

INTEGRALS OF HIGHER BINARY OPTIONS AND DEFAULTABLE BONDS WITH DISCRETE DEFAULT INFORMATION

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ABSTRACT. In this article, we study the problem of pricing defaultable bond with discrete default intensity and barrier under constant risk free short rate using higher order binary options and their integrals. In our credit risk model, the risk free short rate is a constant and the default event occurs in an expected manner when the firm value reaches a given default barrier at predetermined discrete announcing dates or in an unexpected manner at the first jump time of a Poisson process with given default intensity given by a step function of time variable, respectively. We consider both endogenous and exogenous default recovery. Our pricing problem is derived to a solving problem of inhomogeneous or homogeneous Black-Scholes PDEs with different coefficients and terminal value of binary type in every subinterval between the two adjacent announcing dates. In order to deal with the difference of coefficients in subintervals we use a relation between prices of higher order binaries with different coefficients. In our model, due to the inhomogeneous term related to endogenous recovery, our pricing formulae are represented by not only the prices of higher binary options but also the integrals of them. So we consider a special binary option called integral of i -th binary or nothing and then we obtain the pricing formulae of our defaultable corporate bond by using the pricing formulae of higher binary options and integrals of them.

1. INTRODUCTION

The study on defaultable corporate bond and credit risk is now one of the most promising areas of cutting edge in financial mathematics. There are two main approaches to modeling credit risk and pricing defaultable corporate bonds; one is the *structural approach* and the other one is the *reduced form approach*. In the structural method, we think that the default event occurs when the firm value is not enough to repay debt, that is, the firm value reaches a certain lower threshold (*default barrier*) from the above. Such a default can be expected and thus we call it *expected default*. In the reduced-form approach, the default is treated as an unpredictable event governed by a default intensity process. In this case, the

2010 *Mathematics Subject Classification.* 35C15, 35Q91, 91G20, 91G40, 91G50, 91G80.

Key words and phrases. defaultable bond, discrete default intensity, discrete default barrier, structural approach, reduced form approach, expected default, unexpected default, endogenous, exogenous, default recovery, higher order binary options, integral of binary or nothing.

Submitted Oct. 3, 2013; last revised Nov. 22, 2013.

default event can occur without any correlation with the firm value and such a default is called *unexpected default*. In the reduced-form approach, if the default probability in time interval $[t, t + \Delta t]$ is $\lambda \Delta t$, then λ is called *default intensity* [9, 12, 13, 15].

The two approaches have got their own advantages and shortcomings ([3, 12]) and therefore the use of unified models of structural approach and reduced-form approach is a trend. (See [3, 4, 6, 7, 9, 12, 13, 14].) Cathcart et al [6] studied a pricing of corporate bonds in the case when the default intensity is a linear function of the interest rate and gave semi-analytical pricing formulae. Cathcart et al [7] studied a valuation model in the case when the default intensity (hazard rate) is a linear function of the state variable and the interest rate. Realdon [14] studied a pricing of corporate bonds in the case with constant default intensity and gave pricing formulae of the bond using PDE method. Some authors studied the pricing model of defaultable bonds in which the default intensity is given as a stochastic process [3, 4, 12]. In [12], the authors provided analytical pricing formula of corporate defaultable bond with both expected and unexpected default in the case when stochastic default intensity follows Wilmott model where drift and volatility are linear of state variables [15]. Bi et al [4] got the similar result with [12] in the case when stochastic default intensity follows CIR-like model. Ballestra et al [3] proposed a model to price defaultable bonds where default intensity follows Vasicek-like model or CIR-like model coupled with the process of the firm's asset value and provided a closed-form approximate solution to their model. In [3, 4, 6, 7, 12, 14] expected default barrier is given in the whole lifetime of the bond.

On the other hand, in [9, 13] the author studied the pricing problem for defaultable corporate bond under the assumption that we only know the firm value and the default barrier at 2 fixed discrete announcing dates, we don't know about any information of the firm value in another time and the default intensity between the adjoined two announcing dates is a constant determined by its announced firm value at the former announcing date. In that case we encountered a *piecewise constant default intensity*. Generally, *credit rankings* estimated by credit estimation organizations are discretely announced (once in three months, once in a half year and etc.), which reflects the firms default intensity. The computational error in [13] is corrected in [9]. The approach of [9, 13] is a kind of study of defaultable bond under *insufficient information* about the firm and it is interesting to note that Agliardi et al [2] studied bond pricing problem under *imprecise information* with the technique of fuzzy mathematics. The approach of [9, 13] can be seen as a *unified model* of structural model and reduced form model. Agliardi [1] studied a *structural model* for defaultable bond with several (discrete) coupon dates where the default can occur only when the firm value is not large enough to pay its debt and coupon in those *discrete coupon dates*. In that case they encounter *discrete default barriers*. Generally, default barrier reflects the firms debt and the *investor outside of the firm* only can discretely know information of the firm including debt.

Speaking on default recovery, most of authors including [2, 3, 4, 6, 9, 12, 13] have studied the case of *exogenous* default recovery which is independent on firm value whereas [1] have studied the case of *endogenous* recovery which is related to firm value, and [14] studied both cases of exogenous and endogenous recovery.

Here we study the problem of pricing defaultable bond with discrete default intensity and barrier under constant risk free short rate using higher order binary

options and their integrals. In our credit risk model, the default event occurs in an expected manner when the firm value reaches a certain lower threshold - the default barrier at predetermined discrete announcing dates or in an unexpected manner at the first jump time of a Poisson process with given default intensity given by a step function of time variable, respectively. We consider both *endogenous* and *exogenous* default recovery. Our pricing problem is derived to a solving problem of *inhomogeneous* or *homogeneous Black-Scholes PDEs* with *different coefficients* and terminal value of binary type in every subinterval between the two adjacent announcing dates. In order to deal with the difference of coefficients in subintervals we use a *relation* between prices of higher order binaries with different coefficients. In our model, due to the inhomogeneous term related to endogenous recovery, our pricing formulae are represented by not only the prices of higher binary options but also the integrals of them. So we consider a *special binary* option called *integral of i-th binary* or *nothing* and then we obtain the pricing formulae of our defaultable corporate bond by using the pricing formulae of higher binary options and integrals of them.

Our approach to model credit risk is similar with the one of [13, 9]. One of the different points of our model from [9] is that we here consider arbitrary number of announcing dates but [9] consider only 2 announcing dates. Another different point from [9] is that we use constant risk free rate, the purpose of which is to show the applicability of higher order binaries to the pricing of defaultable bonds in the simplest way. Unlikely in [9] we here consider discrete default intensity independent on firm value and it can be seen incompatible with reality but we think our analytical pricing formulae can help the further study on the more realistic situation with discrete default intensity dependent on firm value.

The remainder of the article is organized as follows. In section 2 we give some preliminary knowledge on prices of higher order binary options and their integral on the last expiry date. In section 3 we set our problem for corporate defaultable bonds, provide the pricing formulae in both cases of endogenous and exogenous default recovery and analyze the credit spread. In section 4 we derive the pricing formulae using and higher order binary options and their integral.

2. PRELIMINARIES AND NOTES ON BINARY OPTIONS AND THEIR INTEGRALS

First, we introduce the concept of higher order bond and asset binaries with risk free rate r , dividend rate q and volatility σ and their pricing formulae [5, 10, 11].

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V}{\partial x^2} + (r - q)x \frac{\partial V}{\partial x} - rV = 0, \quad 0 \leq t < T, \quad 0 < x < \infty, \quad (1)$$

$$V(x, T) = x \cdot 1(sx > s\xi), \quad (2)$$

$$V(x, T) = 1(sx > s\xi). \quad (3)$$

The solution to the problem (1) and (2) is called the *asset-or-nothing* binaries (or *asset* binaries) and denoted by $A_\xi^s(x, t; T)$. The solution to the problem (1) and (3) is called the *cash-or-nothing* binaries (or *bond* binaries) and denoted by $B_\xi^s(x, t; T)$. Asset binary and bond binary are called the *first order binary* options. If necessary, we will denote by $A_\xi^s(x, t; T; r, q, \sigma)$ or $B_\xi^s(x, t; T; r, q, \sigma)$ where the coefficients r , q and σ of Black-Scholes equation (1) are explicitly included in the notation.

Let assume that $0 < T_0 < T_1 < \dots < T_{n-1}$ and the $(n-1)$ -th order (asset or bond) binary options $A_{\xi_1 \dots \xi_{n-1}}^{s_1 \dots s_{n-1}}(x, t; T_1, \dots, T_{n-1})$ and $B_{\xi_1 \dots \xi_{n-1}}^{s_1 \dots s_{n-1}}(x, t; T_1, \dots, T_{n-1})$

are already defined. Let

$$V(x, T_0) = A_{\xi_1 \dots \xi_{n-1}}^{s_1 \dots s_{n-1}}(x, T_0; T_1, \dots, T_{n-1}) \cdot 1(s_0 x > s_0 \xi_0), \tag{4}$$

$$V(x, T_0) = B_{\xi_1 \dots \xi_{n-1}}^{s_1 \dots s_{n-1}}(x, T_0; T_1, \dots, T_{n-1}) \cdot 1(s_0 x > s_0 \xi_0). \tag{5}$$

The solution to the problem (1) and (4) is called the *n-th order asset binaries* and denoted by $A_{\xi_0 \xi_1 \dots \xi_{n-1}}^{s_0 s_1 \dots s_{n-1}}(x, t; T_0, T_1, \dots, T_{n-1})$. The solution to the problem (1) and (5) is called the *n-th order bond binaries* and denoted by $B_{\xi_0 \xi_1 \dots \xi_{n-1}}^{s_0 s_1 \dots s_{n-1}}(x, t; T_0, T_1, \dots, T_{n-1})$.

Lemma 1. (The pricing formulae of higher order binary options) [5, 10, 11] *The prices of higher order bond and asset binaries with risk free rate r, dividend rate q and volatility σ are as follows.*

$$\begin{aligned} A_{\xi}^s(x, t; T; r, q, \sigma) &= x e^{-q(T-t)} N(sd^+), \\ B_{\xi}^s(x, t; T; r, q, \sigma) &= e^{-r(T-t)} N_1(sd^-), s = + \text{ or } -. \end{aligned} \tag{6}$$

Here

$$\begin{aligned} N_1(x) &= (\sqrt{2\pi})^{-1} \int_{-\infty}^x \exp(-y^2/2) dy, \\ d^{\pm} &= (\sigma\sqrt{T-t})^{-1} [\ln(x/K) + (r - q \pm \sigma^2/2)(T - t)]. \end{aligned}$$

$$\begin{aligned} A_{K_1 K_2}^{s_1 s_2}(x, t; T_1, T_2; r, q, \sigma) &= x e^{-q(T_2-t)} N_2(s_1 d_1^+, s_2 d_2^+; s_1 s_2 \rho), \\ B_{K_1 K_2}^{s_1 s_2}(x, t; T_1, T_2; r, q, \sigma) &= e^{-r(T_2-t)} N_2(s_1 d_1^-, s_2 d_2^-; s_1 s_2 \rho), s_1, s_2 = + \text{ or } -. \end{aligned} \tag{7}$$

Here

$$\begin{aligned} N_2(a, b; \rho) &= \int_{-\infty}^a \int_{-\infty}^b (2\pi\sqrt{1-\rho^2})^{-1} e^{-\frac{y^2-2\rho yz+z^2}{2(1-\rho^2)}} dy dz, \\ d_i^{\pm} &= (\sigma\sqrt{T_i-t})^{-1} [\ln(x/K_i) + (r - q \pm \sigma^2/2)(T_i - t)], i = 1, 2, \\ \rho &= \sqrt{(T_1-t)/(T_2-t)}. \end{aligned}$$

If $m > 2$ and $s_i = +$ or $-$, $i = 1, \dots, m$, then we have

$$\begin{aligned} A_{K_1 \dots K_m}^{s_1 \dots s_m}(x, t; T_1, \dots, T_m; r, q, \sigma) &= x e^{-q(T_m-t)} N_m(s_1 d_1^+, \dots, s_m d_m^+; A_{s_1 \dots s_m}), \\ B_{K_1 \dots K_m}^{s_1 \dots s_m}(x, t; T_1, \dots, T_m; r, q, \sigma) &= e^{-r(T_m-t)} N_m(s_1 d_1^-, \dots, s_m d_m^-; A_{s_1 \dots s_m}). \end{aligned} \tag{8}$$

Here

$$\begin{aligned} N_m(a_1, \dots, a_m; A) &= \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_m} (\sqrt{2\pi})^{-m} \sqrt{\det A} \exp\left(-\frac{1}{2} y^T A y\right) dy, \\ d_i^{\pm} &= (\sigma\sqrt{T_i-t})^{-1} [\ln(x/K_i) + (r - q \pm \sigma^2/2)(T_i - t)], i = 1, \dots, m, \\ A_{s_1 \dots s_m} &= (s_i s_j a_{ij})_{i,j=1}^m, \quad y^T = (y_1, \dots, y_m), \end{aligned} \tag{9}$$

and the matrix $(a_{i,j})_{i,j=1}^m$ is given as follows:

$$\begin{aligned} a_{11} &= (T_2 - t)/(T_2 - T_1), \quad a_{mm} = (T_m - t)/(T_m - T_{m-1}), \\ a_{ii} &= (T_i - t)/(T_i - T_{i-1}) + (T_i - t)/(T_{i+1} - T_i), \quad 2 \leq i \leq m - 1, \\ a_{i,i+1} &= a_{i+1,i} = -\sqrt{(T_i - t)(T_{i+1} - t)/(T_{i+1} - T_i)}, \quad 1 \leq i \leq m - 1, \\ a_{ij} &= 0 \text{ for another } i, j = 1, \dots, m. \end{aligned} \tag{10}$$

Note that $N_2(a, b; \rho)$ is the cumulative distribution function of bivariate normal distribution with a mean vector $[0, 0]$ and a covariance matrix $[1, \rho; \rho, 1]$ (symbols in **Matlab**), and $N_m(a_1, \dots, a_m; A)$ is the cumulative distribution function of m -variate normal distribution with **zero** mean vector and a covariance matrix $A^{-1} = (r_{ij})_{i,j=1}^m$ where $r_{ij} = \sqrt{(T_i - t)/(T_j - t)}$, $r_{ji} = r_{ij}, i \leq j$. Such special functions can easily be calculated by standard functions supplied in software for mathematical calculation (for example, **Matlab**). Note that $(A_{s_1 \dots s_m})^{-1} = (s_i s_j r_{ij})_{i,j=1}^m$.

Second, we consider a relation between prices of higher order binaries with different risk free rates and dividend rates. From the formulae (6), (7) and (8), we can easily know that the following relations between prices of higher order binaries with different risk free rates and dividend rates hold:

$$\begin{aligned} F_{K_1 \dots K_m}^{s_1 \dots s_m}(x, t; T_1, \dots, T_m; r_1, r_1 + b, \sigma) &= \\ &= e^{-(r_1 - r_2)(T_m - t)} F_{K_1 \dots K_m}^{s_1 \dots s_m}(x, t; T_1, \dots, T_m; r_2, r_2 + b, \sigma). \end{aligned} \tag{11}$$

Here $F = A$ or $F = B$.

Next, we will discuss integrals of the prices of higher order binary options on the last expiry date variable. Let consider (1) with the following two terminal conditions:

$$V(x, T) = f(x, \tau), \tag{12}$$

$$V(x, T) = F(x) := \int_C^D f(x, \tau) d\tau. \tag{13}$$

Lemma 2. Assume that there exist non negative constants M and α such that $|f(x, \tau)| \leq M \cdot x^{\alpha \ln x}$, $x > 0$ and $f(x, \tau)$ is a continuous function of $\tau \in [C, D]$. Then the solution $V_F(x, t)$ to the problem (1) and (13) is given by the integral of the solution $V_f(x, t; \tau)$ to the problem (1) and (12):

$$V_F(x, t) = \int_C^D V_f(x, t; \tau) d\tau. \tag{14}$$

Proof: If we use the proposition 1 at page 249 in [10] and the continuity of f on τ , we can easily get (14).(QED)

Now let consider a special binary option called integral of i -th binary or nothing.

Corollary. Let $g(\tau)$ be a continuous function of $\tau \in [T_{i-1}, T]$ and

$$V(x, T_0) = 1(s_0 x > s_0 K_0) \int_{T_{i-1}}^T g(\tau) F_{K_1 \dots K_{i-1} K_i}^{s_1 \dots s_{i-1} s_i}(x, T_0; T_1, \dots, T_{i-1}, \tau) d\tau. \tag{15}$$

Then the solution of (1) and (15) is given as follows:

$$V(x, t) = \int_{T_{i-1}}^T g(\tau) F_{K_0 K_1 \dots K_{i-1} K_i}^{s_0 s_1 \dots s_{i-1} s_i}(x, t; T_0, T_1, \dots, T_{i-1}, \tau) d\tau, t < T_0. \tag{16}$$

Here $F = A$ or $F = B$.

Proof: We will prove only for bond binary in the case when $i = 1$. The proofs for other cases are the same. By the proposition 1 at page 249 in [10], the solution to (1) with

$$V(x, T_0) = 1(s_0 x > s_0 K_0) \int_{T_0}^T g(\tau) B_{K_1}^{s_1}(x, T_0; \tau) d\tau$$

is given as follows:

$$\begin{aligned}
 U(x, t) &= \\
 &= \frac{e^{-r(T_0-t)}}{\sigma\sqrt{2\pi(T_0-t)}} \int_0^\infty \frac{1}{z} e^{-\frac{[\ln \frac{x}{z} + (r-q-\frac{\sigma^2}{2})(T_0-t)]^2}{2\sigma^2(T_0-t)}} 1(s_0z > s_0K_0) \int_{T_0}^T g(\tau) B_{K_1}^{s_1}(z, T_0; \tau) d\tau dz \\
 &= \int_{T_0}^T g(\tau) \frac{e^{-r(T_0-t)}}{\sigma\sqrt{2\pi(T_0-t)}} \int_0^\infty \frac{1}{z} e^{-\frac{[\ln \frac{x}{z} + (r-q-\frac{\sigma^2}{2})(T_0-t)]^2}{2\sigma^2(T_0-t)}} B_{K_1}^{s_1}(z, T_0; \tau) 1(s_0z > s_0K_0) dz d\tau \\
 &= \int_{T_0}^T g(\tau) B_{K_0K_1}^{s_0s_1}(x, t; T_0, \tau) d\tau. \text{ (QED)}
 \end{aligned}$$

3. THE PROBLEM OF DEFAULTABLE BONDS AND THE PRICING FORMULAE

3.1. The Problem with Endogenous Recovery. Let Assume the followings:

- 1) Short rate r is a constant.
- 2) $0 = t_0 < t_1 < \dots < t_{N_1} < t_N = T$ are announcing dates and T is the maturity of our corporate bond with face value 1 (unit of currency). For every $i = 0, \dots, N-1$ the unexpected default probability in the interval $[t, t + dt] \cap (t_i, t_{i+1})$ is $\lambda_i dt$. Here the *default intensity* λ_i is a constant.
- 3) The firm value $V(t)$ follows a geometric Brownian motion

$$dV(t) = (r - b)dtV(t) + s_V \cdot V(t)dW(t) \tag{17}$$

under the risk neutral martingale measure. Here the volatility s_V of the firm value is a constant and the firm continuously pays out dividend in rate b (constant) for a unit of firm value. Like in [12], the firm value V_t is assumed to consist of m shares of stock S and n sheets of corporate bonds C_t :

$$V_t = mS_t + nC_t. \tag{18}$$

- 4) The expected default barrier is only given at time t_i and the expected default event occurs when

$$V(t_i) \leq K_i e^{-r(T-t_i)}, \quad i = 1, \dots, N. \tag{19}$$

Here K_i is a constant reflecting the quantity of debt and $e^{-r(T-t_i)}$ is default free zero coupon bond price.

- 5) The default recovery R_d is given as the form of endogenous face value

$$R_d = \min\{e^{-r(T-t)}, R \cdot V/n\}. \tag{20}$$

Here *recovery rate* $0 \leq R \leq 1$ is a constant.

- 6) In the subinterval (t_i, t_{i+1}) , the price of our corporate bond is given by a sufficiently smooth function $C_i(V, t)$ ($i = 0, \dots, N - 1$).

Problem: Find the representation of the price function $C_i(V, t)$ ($i = 0, \dots, N - 1$) under the above assumptions.

The Pricing Model. According to [15], under the above assumptions the price C of defaultable bond with a constant default intensity λ and default recovery R_d satisfies the following PDE:

$$\frac{\partial C}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 C}{\partial V^2} + (r - b)V \frac{\partial C}{\partial V} - (r + \lambda)C + \lambda R_d = 0.$$

Therefore if we let $C_N(V, t) \equiv 1$, then the *price model* of our bond is given as follows:

$$\begin{aligned} \frac{\partial C_i}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 C_i}{\partial V^2} + (r - b)V \frac{\partial C_i}{\partial V} - (r + \lambda_i)C_i + \lambda_i \min\{e^{-r(T-t)}, \frac{RV}{n}\} &= 0, \quad t_i < t < t_{i+1}, \\ C_i(t_{i+1}) = C_{i+1}(t_{i+1})1(V > K_{i+1}e^{-r(T-t_{i+1})}) + \min\{e^{-r(T-t_{i+1})}, \frac{RV}{n}\} &1(V \leq K_{i+1}e^{-r(T-t_{i+1})}). \end{aligned} \tag{21}$$

Here $i = 0, 1, \dots, N - 1$.

The Pricing Formulae. Under the assumptions 1)–6), we have the following pricing formulae:

Theorem 1. (endogenous recovery) i) *Assume that $K_i \leq n/R, i = 1, \dots, N$. Under the assumptions 1)–6), the price of our bond, that is, the solution of (21) is represented as follows:*

$$C_i(V, t) = e^{-r(T-t)} u_i(V/e^{-r(T-t)}, t), t_i \leq t < t_{i+1}, \quad i = 0, \dots, N - 1. \tag{22}$$

Here

$$\begin{aligned} u_i(x, t) = & e^{-\lambda_i(t_{i+1}-t)} \left\{ e^{-\sum_{k=i+1}^{N-1} \lambda_k(t_{k+1}-t_k)} B_{K_{i+1} \dots K_N}^+ \dots^+ (x, t; t_{i+1}, \dots, t_N) + \right. \\ & + \frac{R}{n} \sum_{m=i}^{N-1} e^{-\sum_{k=i+1}^m \lambda_k(t_{k+1}-t_k)} A_{K_{i+1} \dots K_m K_{m+1}}^+ \dots^+ \bar{} (x, t; t_{i+1}, \dots, t_m, t_{m+1}) \\ & + \sum_{m=i+1}^{N-1} \lambda_m e^{-\sum_{k=i+1}^{m-1} \lambda_k(t_{k+1}-t_k)} \int_{t_m}^{t_{m+1}} e^{-\lambda_m(\tau-t_m)} \left[B_{K_{i+1} \dots K_m \frac{n}{R}}^+ \dots^+ (x, t; t_{i+1}, \dots, t_m, \tau) \right. \\ & \quad \left. + \frac{R}{n} A_{K_{i+1} \dots K_m \frac{n}{R}}^+ \dots^+ \bar{} (x, t; t_{i+1}, \dots, t_m, \tau) \right] d\tau \left. \right\} \\ & + \lambda_i \int_t^{t_{i+1}} e^{-\lambda_i(\tau-t)} \left[B_{\frac{n}{R}}^+ (x, t; \tau; 0, b, s_V) + \frac{R}{n} A_{\frac{n}{R}}^- (x, t; \tau; 0, b, s_V) \right] d\tau. \end{aligned} \tag{23}$$

ii) *Assume that $K_i > n/R, i = 1, \dots, N$. Under the assumptions 1)–6), the price of our bond, that is, the solution of (21) is represented by (22) with the following*

$u_i(x, t)$:

$$\begin{aligned}
 u_i(x, t) = & e^{-\lambda_i(t_{i+1}-t)} \left\{ \sum_{m=i}^{N-1} e^{-\sum_{k=i+1}^m \lambda_k(t_{k+1}-t_k)} \left[B_{K_{i+1} \dots K_m \frac{n}{R}}^+(x, t; t_{i+1}, \dots, t_m, t_{m+1}) \right. \right. \\
 & \left. \left. + \frac{R}{n} A_{K_{i+1} \dots K_m \frac{n}{R}}^+(x, t; t_{i+1}, \dots, t_m, t_{m+1}) \right] \right. \\
 & - \sum_{m=i}^{N-2} e^{-\sum_{k=i+1}^m \lambda_k(t_{k+1}-t_k)} B_{K_{i+1} \dots K_m K_{m+1}}^+(x, t; t_{i+1}, \dots, t_m, t_{m+1}) \\
 & \left. + \sum_{m=i+1}^{N-1} \lambda_m e^{-\sum_{k=i+1}^{m-1} \lambda_k(t_{k+1}-t_k)} \int_{t_m}^{t_{m+1}} e^{-\lambda_m(\tau-t_m)} \left[B_{K_{i+1} \dots K_m \frac{n}{R}}^+(x, t; t_{i+1}, \dots, t_m, \tau) \right. \right. \\
 & \left. \left. + \frac{R}{n} A_{K_{i+1} \dots K_m \frac{n}{R}}^+(x, t; t_{i+1}, \dots, t_m, \tau) \right] d\tau \right\} \\
 & + \lambda_i \int_t^{t_{i+1}} e^{-\lambda_i(\tau-t)} \left[B_{\frac{n}{R}}^+(x, t; \tau; 0, b, s_V) + \frac{R}{n} A_{\frac{n}{R}}^-(x, t; \tau; 0, b, s_V) \right] d\tau. \tag{24}
 \end{aligned}$$

Here $B_{K_1 \dots K_m}^{s_1 \dots s_m}(x, t; t_1, \dots, t_m)$ and $A_{K_1 \dots K_m}^{s_1 \dots s_m}(x, t; t_1, \dots, t_m)$ are respectively the prices of m -th order bond and asset binaries with 0-risk free rate, b -dividend rate and s_V -volatility. (See lemma 1.)

The proof is not difficult but somewhat complicated. We will prove it in the section 4.

Remark 1. In this theorem, the financial meaning of $u_i(x, t)$ is that it is the relative price of our bond in a subinterval with respect to the risk free zero coupon bond. We can derive the pricing formulae of our bond under other assumptions on the relations between $K_i (i = 1, \dots, N)$ and n/R using the same method.

3.2. The Problem with Exogenous Recovery. Instead of the assumption 5) let assume the following:

7) The default recovery R_d is given as the form of exogenous face value

$$R_d = R e^{-r(T-t)} \quad (0 \leq R \leq 1 \text{ is a constant.}) \tag{25}$$

Then under the assumptions 1), 2), 3), 4), 6) and 7) the pricing model of our bond is given as follows:

$$\begin{aligned}
 \frac{\partial C_i}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 C_i}{\partial V^2} + (r - b) V \frac{\partial C_i}{\partial V} - (r + \lambda_i) C_i + \lambda_i R e^{-r(T-t)} = 0, \quad t_i < t < t_{i+1}, \\
 C_i(t_{i+1}) = C_{i+1}(t_{i+1}) 1(V > K_{i+1} e^{-r(T-t_{i+1})}) + R e^{-r(T-t_{i+1})} 1(V \leq K_{i+1} e^{-r(T-t_{i+1})}). \tag{26}
 \end{aligned}$$

Here $i = 0, 1, \dots, N - 1$ and $C_N(V, t) \equiv 1$.

Theorem 2. (exogenous recovery) Under the assumptions 1), 2), 3), 4), 6) and 7) the price of our bond, that is, the solution of (26) is represented as follows:

$$\begin{aligned}
 C_i(V, t) = W_i(V/e^{-r(T-t)}, t) e^{-r(T-t)} + [1 - W_i(V/e^{-r(T-t)}, t)] R e^{-r(T-t)}, \\
 t_i \leq t < t_{i+1}, \quad i = 0, \dots, N - 1. \tag{27}
 \end{aligned}$$

Here

$$W_i(x, t) = e^{-\lambda_i(t_{i+1}-t) - \sum_{k=i+1}^{N-1} \lambda_k(t_{k+1}-t_k)} B_{K_{i+1} \dots K_N}^+ \dots^+ (x, t; t_{i+1}, \dots, t_N; 0, b, s_V). \quad (28)$$

The proof is done by the same way with that of theorem 1. See the section 4.

Remark 2. The *financial meaning* of the pricing formulae (27) is similar with that of [12]: the price of our defaultable bond at time t can be seen as a probabilistic mean value of the current value $e^{-r(T-t)}$ of the bond in the case when there is no default after time t and the value $Re^{-r(T-t)}$ of the bond in the case when default occurs after time t . So $W_i(V/e^{-r(T-t)}, t)$ is the survival probability after the time $t \in [t_i, t_{i+1})$, that is, the probability with which no default event occurs in the interval $(t, T]$ and $1 - W_i(V/e^{-r(T-t)}, t)$ is the ruin probability after the time $t \in [t_i, t_{i+1})$, that is, the probability with which default event occurs in the interval $(t, T]$ when $t_i \leq t < t_{i+1}$. The formulae (27) can be written as follows:

$$C_i(V, t) = Re^{-r(T-t)} + (1 - R)W_i(V/e^{-r(T-t)}, t)e^{-r(T-t)}, \\ t_i \leq t < t_{i+1}, \quad i = 0, \dots, N - 1. \quad (29)$$

The *financial meaning* of (29) is as follows: the **first term** of (29) is the current price of the part to be given to bond holder *regardless of default occurs or not*, and the **second term** is the *allowance* dependent on the survival probability after time t . If after some moment t , the default is certain ($W = 0$), then the price of the bond at t is exactly the current price of default recovery $Re^{-r(T-t)}$. If the default recovery rate is zero, that is, $R = 0$, then the ratio of the defaultable bond price and default free zero coupon bond price is the very the survival probability after time t . If the default recovery rate is full, that is, $R = 1$, then default event does not effect to the bond price and defaultable bond price is the same with default free zero coupon bond price.

4. THE NUMERICAL RESULTS

4.1. Illustration of the Effect of Parameters on the Bond Price. In this subsection we illustrate the effect of several parameters including recovery rate R , volatility s_V of firm value, $x = V/e^{-r(T-t)}$ (relative price of firm value), default boundary K and default intensity λ on the price of the defaultable bonds. Let $N = 2$, $t_1 = 3, t_2 = 6$ (annum).

Basic data for calculation are as follows: Short rate $r = 0.1$; Firm value: dividend rate $b = 0.05$, volatility $s_V = 1.0$, $x = V/e^{-r(T-t)} = 200$; $\lambda_0 = 0.002$, $\lambda_1 = 0.005$ are respectively default intensities in the intervals $(0, t_1)$, (t_1, t_2) ; $K_1 = K_2 = 100$ is default barrier at time t_1, t_2 ; recovery rate $R = 0.5$.

We will analyze (t, C) -plot changing one of R, s_V, x, K and λ under keeping the remainder of data on as the above. See the following figures 1–9.

Note that figure 1 shows that increase of recovery rate results in increase of bond price. Figure 2 shows that increase of volatility of firm value results in decrease of bond price. The reason is that when s_V increases, the firm value fluctuates more seriously and there are more risks of default, which results in decrease of bond price. Figure 3 shows that increase of firm value results in increase of bond price. Figures 4–9 show the effect of default barrier and default intensity on bond price. In particular, in the figure 5 (or 8) we can see the mixed effect of increase of K_1 (or λ_0) and decrease of K_2 (or λ_1) in the subinterval $[0, 3]$.

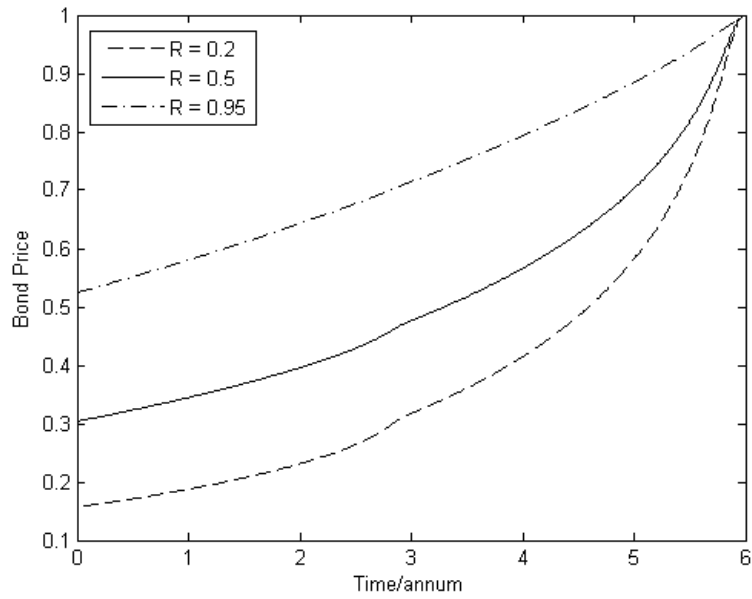


FIGURE 1. Plot $(t : C)$ when $R = 0.2, 0.5, 0.95$

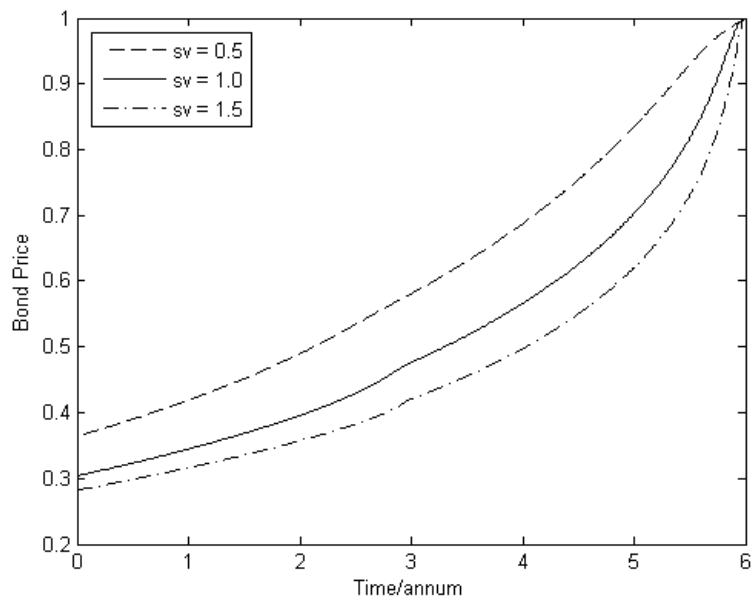


FIGURE 2. Plot $(t : C)$ when $s_V = 0.5, 1.0, 1.5$

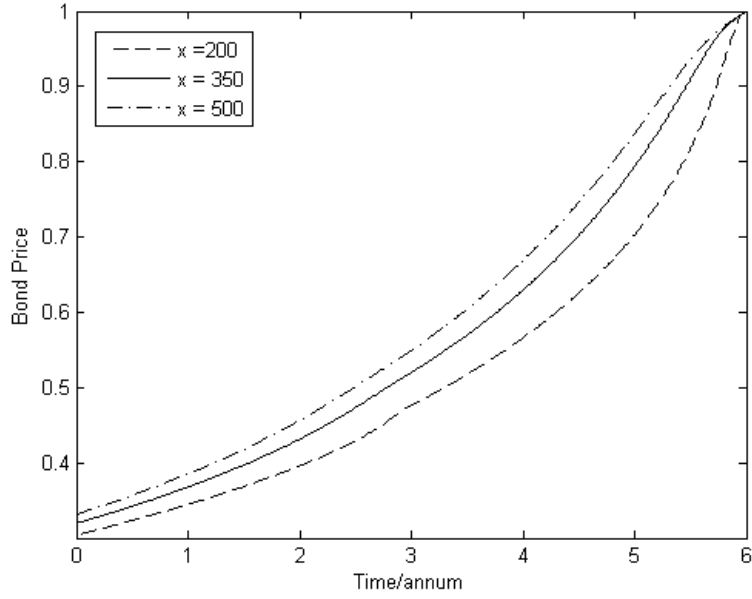


FIGURE 3. Plot $(t : C)$ when $x = V/e^{-r(T-t)} = 200, 350, 500$

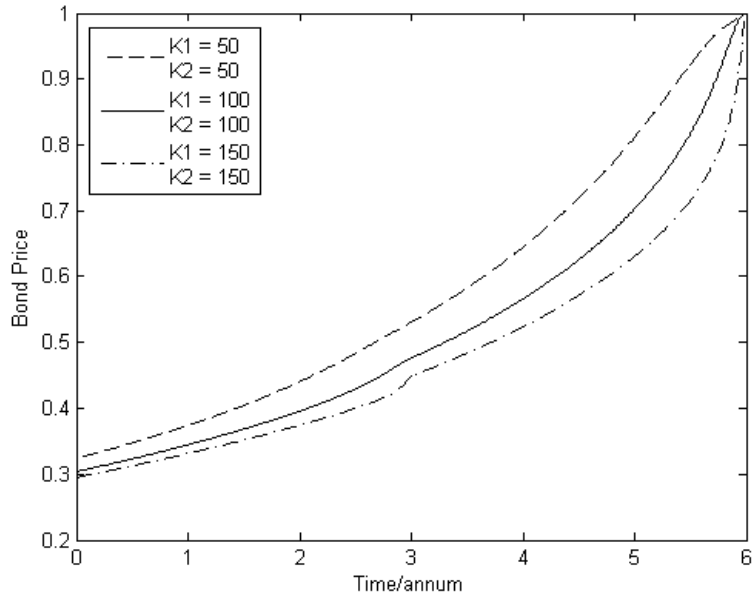


FIGURE 4. Plot $(t : C)$ when $(K_1, K_2) = (50, 50), (100, 100), (150, 150)$

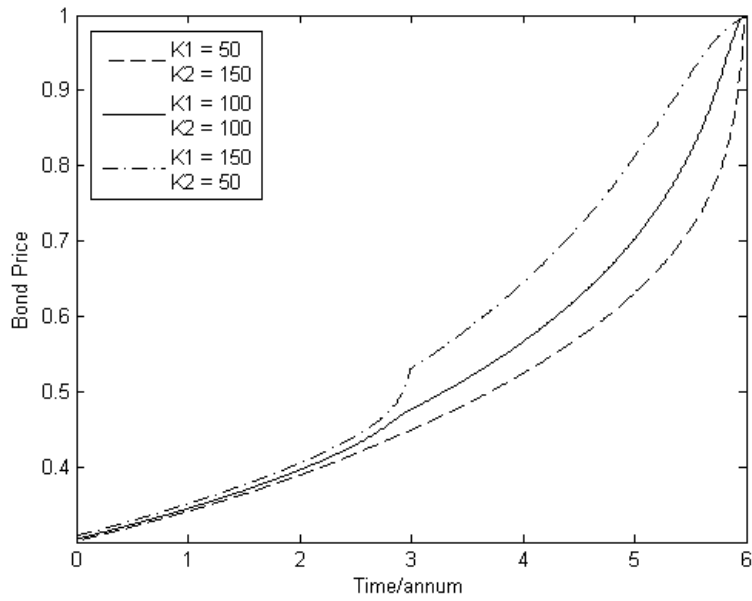


FIGURE 5. Plot $(t : C)$ when $(K_1, K_2) = (50, 150), (100, 100), (150, 50)$

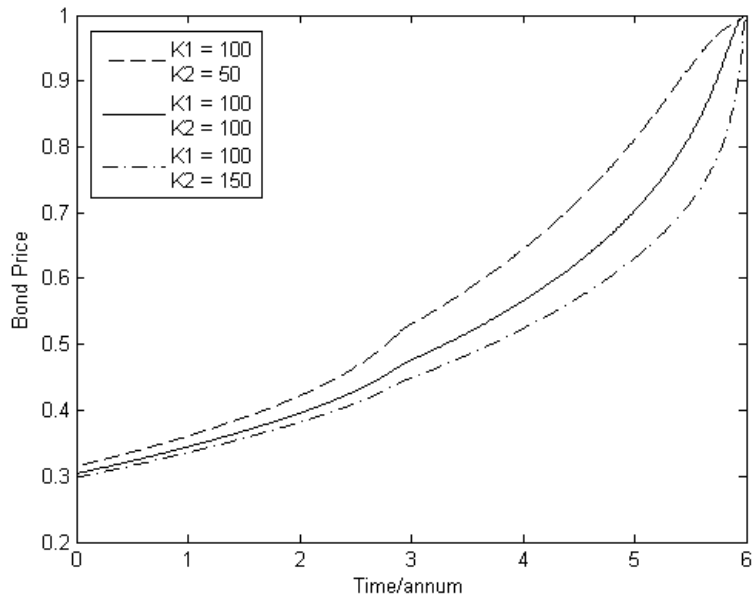


FIGURE 6. Plot $(t : C)$ when $K_1 = 100, K_2 = 50, 100, 150$

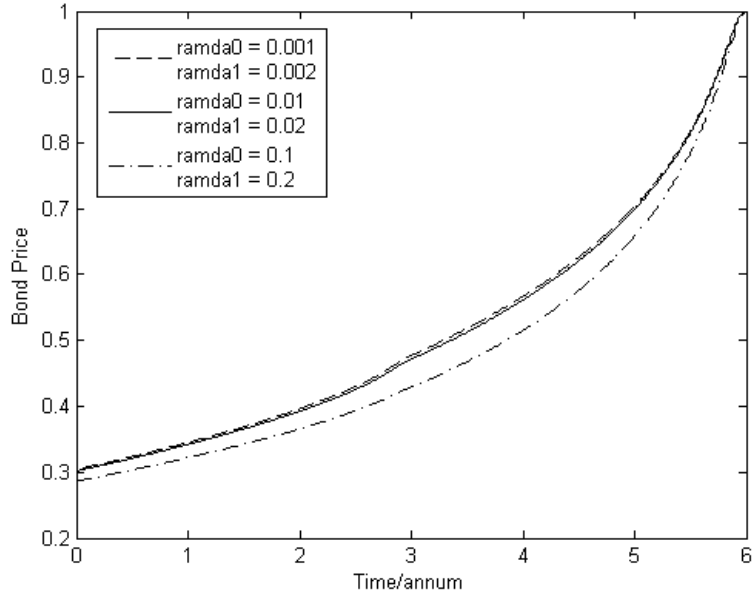


FIGURE 7. Plot $(t : C)$ when $(\lambda_0, \lambda_1) = (0.001, 0.002), (0.01, 0.02), (0.1, 0.2)$

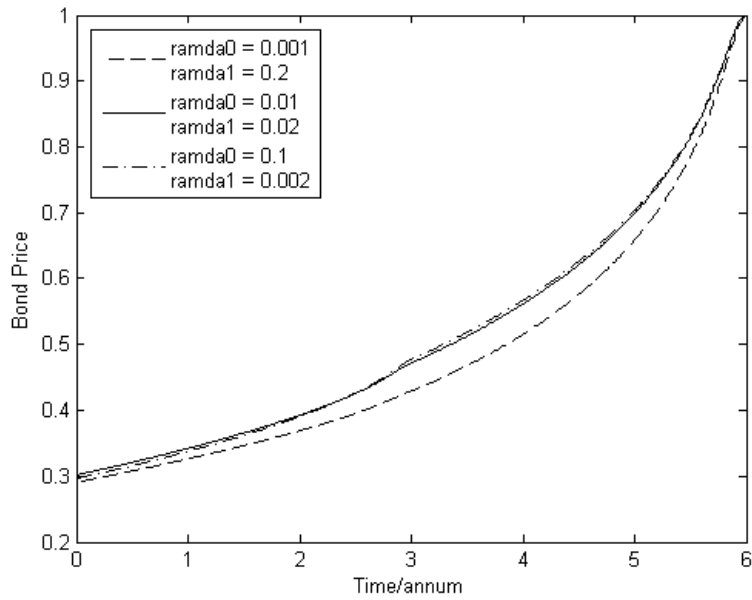


FIGURE 8. Plot $(t : C)$ when $(\lambda_0, \lambda_1) = (0.001, 0.2), (0.01, 0.02), (0.1, 0.002)$

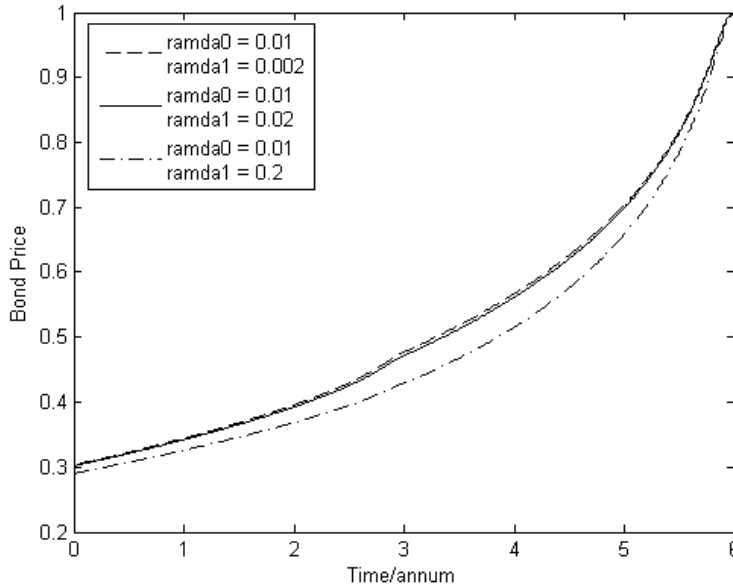


FIGURE 9. Plot $(t : C)$ when $\lambda_0 = 0.01, \lambda_1 = 0.002, 0.02, 0.2$

4.2. Credit Spread Analysis. In this subsection, we illustrate the effect of several parameters including recovery rate R , volatility s_V of firm value, $x = V/e^{-r(T-t)}$, default boundary K and default intensity λ on credit spreads. The *credit spread* is defined using the difference between the yields of the defaultable bond C and the default-free bond $e^{-r(T-t)}$ and is given by the following expression:

$$CS = -\frac{\ln[C/e^{-r(T-t)}]}{T-t}.$$

For simplicity, we only consider the case of exogenous default recovery (theorem 2). Then, the credit spread is differently given in every subinterval as follows:

$$CS_i = -\frac{\ln[R + (1-R)W_i(V/e^{-r(T-t)}, t)]}{T-t}, t_i \leq t < t_{i+1}, i = 0, \dots, N-1. \quad (30)$$

Let $N = 2, t_1 = 3, t_2 = T = 6$ (annum) as in the above.

Basic data for calculation of CS are as follows: Short rate $r = 0.1$; Firm value: dividend rate $b = 0.05$, volatility $s_V = 1.0$, $x = V/e^{-r(T-t)} = 200$; $\lambda_0 = 0.002, \lambda_1 = 0.005$ are respectively default intensities in the intervals $(0, t_1), (t_1, t_2)$; $K_1 = K_2 = 100$ is default barrier at time t_1, t_2 ; recovery rate $R = 0.5$.

We will analyze (t, CS) -plot changing one of R, s_V, x, K and λ under keeping the remainder of data on as the above. In what follows, the figure 10 shows that increase of recovery rate results in decrease of credit spread. Figure 11 shows that increase of volatility of firm value results in increase of credit spread. The reason is that when s_V increases, the firm value fluctuates more seriously and there are more risks of default, which results in increase of credit spread. Figure 12 shows that increase of firm value results in decrease of credit spread. Figures 13–18 show the effect of default barrier and default intensity on credit spread. In particular,

in the figure 14 (or 16) we can see the mixed effect of increase of K_1 (or λ_0) and decrease of K_2 (or λ_1) in the subinterval $[0, 3]$.

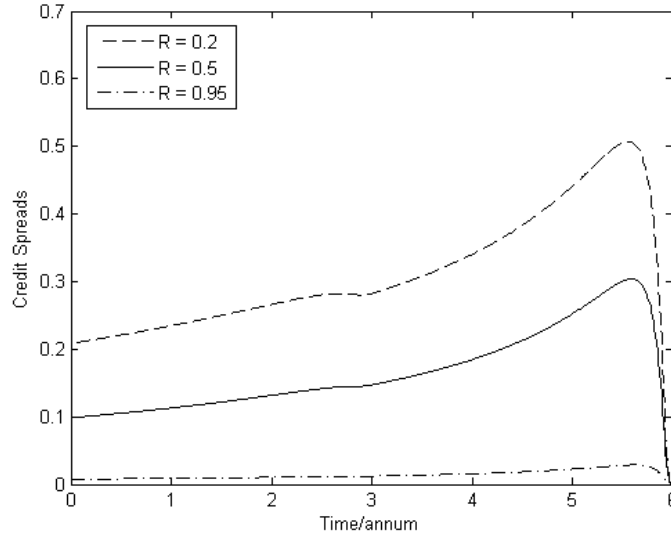


FIGURE 10. Plot $(t : CS)$ when $R = 0.2, 0.5, 0.95$

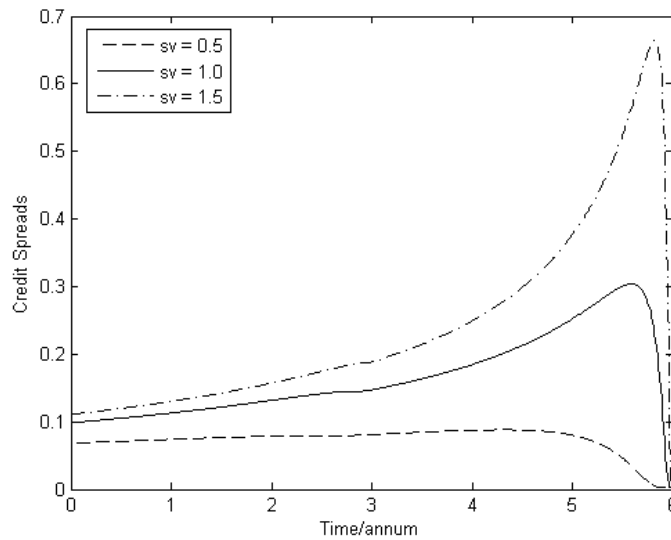


FIGURE 11. Plot $(t : CS)$ when $s_V = 0.5, 1.0, 1.5$

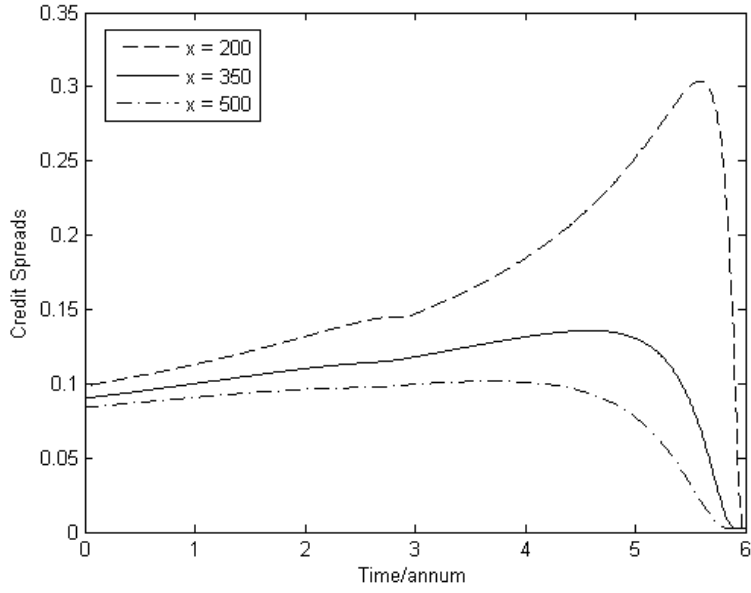


FIGURE 12. Plot $(t : CS)$ when $x = V/e^{-r(T-t)} = 200, 350, 500$

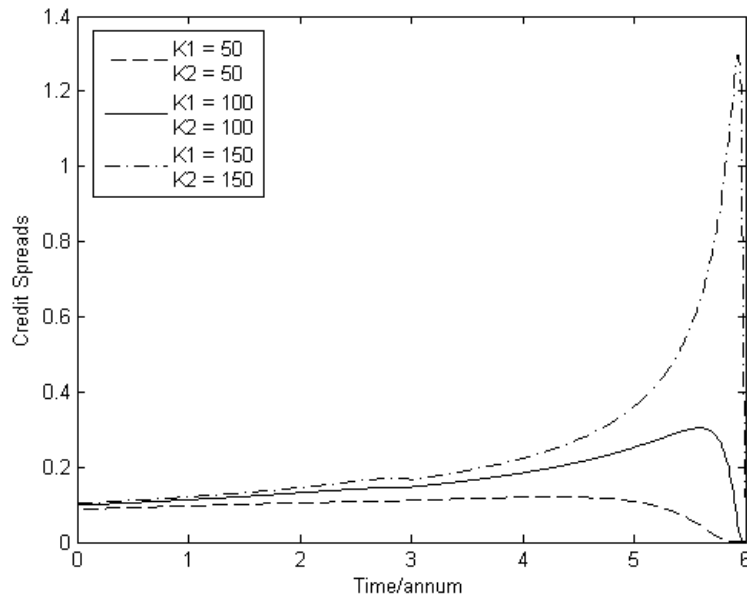


FIGURE 13. Plot $(t : CS)$ when $(K_1, K_2) = (50, 50), (100, 100), (150, 150)$

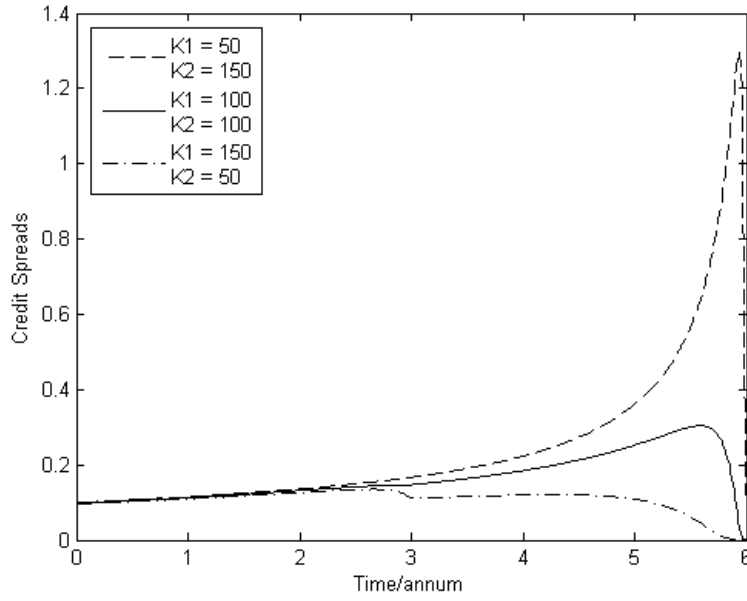


FIGURE 14. Plot $(t : CS)$ when $(K_1, K_2) = (50, 150), (100, 100), (150, 50)$

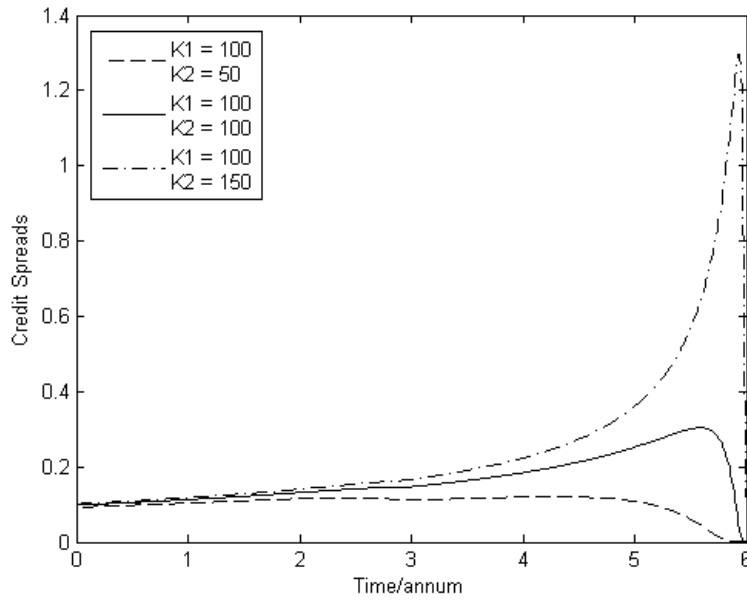


FIGURE 15. Plot $(t : CS)$ when $K_1 = 100, K_2 = 50, 100, 150$

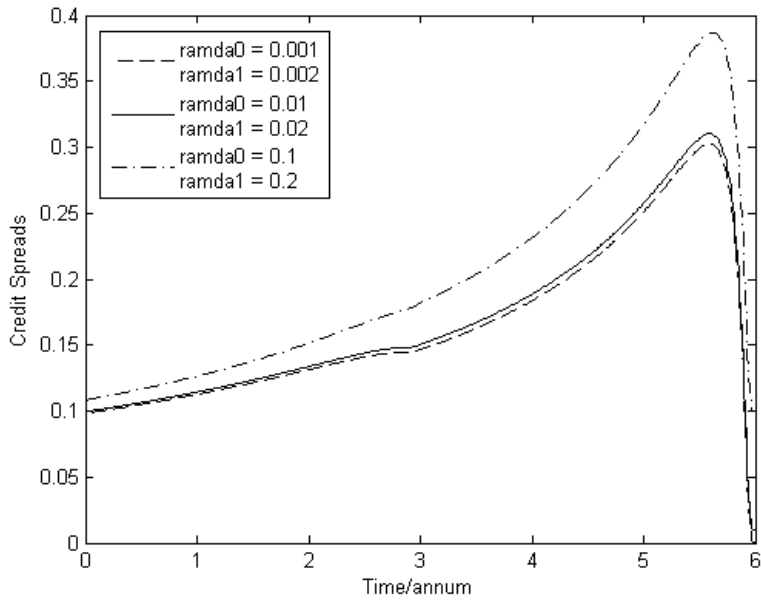


FIGURE 16. Plot($t : CS$) when $(\lambda_0, \lambda_1) = (0.001, 0.002), (0.01, 0.02), (0.1, 0.2)$

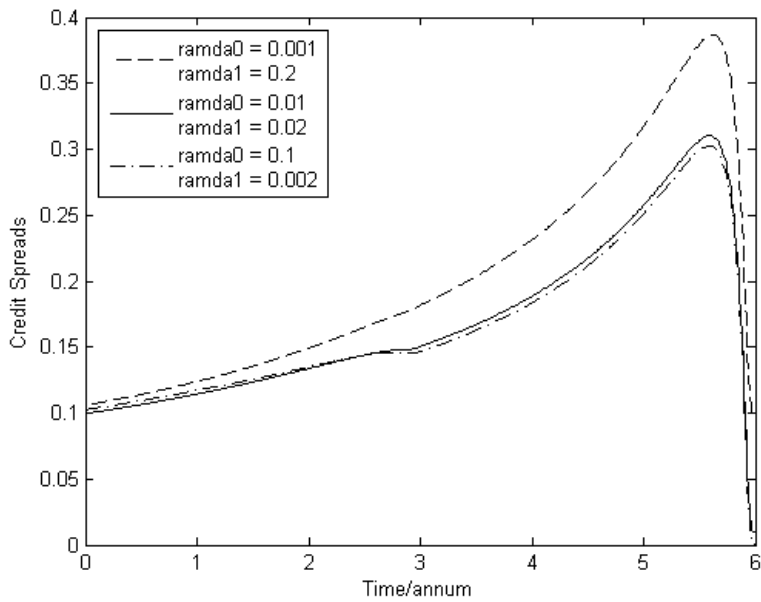


FIGURE 17. Plot($t : CS$) when $(\lambda_0, \lambda_1) = (0.001, 0.2), (0.01, 0.02), (0.1, 0.002)$

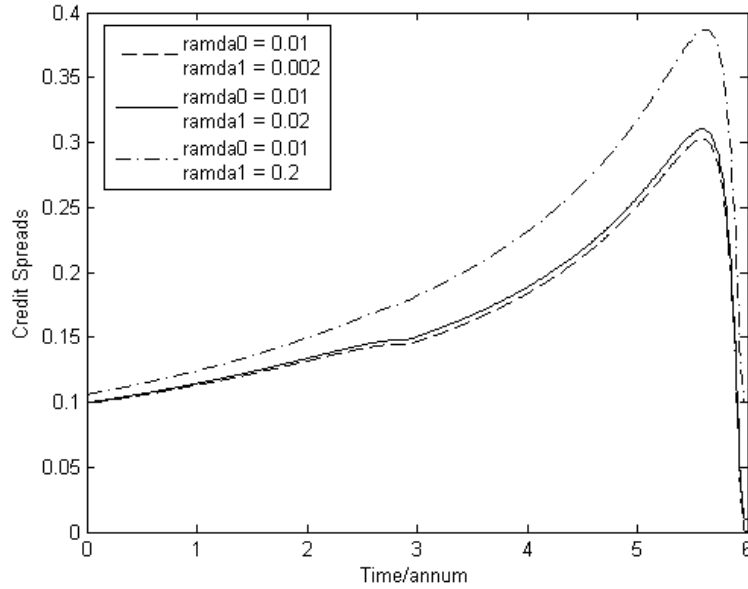


FIGURE 18. Plot $(t : CS)$ when $\lambda_0 = 0.01, \lambda_1 = 0.002, 0.02, 0.2$

5. APPENDIX: PROOFS OF THEOREMS

The Proof of Theorem 1. i) In (21), we use change of numeraire

$$x = \frac{V}{e^{-r(T-t)}}, \quad u_i(x, t) = -\frac{C_i(V, t)}{e^{-r(T-t)}}, \quad t_i \leq t < t_{i+1}, i = 0, \dots, N-1. \quad (31)$$

Substituting (31) into (21) we get

$$\begin{aligned} \frac{\partial u_i}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 u_i}{\partial x^2} - bx \frac{\partial u_i}{\partial x} - \lambda_i u_i + \lambda_i \min\{1, \frac{R}{n}x\} &= 0, \quad t_i < t < t_{i+1}, x > 0, \\ u_i(x, t_{i+1}) &= u_{i+1}(x, t_{i+1})1(x > K_{i+1}) + \min\{1, \frac{R}{n}x\}1(x \leq K_{i+1}), i = 0, \dots, N-1. \end{aligned} \quad (32)$$

Here $u_N(x, t) \equiv 1$. From the assumption

$$K_i \leq n/R, i = 1, \dots, N \quad (33)$$

If $V(t_i) \leq K_i e^{-r(T-t_i)}$, that is, if the default event occurs at time t_i , then $\min\{e^{-r(T-t_i)}, RV(t_i)/n\} = RV(t_i)/n$ and we have

$$\min\{1, \frac{R}{n}x\}1(x \leq K_{i+1}) = \frac{R}{n}x \cdot 1(x \leq K_{i+1}). \quad (34)$$

Then the problem (32) is changed into the following one.

$$\begin{aligned} \frac{\partial u_i}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 u_i}{\partial x^2} - bx \frac{\partial u_i}{\partial x} - \lambda_i u_i + \lambda_i \min\{1, \frac{R}{n}x\} &= 0, \quad t_i < t < t_{i+1}, x > 0, \\ u_i(x, t_{i+1}) &= u_{i+1}(t_{i+1})1(x > K_{i+1}) + \frac{R}{n}x \cdot 1(x \leq K_{i+1}), i = 0, \dots, N-1. \end{aligned} \quad (35)$$

When $i = N - 1$, (35) is as follows:

$$\begin{aligned} \frac{\partial u_{N-1}}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 u_{N-1}}{\partial x^2} - bx \frac{\partial u_{N-1}}{\partial x} - \lambda_{N-1} u_{N-1} + \lambda_{N-1} \min\{1, \frac{R}{n}x\} &= 0, \\ t_{N-1} < t < T, x > 0, \\ u_{N-1}(x, T) = 1(x > K_N) + \frac{R}{n}x \cdot 1(x \leq K_N), \quad x > 0. \end{aligned} \tag{36}$$

This is a terminal value problem for an inhomogenous Black-Scholes equation with coefficients $r = \lambda_{N-1}$, $q = \lambda_{N-1} + b$, $\sigma = s_V$. Let L_{N-1} be the Black-Scholes partial differential operator with coefficients $r = \lambda_{N-1}$, $q = \lambda_{N-1} + b$, $\sigma = s_V$, that is,

$$L_{N-1}u = \frac{\partial u}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 u}{\partial x^2} - bx \frac{\partial u}{\partial x} - \lambda_{N-1}u.$$

Then the solution of (36) is provided by sum of the solutions U_1 and U_2 to the two following problems:

$$\begin{aligned} L_{N-1}U_1 &= 0, & t_{N-1} < t < T, x > 0, \\ U_1(x, T) &= 1(x > K_N) + \frac{R}{n}x \cdot 1(x \leq K_N), \quad x > 0. \end{aligned} \tag{37}$$

$$\begin{aligned} L_{N-1}U_2 + \lambda_{N-1} \min\{1, \frac{R}{n}x\} &= 0, \quad t_{N-1} < t < T, x > 0, \\ U_2(x, T) &= 0, \quad x > 0. \end{aligned} \tag{38}$$

The terminal payoff of (37) is linear combination of the terminal payoffs of bond and asset binaries (refer to section 2) and thus the solution to (37) is given as follows:

$$U_1 = B_{K_N}^+(x, t; T; \lambda_{N-1}, \lambda_{N-1} + b, s_V) + \frac{R}{n} A_{K_N}^-(x, t; T; \lambda_{N-1}, \lambda_{N-1} + b, s_V), \quad t_{N-1} \leq t < T.$$

The problem (38) is a 0-terminal value problem of an inhomogeneous equation and thus we can use the *Duhamel's principle* to solve it. Fix $\tau \in (t_{N-1}, T]$ and let $W(x, t; \tau)$ be the solution to the following terminal value problem:

$$\begin{aligned} L_{N-1}W &= 0, & t_{N-1} < t < \tau, x > 0, \\ W(x, \tau; \tau) &= \lambda_{N-1} \min\{1, \frac{R}{n}x\}, \quad x > 0. \end{aligned}$$

Since $\lambda_{N-1} \min\{1, \frac{R}{n}x\} = \lambda_{N-1} [1(x > n/R) + \frac{R}{n}x \cdot 1(x < n/R)]$, the solution is given as follows:

$$\begin{aligned} W(x, t; \tau) &= \lambda_{N-1} \left[B_{n/R}^+(x, t; \tau; \lambda_{N-1}, \lambda_{N-1} + b, s_V) + \right. \\ &\quad \left. + \frac{R}{n} A_{n/R}^-(x, t; \tau; \lambda_{N-1}, \lambda_{N-1} + b, s_V) \right], \quad t_{N-1} \leq t < \tau, x > 0. \end{aligned}$$

Then the solution U_2 to (38) is given as follows:

$$\begin{aligned} U_2 &= \int_t^T W(x, t; \tau) d\tau = \\ &= \lambda_{N-1} \int_t^T \left[B_{n/R}^+(x, t; \tau; \lambda_{N-1}, \lambda_{N-1} + b, s_V) + \right. \\ &\quad \left. + \frac{R}{n} A_{n/R}^-(x, t; \tau; \lambda_{N-1}, \lambda_{N-1} + b, s_V) \right] d\tau, \quad t_{N-1} \leq t < T, x > 0. \end{aligned}$$

Thus the solution to (36) is provided by $u_{N-1}(x, t) = U_1 + U_2$, that is,

$$\begin{aligned} u_{N-1}(x, t) &= \\ &= B_{K_N}^+(x, t; T; \lambda_{N-1}, \lambda_{N-1} + b, s_V) + \frac{R}{n} A_{K_N}^-(x, t; T; \lambda_{N-1}, \lambda_{N-1} + b, s_V) + \\ &+ \lambda_{N-1} \int_t^T \left[B_{n/R}^+(x, t; \tau; \lambda_{N-1}, \lambda_{N-1} + b, s_V) + \right. \\ &\quad \left. + \frac{R}{n} A_{n/R}^-(x, t; \tau; \lambda_{N-1}, \lambda_{N-1} + b, s_V) \right] d\tau, \quad t_{N-1} \leq t < T, \quad x > 0. \quad (39) \end{aligned}$$

For our further purpose, using the relations (11) we rewrite (39) by the price of bond and asset binaries with the coefficients $r = 0, q = b, \sigma = s_V$:

$$\begin{aligned} u_{N-1}(x, t) &= e^{-\lambda_{N-1}(T-t)} \left[B_{K_N}^+(x, t; T; 0, b, s_V) + \frac{R}{n} A_{K_N}^-(x, t; T; 0, b, s_V) \right] + \\ &+ \lambda_{N-1} \int_t^T e^{-\lambda_{N-1}(\tau-t)} \left[B_{n/R}^+(x, t; \tau; 0, b, s_V) + \frac{R}{n} A_{n/R}^-(x, t; \tau; 0, b, s_V) \right] d\tau, \\ &\quad t_{N-1} \leq t < T, \quad x > 0. \quad (40) \end{aligned}$$

Now solve (35) when $i = N - 2$. In this case (35) is as follows:

$$\begin{aligned} \frac{\partial u_{N-2}}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 u_{N-2}}{\partial x^2} - bx \frac{\partial u_{N-2}}{\partial x} - \lambda_{N-2} u_{N-1} + \lambda_{N-2} \min\left\{1, \frac{R}{n} x\right\} &= 0, \\ t_{N-2} < t < t_{N-1}, \quad x > 0, \end{aligned}$$

$$u_{N-2}(x, t_{N-1}) = u_{N-1}(x, t_{N-1}) 1(x > K_{N-1}) + \frac{R}{n} x \cdot 1(x \leq K_{N-1}). \quad (41)$$

This is a terminal value problem of the inhomogeneous Black-Scholes equation with coefficients $r = \lambda_{N-2}, q = \lambda_{N-2} + b, \sigma = s_V$.

Remark 3. If we consider (39), then the expiry payoff of (41) is the linear combination of first order binaries or zero and integrals of first order binaries or zero and therefore you could think that it is natural to solve (41) using the pricing formulae of second order binaries and their integrals. But we must *note* that the coefficients of (41) are *different* from those of (36) and (39) and thus we can't directly apply the pricing formulae of second order binaries here. Fortunately, the *differences* between risk free rates and dividend rates in adjacent subintervals are *all a constant* $-b$ and *volatility is not changed* in whole time interval and thus we can carefully use the pricing formulae of second order binaries with (11) together to give a representation of the solution to (41).

If we rewrite the terminal payoff of (41) into prices of binaries with the coefficients $r = \lambda_{N-2}, q = \lambda_{N-2} + b, \sigma = s_V$ using (11), then from (39) we get:

$$\begin{aligned} u_{N-1}(x, t_{N-1}) &= e^{-(\lambda_{N-1} - \lambda_{N-2})(T - t_{N-1})} \left[B_{K_N}^+(x, t; T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\ &\quad \left. + \frac{R}{n} A_{K_N}^-(x, t; T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] + \\ &+ \lambda_{N-1} \int_t^T e^{-(\lambda_{N-1} - \lambda_{N-2})(\tau - t_{N-1})} \left[B_{n/R}^+(x, t; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\ &\quad \left. + \frac{R}{n} A_{n/R}^-(x, t; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] d\tau. \end{aligned}$$

Let L_{N-2} be the Black-Scholes partial differential operator with coefficients $r = \lambda_{N-2}, q = \lambda_{N-2} + b, \sigma = s_V$. Then the solution to (41) is the sum $U_1 + U_2 + U_3$

of the solutions to the following three problems:

$$\begin{aligned}
 L_{N-2}U_1 &= 0, & t_{N-2} < t < t_{N-1}, x > 0, \\
 U_1(x, t_{N-1}) &= e^{-(\lambda_{N-1}-\lambda_{N-2})(T-t_{N-1})} \left[B_{K_N}^+(x, t_{N-1}; T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
 &\quad \left. + \frac{R}{n} A_{K_N}^-(x, t_{N-1}; T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] 1(x > K_{N-1}) + \\
 &\quad + \frac{R}{n} x \cdot 1(x \leq K_{N-1}), \tag{42}
 \end{aligned}$$

$$\begin{aligned}
 L_{N-2}U_2 &= 0, & t_{N-2} < t < t_{N-1}, x > 0, \\
 U_2(x, t_{N-1}) &= \\
 &= \lambda_{N-1} \int_{t_{N-1}}^T e^{-(\lambda_{N-1}-\lambda_{N-2})(\tau-t_{N-1})} \left[B_{\frac{n}{R}}^+(x, t_{N-1}; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
 &\quad \left. + \frac{R}{n} A_{\frac{n}{R}}^-(x, t_{N-1}; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] d\tau \cdot 1(x > K_{N-1}). \tag{43}
 \end{aligned}$$

$$\begin{aligned}
 L_{N-2}U_3 + \lambda_{N-2} \min\left\{1, \frac{R}{n}x\right\} &= 0, & t_{N-2} < t < t_{N-1}, x > 0, \\
 U_3(x, t_{N-1}) &= 0, & x > 0. \tag{44}
 \end{aligned}$$

Using the prices of first and second order binaries (6) and (7), the solution to (42) is given as follows:

$$\begin{aligned}
 U_1(x, t) &= \\
 &= e^{-(\lambda_{N-1}-\lambda_{N-2})(T-t_{N-1})} \left[B_{K_{N-1}K_N}^+(x, t; t_{N-1}, T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
 &\quad \left. + \frac{R}{n} A_{K_{N-1}K_N}^-(x, t; t_{N-1}, T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] + \\
 &\quad + \frac{R}{n} A_{K_{N-1}}^-(x, t; t_{N-1}; \lambda_{N-2}, \lambda_{N-2} + b, s_V), \quad t_{N-2} \leq t < t_{N-1}. \tag{45}
 \end{aligned}$$

From the corollary of Lemma 2, the solution to (43) is given as follows:

$$\begin{aligned}
 U_2(x, t) &= \\
 &= \lambda_{N-1} \int_{t_{N-1}}^T e^{-(\lambda_{N-1}-\lambda_{N-2})(\tau-t_{N-1})} \left[B_{K_{N-1}\frac{n}{R}}^+(x, t, t_{N-1}, \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
 &\quad \left. + \frac{R}{n} A_{K_{N-1}\frac{n}{R}}^-(x, t; t_{N-1}, \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] d\tau, \quad t_{N-2} \leq t < t_{N-1}. \tag{46}
 \end{aligned}$$

(44) is a 0-terminal value problem of an inhomogeneous equation just like (38), so its solution is given by

$$\begin{aligned}
 U_3(x, t) &= \lambda_{N-2} \int_t^{t_{N-1}} \left[B_{\frac{n}{R}}^+(x, t; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
 &\quad \left. + \frac{R}{n} A_{\frac{n}{R}}^-(x, t; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] d\tau, \quad t_{N-2} \leq t < t_{N-1}. \tag{47}
 \end{aligned}$$

Thus we obtain the representation of $u_{N-2}(x, t) = U_1 + U_2 + U_3$, that is,

$$\begin{aligned}
u_{N-2}(x, t) &= \\
&= e^{-(\lambda_{N-1}-\lambda_{N-2})(T-t_{N-1})} \left[B_{K_{N-1}K_N}^+ (x, t; t_{N-1}, T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
&\quad \left. + \frac{R}{n} A_{K_{N-1}K_N}^+ (x, t; t_{N-1}, T; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] + \\
&+ \frac{R}{n} A_{K_{N-1}}^- (x, t; t_{N-1}; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&+ \lambda_{N-1} \int_{t_{N-1}}^T e^{-(\lambda_{N-1}-\lambda_{N-2})(\tau-t_{N-1})} \left[B_{K_{N-1}\frac{n}{R}}^+ (x, t, t_{N-1}, \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
&\quad \left. + \frac{R}{n} A_{K_{N-1}\frac{n}{R}}^+ (x, t; t_{N-1}, \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] d\tau \\
&+ \lambda_{N-2} \int_t^{t_{N-1}} \left[B_{n/R}^+ (x, t; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) + \right. \\
&\quad \left. + \frac{R}{n} A_{n/R}^- (x, t; \tau; \lambda_{N-2}, \lambda_{N-2} + b, s_V) \right] d\tau, \quad t_{N-2} \leq t < t_{N-1}. \quad (48)
\end{aligned}$$

For our further purpose, using the relations (11) we rewrite (48) by the price of bond and asset binaries with the coefficients $r = 0$, $q = b$, $\sigma = s_V$ to get

$$\begin{aligned}
u_{N-2}(x, t) &= \\
&= e^{-\lambda_{N-2}(T-t)-(\lambda_{N-1}-\lambda_{N-2})(T-t_{N-1})} \left[B_{K_{N-1}K_N}^+ (x, t; t_{N-1}, T; 0, b, s_V) + \right. \\
&\quad \left. + \frac{R}{n} A_{K_{N-1}K_N}^+ (x, t; t_{N-1}, T; 0, b, s_V) \right] + \\
&+ e^{-\lambda_{N-2}(t_{N-1}-t)} \frac{R}{n} A_{K_{N-1}}^- (x, t; t_{N-1}; 0, b, s_V) + \\
&+ \lambda_{N-1} \int_{t_{N-1}}^T e^{-\lambda_{N-2}(\tau-t)-(\lambda_{N-1}-\lambda_{N-2})(\tau-t_{N-1})} \left[B_{K_{N-1}\frac{n}{R}}^+ (x, t, t_{N-1}, \tau; 0, b, s_V) + \right. \\
&\quad \left. + \frac{R}{n} A_{K_{N-1}\frac{n}{R}}^+ (x, t; t_{N-1}, \tau; 0, b, s_V) \right] d\tau + \\
&+ \lambda_{N-2} \int_t^{t_{N-1}} e^{-\lambda_{N-2}(\tau-t)} \left[B_{\frac{n}{R}}^+ (x, t; \tau; 0, b, s_V) + \frac{R}{n} A_{\frac{n}{R}}^- (x, t; \tau; 0, b, s_V) \right] d\tau, \\
&\quad t_{N-2} \leq t < t_{N-1}. \quad (49)
\end{aligned}$$

By induction we can obtain the representations of all $u_i(x, t) (i = 0, \dots, N-1)$. If in every representation of $u_i(x, t)$ we replace the higher order binaries with the coefficients $r = \lambda_i$, $q = \lambda_i + b$, $\sigma = s_V$ into the higher order binaries with the coefficients $r = 0$, $q = b$, $\sigma = s_V$ using the relation (11) and arrange the exponents properly, we soon obtain (23). If we return to the original variable V and the unknown function C using (31), then we soon obtain (22).

The proof of ii) is similar with the proof of i) and we will give the sketch. From the assumption $K_i > n/R$, $i = 1, 2, \dots, N$. Then we have

$$\min \left\{ 1, \frac{R}{n} x \right\} \cdot 1\{x \leq K_{i+1}\} = \frac{R}{n} x \cdot 1\left\{x \leq \frac{n}{R}\right\} + 1\left\{x > \frac{n}{R}\right\} - 1\{x > K_i\}. \quad (50)$$

Then the problem (32) is changed into the following one.

$$\begin{aligned} \frac{\partial u_i}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 u_i}{\partial x^2} - bx \frac{\partial u_i}{\partial x} - \lambda_i u_i + \lambda_i \min\{1, \frac{R}{n}x\} &= 0, \quad t_i < t < t_{i+1}, x > 0, \\ u_i(x, t_{i+1}) &= [u_{i+1}(t_{i+1}) - 1] \cdot 1(x > K_{i+1}) + \frac{R}{n} x \cdot 1\left\{x \leq \frac{n}{R}\right\} + 1\left\{x > \frac{n}{R}\right\}. \end{aligned} \tag{51}$$

Here $i = 0, \dots, N - 1$. This problems can be solved in the same way with the method of solving (35). The details are omitted. (QED)

The proof of Theorem 2. In (26), if we use change of numeraire (31), then we have

$$\begin{aligned} \frac{\partial u_i}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 u_i}{\partial x^2} - bx \frac{\partial u_i}{\partial x} - \lambda_i u_i + \lambda_i R &= 0, \quad t_i < t < t_{i+1}, x > 0, \\ u_i(x, t_{i+1}) &= u_{i+1}(x, t_{i+1})1(x > K_{i+1}) + R \cdot 1(x \leq K_{i+1}), i = 0, \dots, N - 1. \end{aligned} \tag{52}$$

Here $u_N(x, t) \equiv 1$. We use the change of unknown function

$$u_i = (1 - R)W_i + R, i = 0, \dots, N - 1. \tag{53}$$

Then the problem (52) is changed into the following one.

$$\begin{aligned} \frac{\partial W_i}{\partial t} + \frac{s_V^2}{2} x^2 \frac{\partial^2 W_i}{\partial x^2} - bx \frac{\partial W_i}{\partial x} - \lambda_i W_i &= 0, \quad t_i < t < t_{i+1}, x > 0, \\ W_i(x, t_{i+1}) &= W_{i+1}(x, t_{i+1})1(x > K_{i+1}), \quad x > 0, \quad i = 0, \dots, N - 1. \end{aligned} \tag{54}$$

Here $W_N(x, t) \equiv 1$. These equations are simpler than ones in theorem 1 (note that (54) are *homogenous* Black-Scholes equations) and we can easily solve them with the same method in the above to get (28) and (27).

6. CONCLUSIONS

In this paper we studied the pricing of defaultable bond with discrete default intensity and barrier under constant risk free short rate using higher order binary options ([5, 10, 11]) and their integrals. We considered both *endogenous* and *exogenous* default recovery. Our pricing problem is derived to a solving problem of *inhomogeneous* or *homogeneous Black-Scholes PDEs* with *different coefficients* and terminal value of binary type in every subinterval between the two adjacent announcing dates. See (21) and (26). In order to deal with the difference of coefficients in subintervals we used a *relation* (11) between prices of higher order binaries with different coefficients. In our model, due to the inhomogenous term related to endogenous recovery, our bond prices are represented by *not only* the prices of higher binary options *but also* the *integrals* of them. See the formulae (23) and (24)(3.8). So first we provided the pricing formulae (corollary of lemma 2) of a *special binary option* called *integral of i-th binary or nothing* and then we obtain the pricing formulae of our defaultable corporate bond by using the pricing formulae of higher binary options and integrals of them and provided illasration of the effect of parameters on the price of corporate bond and the credit spread.

Acknowledgment Authors would like to thank Prof. Ahmed El-Sayed and anonymous reviewers for their help and precious advice.

REFERENCES

- [1] Agliardi R., *A Comprehensive Structural Model for Defaultable Fixed-income Bonds*, Quantitative Finance, **11:5**, 2011, 749-762, DOI:10.1080/14697680903222451
- [2] Agliardi E. and Agliardi R., *Bond Pricing under Imprecise Information*, Operational Research International Journal, **11**, 299-309, 2011, DOI: 10.1007/s12351-010-0087-x
- [3] Ballestra, L., G Pacelli, *Pricing defaultable bonds: a new model combining structural information with the reduced-form approach*, March 11, 2008, Available at SSRN, DOI: 10.2139/ssrn.1492665
- [4] BI, Yusheng and BIAN, Baojun, *Pricing Corporate Bonds with Both Expected and Unexpected Defaults*, Journal of Tongji University(Natural Science) Vol. **35** No. **7**, 989-993, July, 2007
- [5] Buchen, P.; *The Pricing of dual-expiry exotics*, Quantitative Finance, **4**, 2004, 101-108, DOI: 10.1088/1469-7688/4/1/009
- [6] Cathcart, L., L El-Jahel, *Semi-analytical pricing of defaultable bonds in a signaling jump-default model*, Journal of Computational Finance, **6, 3**, 91-108, 2003, Crossref
- [7] Cathcart, L., Lina El-Jahel, *Pricing defaultable bonds: a middle-way approach between structural and reduced-form models*, Quantitative Finance, Volume **6**, Issue **3**, 243-253, 2006, DOI: 10.1080/14697680600670754
- [8] Jiang, Li-shang, *Mathematical Models and Methods of Option Pricing*, World Scientific, 1-329, 2005
- [9] O, Hyong-chol et al., *Pricing Corporate Defaultable Bond using Declared Firm Value*, Electronic Journal of Mathematical Analysis and Applications, Vol.**2(1)** Jan 2014, pp.1-11, cross-ref, arXiv preprint, pp.1-12, arXiv:1302.3654[q-fin.PR]
- [10] O, Hyong-chol and Kim, Mun-chol ; *Higher Order Binary Options and Multiple Expiry Exotics*, Electronic Journal of Mathematical Analysis and Applications, Vol. **1(2)** July 2013, pp. 247-259, cross-ref
- [11] O, Hyong-chol and Kim, Mun-chol; *The Pricing of Multiple Expiry Exotics*, arXiv preprint, pp 1-16, arXiv:1302.3319[q-fin.PR]
- [12] O, Hyong-chol and WAN, Ning, *Analytical Pricing of Defaultable Bond with Stochastic Default Intensity*, Derivatives eJournal, 05/2005, 1-35, DOI:10.2139/ssrn.723601, arXiv preprint, arXiv:1303.1298[q-fin.PR]
- [13] O, Hyong-Chol, Wan, Ning and Ren, Xue-Min, *Pricing Corporate Defaultable Bond with Fixed Discrete Declaration Time of Firm Value*, Derivatives eJournal 05/2005, 1-14, DOI: 10.2139/ssrn.723562
- [14] Realdon, M., *Credit risk pricing with both expected and unexpected default*, Applied Financial Economics Letters, Vol. **3**, No.**4**, 225-230, 2007, DOI:10.1080/17446540600993837
- [15] Wilmott, P., *Derivatives: the Theory and Practice of Financial Engineering*, John Wiley & Sons. Inc., 360-583, 1998

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