{\Delta t}{(\Delta x)^2}\leq\frac{1}{2}$$

hence

$$\mathbb{E}\left|u_k^{n+1}\right|^2\leq(1+\sigma^2\Delta t)\sup_k\mathbb{E}\left|u_k^n\right|^2$$

and by substituting:$\Delta t=\frac{t}{n+1}$ we get

$$\mathbb{E}\left|u_k^{n+1}\right|^2\leq(1+\frac{\sigma^2 t}{n+1})^{n+1}\mathbb{E}\left|u_k^0\right|^2=\mathbb{E}^{\sigma^2 t}\mathbb{E}\left|u_k^0\right|^2$$

Hence: the scheme is conditionally stable with $k=1$ and $b=\sigma^2$ in mean square sense with the condition

$$0\leq r\beta=\frac{\Delta t}{(\Delta x)^2}\leq\frac{1}{2},\ \beta>0$$

□

3.1.3. ***Convergence of (RFDS)***. [[6],[9],[11]] . A random difference scheme $L_k^n u_k^n = G_k^n$approximating RPDE $Lv=G$ is convergent in mean square at time $t=(n+1)\Delta t$, if

$$\mathbb{E}\left|u_k^n-u\right|^2\to 0$$

as $\Delta t\to 0$, $\Delta x\to 0$ and $(k\Delta x,n\Delta t)\to(x,t)$ or as $n\to 0$, $k\to 0$ and $(k\Delta x,n\Delta t)\to(x,t)$

Theorem 3. ***(A Stochastic Version of Lax-Richtmyer)*** [11]

A random difference scheme$L_k^n u_k^n=G_k^n$ approximating SPDE $Lv=G$ is convergent in mean square at time $t=(n+1)\Delta t$, if it is consistent and stable.

Theorem 4. *The random difference scheme (7) is convergent in mean square sense*

Proof.

$$\mathbb{E}\left|u_k^n-u\right|^2=\mathbb{E}\left|(L_k^n)^{-1}(L_k^n u_k^n-L_k^n u)\right|^2$$

Since the scheme is consistent then we have:

$$L_k^n u_k^n\overset{m.s}{\to}Lu$$

Then we obtain

$$\mathbb{E}\left|(L_k^n)^{-1}(L_k^n u_k^n-L_k^n u)\right|^2\to 0$$

as $\Delta t \to 0$, $\Delta x \to 0$, and $(k\Delta x, n\Delta t) \to (x,t)$. And since the scheme is stable then $(L_k^n)^{-1}$is bounded, Hence $\mathbb{E}\left|u_k^n - u\right|^2 \to 0$ as $\Delta t \to 0$, $\Delta x \to 0$. Then the random difference scheme (7) is convergent in mean square sense. □

3.2. **Stochastic difference scheme (with five points).** For difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis. Notational, u_k^n will be approximate of $u(x,t)$ at point $(k\Delta x, n\Delta t)$ Hence for 1 the difference scheme is:

$$u_k^{n+1} = u_k^n + r\beta(\frac{-1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n) + \sigma u_k^n(W_k^{n+1} - W_k^n) \tag{8}$$

$$u_k^0 = u_0(x_k) \tag{9}$$

$$u_0^n = u_X^n = 0 \tag{10}$$

Where $r = \frac{\Delta t}{\Delta x^2}$, and it can be written in the form:

$$u_k^{n+1} = (1 - \frac{5}{2}r\beta)u_k^n + r\beta(\frac{-1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n) + \sigma u_k^n(W_k^{n+1} - W_k^n) \tag{11}$$

Now Possible to prove the consistency, stability and the convergence of SFDS (11)) according to definitions (1, 2, 3) as follows:

3.2.1. ***Consistency Of (SFDS)***.

Theorem 5. *The random difference scheme (11) is consistent in mean square sense*

3.2.2. ***Stability Of (RFDS)***. In this sectin we will discuss the stability of (11).

Theorem 6. *The stochastic difference scheme (11) is stable in mean square sense.*

Proof. Since:

$$u_k^{n+1} = (1 - \frac{5}{2}r\beta)u_k^n + r\beta(\frac{-1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n) + \sigma u_k^n(W_k^{n+1} - W_k^n)$$

then:

$$\mathbb{E}\left|u_k^{n+1}\right|^2 = \mathbb{E}\left|(1 - \frac{5}{2}r\beta)u_k^n + r\beta(\frac{-1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n) + \sigma u_k^n(W_k^{n+1} - W_k^n)\right|^2$$

Since: $\{W(..,t) - W(..,s)\}$ is normally distributed with mean zero and variance $t-s$, increments of the Wiener process are independent of u_k^n and$(W_k^{n+1} - W_k^n) =$

$(W(k\Delta x,(n+1)\Delta t)-W(k\Delta x,n\Delta t))$ and: we will have::

$$\begin{aligned}
\mathbb{E}\left|u_k^{n+1}\right|^2 &= [(1-\frac{5}{2}r\beta)^2\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta\right|\left|1-\frac{5}{2}r\beta\right|(\frac{1}{12}\mathbb{E}\left|u_k^n(u_{k-2}^n+u_{k+2}^n)\right| \\
&+\frac{4}{3}\mathbb{E}\left|u_k^n(u_{k-1}^n+u_{k+1}^n)\right|)+\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2 \\
&+(r\beta)^2\mathbb{E}\left|-\frac{1}{12}(u_{k-2}^n+u_{k+2}^n)+\frac{4}{3}(u_{k-1}^n+u_{k+1}^n)\right|^2 \\
&= (1-\frac{5}{2}r\beta)^2\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta\right|\left|1-\frac{5}{2}r\beta\right|(\frac{1}{12}\mathbb{E}\left|u_k^n(u_{k-2}^n+u_{k+2}^n)\right| \\
&+\frac{4}{3}\mathbb{E}\left|u_k^n(u_{k-1}^n+u_{k+1}^n)\right|)+(r\beta)^2[\frac{1}{144}\mathbb{E}\left|(u_{k-2}^n+u_{k+2}^n)\right|^2+\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{16}{9}\mathbb{E}\left|(u_{k-1}^n+u_{k+1}^n)\right|^2+\frac{8}{36}\mathbb{E}\left|u_{k-2}^n(u_{k-1}^n+u_{k+1}^n)+u_{k+2}^n(u_{k-1}^n+u_{k+1}^n)\right|]
\end{aligned}$$

Since:$\left|r\beta\right|\left|1-\frac{5}{2}r\beta\right|=\left|r\beta(\frac{5}{2}r\beta-1)\right|$ then we get:

$$\begin{aligned}
\mathbb{E}\left|u_k^{n+1}\right|^2 &= (1-\frac{5}{2}r\beta)^2\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta(\frac{5}{2}r\beta-1)\right|(\frac{1}{12}\mathbb{E}\left|u_k^n(u_{k-2}^n+u_{k+2}^n)\right| \\
&+\frac{4}{3}\mathbb{E}\left|u_k^n(u_{k-1}^n+u_{k+1}^n)\right|)+(r\beta)^2[\frac{1}{144}\mathbb{E}\left|(u_{k-2}^n+u_{k+2}^n)\right|^2+\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{16}{9}\mathbb{E}\left|(u_{k-1}^n+u_{k+1}^n)\right|^2+\frac{8}{36}\mathbb{E}\left|u_{k-2}^n(u_{k-1}^n+u_{k+1}^n)+u_{k+2}^n(u_{k-1}^n+u_{k+1}^n)\right|]
\end{aligned}$$

Scinc:$\mathbb{E}\left|X+Y\right|^2\leq k(\mathbb{E}\left|X\right|^2+\mathbb{E}\left|Y\right|^2)$, $k=1$ at $s\leq 1$ and $k=2^{s-1}$ at $s\geq 1$ then we have:

$$\begin{aligned}
\mathbb{E}\left|u_k^{n+1}\right|^2 &= (1-\frac{5}{2}r\beta)^2\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta(\frac{5}{2}r\beta-1)\right|(\frac{1}{12}[\mathbb{E}\left|u_k^n u_{k+2}^n\right| \\
&+\mathbb{E}\left|u_k^n u_{k-2}^n\right|]+\frac{4}{3}[\mathbb{E}\left|u_k^n u_{k+1}^n\right|+\mathbb{E}\left|u_k^n u_{k-1}^n\right|] \\
&+(r\beta)^2[\frac{2}{144}(\mathbb{E}\left|u_{k-2}^n\right|^2+\mathbb{E}\left|u_{k+2}^n\right|^2)+\mathbb{E}\left|u_{k-2}^n u_{k+1}^n\right| \\
&+\frac{32}{9}(\mathbb{E}\left|u_{k+1}^n\right|^2+\mathbb{E}\left|u_{k-1}^n\right|^2)+\frac{8}{36}[\mathbb{E}\left|u_{k-2}^n u_{k+1}^n\right| \\
&+\mathbb{E}\left|u_{k+2}^n u_{k+1}^n\right|+\mathbb{E}\left|u_{k+2}^n u_{k-1}^n\right|]]+\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2 \\
&\leq (1-\frac{5}{2}r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta(\frac{5}{2}r\beta-1)\right|(\frac{2}{12}\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{8}{3}\sup_k\mathbb{E}\left|u_k^n\right|^2)+(r\beta)^2[\frac{4}{144}\sup_k\mathbb{E}\left|u_k^n\right|^2+\frac{64}{9}\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{32}{36}\sup_k\mathbb{E}\left|u_k^n\right|^2]+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&\leq (1-\frac{5}{2}r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\frac{34}{6}\left|r\beta(\frac{5}{2}r\beta-1)\right|\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{289}{36}(r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&= [\left|(1-\frac{5}{2}r\beta)\right|+\frac{17}{6}\left|r\beta\right|]^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left|u_k^{n+1}\right|^2 &= (1-\frac{5}{2}r\beta)^2\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta(\frac{5}{2}r\beta-1)\right|(\frac{1}{12}[\mathbb{E}\left|u_k^n u_{k+2}^n\right| \\
&+\mathbb{E}\left|u_k^n u_{k-2}^n\right|]+\frac{4}{3}[\mathbb{E}\left|u_k^n u_{k+1}^n\right|+\mathbb{E}\left|u_k^n u_{k-1}^n\right|]+\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2 \\
&+(r\beta)^2[\frac{2}{144}(\mathbb{E}\left|u_{k-2}^n\right|^2+\mathbb{E}\left|u_{k+2}^n\right|^2)+\frac{32}{9}(\mathbb{E}\left|u_{k+1}^n\right|^2+\mathbb{E}\left|u_{k-1}^n\right|^2) \\
&+\frac{8}{36}[\mathbb{E}\left|u_{k-2}^n u_{k+1}^n\right|+\mathbb{E}\left|u_{k-2}^n u_{k+1}^n\right|+\mathbb{E}\left|u_{k+2}^n u_{k+1}^n\right|+\mathbb{E}\left|u_{k+2}^n u_{k-1}^n\right|]] \\
&\leq (1-\frac{5}{2}r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+2\left|r\beta(\frac{5}{2}r\beta-1)\right|(\frac{2}{12}\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{8}{3}\sup_k\mathbb{E}\left|u_k^n\right|^2)+(r\beta)^2[\frac{4}{144}\sup_k\mathbb{E}\left|u_k^n\right|^2+\frac{64}{9}\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{32}{36}\sup_k\mathbb{E}\left|u_k^n\right|^2]+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&\leq (1-\frac{5}{2}r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\frac{34}{6}\left|r\beta(\frac{5}{2}r\beta-1)\right|\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&+\frac{289}{36}(r\beta)^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&= [\left|(1-\frac{5}{2}r\beta)\right|+\frac{17}{6}\left|r\beta\right|]^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2
\end{aligned}$$

Now with:$0\leq r\beta=\frac{\Delta t}{(\Delta x)^2}\leq\frac{2}{5}$ then: $\left|1-\frac{5}{2}(r\beta)\right|=1-\frac{5}{2}\left|(r\beta)\right|$, hence:

$$\begin{aligned}
\mathbb{E}\left|u_k^{n+1}\right|^2 &\leq [1+\frac{1}{3}\left|r\beta\right|]^2\sup_k\mathbb{E}\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k\mathbb{E}\left|u_k^n\right|^2 \\
&= [1+\frac{1}{9}\left|r\beta\right|^2+\frac{2}{3}\left|r\beta\right|+\sigma^2\Delta t]\sup_k\mathbb{E}\left|u_k^n\right|^2,
\end{aligned}$$

it is enough to select λ such that:$1+\frac{1}{9}\left|r\beta\right|^2+\frac{2}{3}\left|r\beta\right|+\sigma^2\Delta t\leq\lambda^2\Delta t$ for all k then we have:and by substituting:$\Delta t=\frac{t}{n+1}$ we get:

$$\mathbb{E}\left|u_k^{n+1}\right|^2\leq(1+\frac{\lambda^2 t}{n+1})^{n+1}\mathbb{E}\left|u_k^0\right|^2=\mathbb{E}^{\lambda^2 t}\mathbb{E}\left|u_k^0\right|^2$$

Hence: the scheme is conditionally stable with $k=1$ and $b=\lambda^2$ in mean square sense with the condition : $0\leq r\beta=\frac{\Delta t}{(\Delta x)^2}\leq\frac{5}{2}$, $\beta>0$. □

3.2.3. ***Convergence of (RFDS)***.

Theorem 7. *The random difference scheme (11) is convergent in mean square sense*

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