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FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper, we obtain Fekete-Szegö inequalities for a certain class of analytic functions $f(z)$ for which

$$1+\frac{1}{b}\left[\frac{z\left(f*g\right)'(z)+\gamma z^{2}\left(f*g\right)''(z)}{\left(1-\gamma\right)\left(f*g\right)(z)+\gamma z\left(f*g\right)'(z)}-1\right]\prec\Phi(z)$$

.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_{k}z^{k}\ \left(z\in U=\{z:z\in\mathbb{C}\text{ and }|z|<1\}\right) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$, which are univalent functions.

Let $g(z)\in\mathcal{S}$, be given by

$$g(z)=z+\sum_{k=2}^{\infty}b_{k}z^{k}, \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_{k}b_{k}z^{k}=(g*f)(z). \tag{1.3}$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$ if there exists a Schwarz function w, which is analytic in U with $w(0)=0$ and $|w(z)|<1$ for all $z\in U$, such that $f(z)=g(w(z))$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [2] and [11]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)\text{ and }f(U)\subset g(U).$$

For complex parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ $(\beta_j\notin\mathbb{Z}_0^-=\{0,-1,-2,...\}$; $j=1,2,...,s)$, we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,$

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$..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, [18, p.19])

$$ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k} . \frac{z^k}{k!} $$

$$ (q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, ...\}; z \in U), $$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$ (\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0;\ \theta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}), \\ \theta(\theta+1)....(\theta+\nu-1) & (\nu \in \mathbb{N};\ \theta \in \mathbb{C}). \end{cases} \tag{1.4} $$

Corresponding to the function $h_{q,s}(\alpha_1, \beta_1; z) = h(\alpha_1, ...\alpha_q; \beta_1 ..., \beta_s; z)$, defined by

$$ \begin{aligned} h_{q,s}(\alpha_1, \beta_1; z) &= z {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \\ &= z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k, \end{aligned} \tag{1.5} $$

where

$$ \Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} ...... (\alpha_q)_{k-1}}{(\beta_1)_{k-1} ... (\beta_s)_{k-1} (k-1)!}. \tag{1.6} $$

In [8] El-Ashwah and Aouf defined the operator $I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1) f(z)$ as follows:

$$ I_{q,s,\lambda}^{0,\ell}(\alpha_1, \beta_1) f(z) = f(z) * h_{q,s}(\alpha_1, \beta_1; z); $$

$$ I_{q,s,\lambda}^{1,\ell}(\alpha_1, \beta_1) f(z) = (1-\lambda)(f(z) * h_{q,s}(\alpha_1, \beta_1; z)) + \frac{\lambda}{(1+\ell) z^{\ell-1}} \left[z^\ell \left(f(z) * h_{q,s}(\alpha_1, \beta_1; z) \right) \right]'; $$

and

$$ I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1) f(z) = I_{q,s,\lambda}^{1,\ell}(I_{q,s,\lambda}^{m-1,\ell}(\alpha_1, \beta_1) f(z)). \tag{1.7} $$

If $f \in A$, then from (1.1) and (1.7), we can easily see that

$$ I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1) f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell} \right]^m \Gamma_k(\alpha_1) a_k z^k, \tag{1.8} $$

where $m \in \mathbb{Z} = \{0, \pm 1, ...\}, \ell \geq 0$ and $\lambda \geq 0$.

We note that when $\ell = 0$, the operator $I_{q,s,\lambda}^{m,0}(\alpha_1, \beta_1) f(z) = D_\lambda^m(\alpha_1, \beta_1) f(z)$ was studied by Selvaraj and Karthikeyan [16].

We also note that:

(i) $I_{q,s,\lambda}^{0,\ell} f(z) = H_{q,s}(\alpha_1, \beta_1) f(z)$ (see Dziok and Srivastava [6,7]);

(ii) For $q = s+1,\ \alpha_i = 1 (i = 1, ..., s+1)$ and $\beta_j = 1 (j = 1, ..., s)$, we get the operator $I(m, \lambda, \ell)$ (see Catas [3], Prajapat [12] and El-Ashwah and Aouf [9]);

(iii) For $q = s+1,\ \alpha_i = 1 (i = 1, ..., s+1),\ \beta_j = 1 (j = 1, ..., s),\ \lambda = 1$ and $\ell = 0$, we obtain the Salagean operator D^m (see Salagean [15]);

(iv) For $q = s+1,\ \alpha_i = 1 (i = 1, ..., s+1),\ \beta_j = 1 (j = 1, ..., s)$ and $\lambda = 1$, we get the operator I_ℓ^m(see Cho and Srivastava [4] and Cho and Kim [5]).

(v) For $q = s+1, \alpha_i = 1 (i = 1, ..., s+1), \beta_j = 1 (j = 1, ..., s)$ and $\ell = 0$, we obtain the operator D_λ^m(see Al-Oboudi [1]).

By specializing the parameters $m, \lambda, \ell, q, s, \alpha_i$ $(i = 1, .., q)$ and β_j $(j = 1, ..., s)$, we obtain:

(i) $I_{2,1,\lambda}^{m,\ell}(n+1,1;1)f(z) = I_{\lambda}^{m,\ell}(n)f(z) = z+\sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(n+1)_{k-1}}{(1)_{k-1}}a_k z^k$ $(n > -1)$;

(ii) $I_{2,1,\lambda}^{m,\ell}(a,1;c)f(z) = I_{\lambda}^{m,\ell}(a;c)f(z) = z+\sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k$ $(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-)$;

(iii) $I_{2,1,\lambda}^{m,\ell}(2,1;n+1)f(z) = I_{\lambda,n}^{m,\ell}f(z) = z+\sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(2)_{k-1}}{(n+1)_{k-1}}a_k z^k$ $(n \in \mathbb{Z}; n > -1)$.

In this paper, we define the following class $N^{\gamma}(g,b;\Phi)$ $(b \in \mathbb{C}^*;\ 0 \le \gamma \le 1)$ as follows:

Definition 1. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ...$ be univaent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis. Let $b \in \mathbb{C}^*$, $B_1 > 0$ and $g(z)$ be given by (1.2). Then functions $f(z) \in \mathcal{A}$ is in the class $N^{\gamma}(g,b;\Phi)$ if

$$1+\frac{1}{b}\left[\frac{z\left(f*g\right)^{'}(z)+\gamma z^2\left(f*g\right)^{''}(z)}{(1-\gamma)\left(f*g\right)(z)+\gamma z\left(f*g\right)^{'}(z)}-1\right] \prec \Phi(z)\ \ (b \in \mathbb{C}^*;\ 0 \le \gamma \le 1;\ z \in U). \tag{1.9}$$

We note that for suitable choices of $b,\ \gamma,\ g(z)$ and $\Phi(z)$ we obtain the following subclasses:

(i) $N^0\left(\frac{z}{1-z},b;\Phi\right) = S_b^*(\Phi)$ and $N^1\left(b,\frac{z}{1-z};\Phi\right) = C_b(\Phi)$ $(b \in \mathbb{C}^*)$ (see Ravichandran et al. [14]);

(ii) $N^0\left(z+\sum_{k=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(k-1)+1\right]^m z^k,1;\Phi\right) = M_{\alpha,\beta,\lambda,\delta}^m(\Phi)$ $(\alpha,\beta,\lambda,$ $\delta \ge 0,\ \lambda > \delta,\ \beta > \alpha,\ m \in \mathbb{N})$ (see Ramadan and Darus [13]);

(iii) $N^{\gamma}\left(\frac{z}{1-z},1;\Phi\right) = M_{\gamma}(\Phi)$ $(0 \le \gamma \le 1)$ (see Shanmugam and Sivasubramanian [17]).

Also, we note that:

(i) $N^{\gamma}(z+\sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)z^k,b;\Phi) = N_b^{\gamma}(\lambda,\ell,m,q,s,\alpha_1,\beta_1;\Phi)$

$$=\left\{f(z)\in\mathcal{A}:1+\frac{1}{b}\left[\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)\right)^{'}+\gamma z^2\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)\right)^{''}}{(1-\gamma)\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)\right)+\gamma z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)\right)^{'}}-1\right]\prec\Phi(z),\right.$$

$$\left.(b \in \mathbb{C}^*;\ 0 \le \gamma \le 1;\ m \in \mathbb{N}_0;\ \ell \ge 0;\ \lambda \ge 0;\ q \le s+1;\ q,s \in \mathbb{N}_0;\ z \in U)\right\};$$

(ii) $N^{\gamma}(z+\sum_{k=2}^{\infty}\left[\frac{1+\ell}{1+\ell+\lambda(k-1)}\right]^m z^k,b;\Phi) = N_b^{\gamma}(\lambda,\ell,m;\Phi)$

$$=\left\{f(z)\in\mathcal{A}:1+\frac{1}{b}\left[\frac{z\left(J^m(\lambda,\ell)f(z)\right)^{'}+\gamma z^2\left(J^m(\lambda,\ell)f(z)\right)^{''}}{(1-\gamma)\left(J^m(\lambda,\ell)f(z)\right)+\gamma z\left(J^m(\lambda,\ell)f(z)\right)^{'}}-1\right]\prec\Phi(z),\right.$$

$$\left.(b \in \mathbb{C}^*;\ 0 \le \gamma \le 1;\ m \in \mathbb{N}_0;\ \ell \ge 0;\ \lambda \ge 0;\ z \in U)\right\};$$

(iii) $N^{\gamma}\left(g,(1-\rho)\cos\eta e^{-i\eta};\frac{1+Az}{1+Bz}\right)=N^{\gamma}\left[\rho,\eta,A,B,g\right]$

$$=\left\{f(z)\in\mathcal{A}:e^{i\eta}\left[\frac{z\left(f*g\right)^{'}(z)+\gamma z^{2}\left(f*g\right)^{''}(z)}{(1-\gamma)\left(f*g\right)(z)+\gamma z\left(f*g\right)^{'}(z)}\right]\prec(1-\rho)\cos\eta\cdot\frac{1+Az}{1+Bz}+\rho\cos\eta+i\sin\eta,\right.$$

$$\left.\left(|\eta|<\frac{\pi}{2};\ 0\leq\gamma\leq1;\ 0\leq\rho<1;\ -1\leq B<A\leq1;\ z\in U\right)\right\};$$

(iv) $N^{\gamma}(z+\sum\limits_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^{m}\Gamma_{k}(\alpha_{1})z^{k},(1-\rho)\cos\eta e^{-i\eta};\frac{1+Az}{1+Bz})=N_{\lambda,\ell,q,s}^{\gamma,m}\left[\rho,\eta,A,B,\alpha_{1},\beta_{1}\right]$

$$=\left\{f(z)\in\mathcal{A}:e^{i\eta}\left[\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_{1},\beta_{1})f(z)\right)^{'}+\gamma z^{2}\left(I_{q,s,\lambda}^{m,\ell}(\alpha_{1},\beta_{1})f(z)\right)^{''}}{(1-\gamma)\left(I_{q,s,\lambda}^{m,\ell}(\alpha_{1},\beta_{1})f(z)\right)+\gamma z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_{1},\beta_{1})f(z)\right)^{'}}\right]\prec(1-\rho)\cos\eta\cdot\frac{1+Az}{1+Bz}+\rho\cos\eta+i\sin\eta,\right.$$

$$\left.\left(|\eta|<\frac{\pi}{2};0\leq\gamma\leq1;0\leq\rho<1;-1\leq B<A\leq1;m\in\mathbb{N}_{0};\ell\geq0\ \lambda\geq0;q,s\in\mathbb{N}_{0};z\in U\right)\right\};$$

(v) $N^{0}\left(g,b;\Phi\right)=S_{b}^{*}\left(g;\Phi\right)$

$$1+\frac{1}{b}\left[\frac{z\left(f*g\right)^{'}(z)}{\left(f*g\right)(z)}-1\right]\prec\Phi(z)\quad\left(b\in\mathbb{C}^{*};\ z\in U\right);$$

(vi) $N^{1}\left(g,b;\Phi\right)=C_{b}\left(g;\Phi\right)$

$$1+\frac{1}{b}\frac{z\left(f*g\right)^{''}(z)}{\left(f*g\right)^{'}(z)}\prec\Phi(z)\quad\left(b\in\mathbb{C}^{*};\ z\in U\right).$$

In this paper, we obtain the Fekete-Szegö inequalities for the functions in the class $N^{\gamma}\left(g,b;\Phi\right)$.

2. Fekete-Szegö problem

To prove our results, we need the following lemmas.

Lemma 1 [10]. If $p(z)=1+c_{1}z+c_{2}z^{2}+.....$ is an analytic function with positive real part in U and ν is a complex number, then

$$\left|c_{2}-\nu c_{1}^{2}\right|\leq2\max\{1;|2\nu-1|\}.$$

The result is sharp for the functions given by

$$p(z)=\frac{1+z^{2}}{1-z^{2}}\text{ and }p(z)=\frac{1+z}{1-z}.$$

Lemma 2 [10]. If $p(z)=1+c_{1}z+c_{2}z^{2}+.....$ is an analytic function with positive real part in U, then

$$\left|c_{2}-\nu c_{1}^{2}\right|\leq\begin{cases}-4\nu+2, & if\ \nu\leq0,\\ 2, & if\ 0\leq\nu\leq1,\\ 4\nu-2, & if\ \nu\geq1.\end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z) = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = (\frac{1}{2} + \frac{1}{2}\vartheta)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\vartheta)\frac{1-z}{1+z} \ (0 \le \vartheta \le 1),$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if

$$\frac{1}{p(z)} = (\frac{1}{2} + \frac{1}{2}\vartheta)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\vartheta)\frac{1-z}{1+z} \ (0 \le \vartheta \le 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \le 2 \qquad (0 < \nu < \frac{1}{2}),$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \le 2 \quad (\frac{1}{2} < \nu < 1).$$

Unless otherwise mentioned, we assume throughout this paper that :
$0 \le \gamma \le 1$, $z \in U$ and $g(z)$ given by (1.2).

Theorem 1. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \ldots\ldots\ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $N^{\gamma}(g, b; \Phi)$ and $b \in \mathbb{C}^*$, then

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b|\, B_1}{2(1+2\gamma) b_3} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + bB_1 - \frac{2\mu b B_1 (1+2\gamma) b_3}{(1+\gamma)^2 b_2^2}\right|\right\}. \tag{2.1}$$

The result is sharp

Proof. Let $f(z) \in N^{\gamma}(g, b; \Phi)$, then there is a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b}\left[\frac{z(f*g)'(z) + \gamma z^2 (f*g)''(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)'(z)} - 1\right] = \Phi(w(z)). \tag{2.2}$$

If the function $p_1(z)$ is analytic and has positive real part in U and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots\ldots. \tag{2.3}$$

Since $w(z)$ is a Schwarz function. Define

$$p(z) = 1 + \frac{1}{b}\left[\frac{z(f*g)'(z) + \gamma z^2 (f*g)''(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)'(z)} - 1\right] = 1 + d_1 z + d_2 z^2 + \ldots\ldots (z \in U). \tag{2.4}$$

In view of the equations (2.2) and (2.3), we have

$$p(z) = \Phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right).$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\left(c_3+\frac{c_1^3}{4}-c_1c_2\right)z^3+......\right].$$

Therefore

$$\Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right]z^2+......, \tag{2.5}$$

and from (2.5), we obtain

$$d_1=\tfrac{1}{2}B_1c_1 \quad \text{and} \quad d_2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2.$$

Then, from (2.4), we see that

$$d_1=\frac{(1+\gamma)\,a_2b_2}{b} \quad \text{and} \quad d_2=\frac{2\,(1+2\gamma)\,a_3b_3}{b}-\frac{(1+\gamma)^2\,a_2^2b_2^2}{b}. \tag{2.6}$$

Now from (2.4), (2.5) and (2.6), we have

$$a_2=\frac{bB_1c_1}{2\,(1+\gamma)\,b_2},$$

and

$$a_3=\frac{bB_1}{4\,(1+2\gamma)\,b_3}\left\{c_2-\frac{c_1^2}{2}+\frac{1}{2}\frac{B_2}{B_1}c_1^2+\frac{1}{2}bB_1c_1^2\right\}.$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{bB_1}{4\,(1+2\gamma)\,b_3}\left\{c_2-\nu c_1^2\right\}, \tag{2.7}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}-bB_1+\frac{2\mu bB_1\,(1+2\gamma)\,b_3}{(1+\gamma)^2\,b_2^2}\right]. \tag{2.8}$$

Our result now follows by an application of Lemma 1. The result is sharp for the functions

$$1+\frac{1}{b}\left[\frac{z\,(f*g)^{'}(z)+\gamma z^2\,(f*g)^{''}(z)}{(1-\gamma)\,(f*g)\,(z)+\gamma z\,(f*g)^{'}(z)}-1\right]=\Phi(z^2) \tag{2.9}$$

and

$$1+\frac{1}{b}\left[\frac{z\,(f*g)^{'}(z)+\gamma z^2\,(f*g)^{''}(z)}{(1-\gamma)\,(f*g)\,(z)+\gamma z\,(f*g)^{'}(z)}-1\right]=\Phi(z) \tag{2.10}$$

This completes the proof of Theorem 1.

Putting $g(z)=\frac{z}{1-z}$ and $\gamma=0$ in Theorem 1, we obtain the following corollary improving the result obtained by Ravichandran et al [14, Theorem 4.1].

Corollary 1. Let $\Phi(z)=1+B_1z+B_2z^2+.....\ (B_1>0)$. If $f(z)$ given by (1.1) belongs to the class $S_b^*(\Phi)$ and $b\in\mathbb{C}^*$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{|b|\,B_1}{2}\cdot\max\left\{1,\left|\frac{B_2}{B_1}+(1-2\mu)\,bB_1\right|\right\}. \tag{2.11}$$

The result is sharp.

Putting $g(z) = \frac{z}{1-z}$ and $\gamma = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ... \ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $C_b(\Phi)$ and $b \in \mathbb{C}^*$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b| B_1}{6} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{3}{2}\mu\right) b B_1\right|\right\}. \tag{2.12}$$

The result is sharp.

Putting $g(z) = \frac{z}{1-z}$ and $b = 1$ in Theorem 1, we obtain the following corollary:

Corollary 3. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ... \ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $M_\gamma(\Phi)$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{2(1+2\gamma)} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + B_1 - \frac{2\mu B_1 (1+2\gamma)}{(1+\gamma)^2}\right|\right\}. \tag{2.13}$$

The result is sharp.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[(\lambda - \delta)(\beta - \alpha)(k-1) + 1\right]^m z^k \ (\alpha, \beta, \lambda, \delta \geq 0, \ \lambda > \delta, \ \beta > \alpha, \ m \in \mathbb{N}_0)$, $b = 1$ and $\gamma = 0$ in Theorem 1, we obtain the following corollary.

Corollary 4. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ... \ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $M^m_{\alpha,\beta,\lambda,\delta}(\Phi)$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^m} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{2\mu\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^m}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2m}}\right) B_1\right|\right\}. \tag{2.14}$$

The result is sharp.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) z^k \ (m \in \mathbb{Z}; \ \ell \geq 0; \ \lambda \geq 0;$ $q \leq s+1; \ q, s \in \mathbb{N}_0$ and $\Gamma_k(\alpha_1)$ be given by (1.6)) in Theorem 1, we obtain the following corollary:

Corollary 5. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ... \ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $N_b^\gamma(\lambda, \ell, m, q, s, \alpha_1, \beta_1; \Phi)$ and $b \in \mathbb{C}^*$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b| B_1 (1+\ell)^m}{2(1+2\gamma)(1+\ell+2\lambda)^m \Gamma_3(\alpha_1)} \cdot$$
$$\cdot \max\left\{1, \left|\frac{B_2}{B_1} + bB_1 - \frac{2\mu b B_1 (1+2\gamma)}{(1+\gamma)^2} \cdot \frac{(1+\ell+2\lambda)^m (1+\ell)^m \Gamma_3(\alpha_1)}{(1+\ell+\lambda)^{2m} \left(\Gamma_2(\alpha_1)\right)^2}\right|\right\}. \tag{2.15}$$

The result is sharp.

Putting $\gamma = 0$ in Theorem 1, we obtain the following corollary.

Corollary 6. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ... \ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $S_b^*(g; \Phi)$ and $b \in \mathbb{C}^*$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b| B_1}{2b_3} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{2\mu b_3}{b_2^2}\right) b B_1\right|\right\}. \tag{2.16}$$

The result is sharp.

Putting $\gamma = 1$ in Theorem 1, we obtain the following corollary.

Corollary 7. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ...\ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $C_b\,(g;\Phi)$ and $b \in \mathbb{C}^*$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,B_1}{6b_3} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{3\mu b_3}{2b_2^2}\right) bB_1\right|\right\}. \tag{2.17}$$

The result is sharp.

Putting $b = (1-\rho)\,e^{-i\eta}\cos\eta\ \left(|\eta| < \frac{\pi}{2},\ 0 \leq \rho < 1\right)$ and $\Phi(z) = \frac{1+Az}{1+Bz}\ (-1 \leq B < A \leq 1)$ in Theorem 1, we obtain the following corollary:

Corollary 8. If $f(z)$ given by (1.1) belongs to the class $N^{\gamma}\,[\rho,\eta,A,B,g]$, then

$$\left|a_3 - a_2^2\right| \leq \frac{(A-B)\,(1-\rho)\cos\eta}{2\,(1+2\gamma)\,b_3} \cdot \tag{2.18}$$

$$\cdot \max\left\{1, \left|-B + (A-B)\,(1-\rho)\,e^{-i\eta}\cos\eta - \frac{2\mu(A-B)\,(1-\rho)\,e^{-i\eta}\cos\eta\,(1+2\gamma)\,b_3}{(1+\gamma)^2\,b_2^2}\right|\right\}.$$

The result is sharp.

By using Lemma 2, we can obtain the following theorem.

Theorem 2. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ...\ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $N^{\gamma}\,(g,b;\Phi)$ and $b > 0$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{bB_1}{2\,(1+2\gamma)\,b_3}\left[\dfrac{B_2}{B_1} + bB_1 - \dfrac{2\mu bB_1\,(1+2\gamma)\,b_3}{(1+\gamma)^2\,b_2^2}\right] & \text{if } \mu \leq \sigma_1, \\ \dfrac{bB_1}{2\,(1+2\gamma)\,b_3} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{-bB_1}{2\,(1+2\gamma)\,b_3}\left[\dfrac{B_2}{B_1} + bB_1 - \dfrac{2\mu bB_1\,(1+2\gamma)\,b_3}{(1+\gamma)^2\,b_2^2}\right] & \text{if } \mu \geq \sigma_2, \end{cases} \tag{2.19}$$

where

$$\sigma_1 = \frac{(1+\gamma)^2\,b_2^2}{2bB_1\,(1+2\gamma)\,b_3}\left[-1 + \frac{B_2}{B_1} + bB_1\right],$$

and

$$\sigma_2 = \frac{(1+\gamma)^2\,b_2^2}{2bB_1\,(1+2\gamma)\,b_3}\left[1 + \frac{B_2}{B_1} + bB_1\right].$$

The result is sharp.

Proof. To show that the bounds are sharp, we define the functions $K_{\varphi_\delta}(\delta \geq 2)$ by

$$1+\frac{1}{b}\left[\frac{z\,(K_{\varphi_\delta} * g)^{'}\,(z) + \gamma z^2\,(K_{\varphi_\delta} * g)^{''}\,(z)}{(1-\gamma)\,(K_{\varphi_\delta} * g)\,(z) + \gamma z\,(K_{\varphi_\delta} * g)^{'}\,(z)} - 1\right] = \Phi(z^{\delta-1}),\ K_{\varphi_\delta}(0) = 0 = K_{\varphi_\delta}^{'}(0) - 1,$$

and the functions F_ρ and G_ρ $(0 \leq \rho \leq 1)$ by

$$1+\frac{1}{b}\left[\frac{z\,(F_\rho * g)^{'}\,(z) + \gamma z^2\,(F_\rho * g)^{''}\,(z)}{(1-\gamma)\,(F_\rho * g)\,(z) + \gamma z\,(F_\rho * g)^{'}\,(z)} - 1\right] = \Phi\left(\frac{z(z+\rho)}{1+\rho z}\right),\ F_\rho(0) = 0 = F_\rho^{'}(0) - 1,$$

and

$$1+\frac{1}{b}\left[\frac{z\left(G_\rho * g\right)^{'}(z)+\gamma z^2\left(G_\rho * g\right)^{''}(z)}{(1-\gamma)\left(G_\rho * g\right)(z)+\gamma z\left(G_\rho * g\right)^{'}(z)}-1\right]=\Phi\left(-\frac{z(z+\rho)}{1+\rho z}\right),\ G_\rho(0)=0=G_\rho^{'}(0)-1.$$

Cleary the functions $K_{\varphi_\delta}, F_\rho$ and $G_\rho \in N^\gamma(g, b; \Phi)$. Also we write $K_\varphi = K_{\varphi_2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is K_{φ_3} or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_ρ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_ρ or one of its rotations. If $\sigma_1 \le \mu \le \sigma_2$, in view of Lemma. This completes the proof of Theorem 2.

Remark 1. Putting $g(z) = z + \sum_{k=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(k-1)+1\right]^m z^k(\alpha, \beta, \lambda,$ $\delta \ge 0, \lambda > \delta, \beta > \alpha, m \in \mathbb{N})$, $b = 1$ and $\gamma = 0$ in Theorem 2, we obtain the result obtained by Ramadan and Darus [13,Theorem 1].

Putting $g(z) = \frac{z}{1-z}$ and $b = 1$ in Theorem 2, we obtain the following corollary improving the result obtained by Shanmugam and Sivasubramanian [17, Theorem 2.1].

Corollary 9. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \ldots..\ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $M_\gamma(\Phi)$, then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{B_1}{2(1+2\gamma)}\left[\dfrac{B_2}{B_1}+B_1-\dfrac{2\mu B_1(1+2\gamma)}{(1+\gamma)^2}\right] & \text{if } \mu \le \eta_1, \\ \dfrac{B_1}{2(1+2\gamma)} & \text{if } \eta_1 \le \mu \le \eta_2, \\ \dfrac{-B_1}{2(1+2\gamma)}\left[\dfrac{B_2}{B_1}+B_1-\dfrac{2\mu B_1(1+2\gamma)}{(1+\gamma)^2}\right] & \text{if } \mu \ge \eta_2, \end{cases}$$

where

$$\eta_1 = \frac{(1+\gamma)^2}{2B_1(1+2\gamma)}\left[-1+\frac{B_2}{B_1}+B_1\right],$$

and

$$\eta_2 = \frac{(1+\gamma)^2}{2B_1(1+2\gamma)}\left[1+\frac{B_2}{B_1}+B_1\right].$$

The result is sharp.

Putting $g(z) = \frac{z}{1-z}$ and $\gamma = 0$ in Theorem 2, we obtain the following corollary:

Corollary 10. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \ldots..\ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $S_b^*(\Phi)$ and $b > 0$, then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{bB_1}{2}\left[\dfrac{B_2}{B_1}+(1-2\mu)bB_1\right] & \text{if } \mu \le \sigma_1^*, \\ \dfrac{bB_1}{2} & \text{if } \sigma_1^* \le \mu \le \sigma_2^*, \\ \dfrac{-bB_1}{2}\left[\dfrac{B_2}{B_1}+(1-2\mu)bB_1\right] & \text{if } \mu \ge \sigma_2^*, \end{cases}$$

where

$$\sigma_1^* = \frac{1}{2bB_1}\left[-1+\frac{B_2}{B_1}+bB_1\right],$$

and

$$\sigma_2^* = \frac{1}{2bB_1}\left[1 + \frac{B_2}{B_1} + bB_1\right].$$

The result is sharp.

Putting $g(z) = \frac{z}{1-z}$ and $\gamma = 1$ in Theorem 2, we obtain the following corollary:

Corollary 11. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ...\ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $C_b(\Phi)$ and $b > 0$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{bB_1}{6}\left[\dfrac{B_2}{B_1} + \left(1 - \frac{3}{2}\mu\right) bB_1\right] & \text{if } \mu \le \sigma_1^{**}, \\ \dfrac{bB_1}{6} & \text{if } \sigma_1^{**} \le \mu \le \sigma_2^{**}, \\ \dfrac{-bB_1}{6}\left[\dfrac{B_2}{B_1} + \left(1 - \frac{3}{2}\mu\right) bB_1\right] & \text{if } \mu \ge \sigma_2^{**}, \end{cases}$$

where

$$\sigma_1^{**} = \frac{2}{3bB_1}\left[-1 + \frac{B_2}{B_1} + bB_1\right],$$

and

$$\sigma_2^{**} = \frac{2}{3bB_1}\left[1 + \frac{B_2}{B_1} + bB_1\right].$$

The result is sharp.

Putting $g(z) = z + \sum\limits_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) z^k$ $(m \in \mathbb{Z};\ \ell \ge 0;\ \lambda \ge 0;$ $q \le s+1;\ q, s \in \mathbb{N}_0)$ and $\Gamma_k(\alpha_1)$ be given by (1.6) in Theorem 2, we obtain the following corollary:

Corollary 12. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ...\ (B_1 > 0)$. If $f(z)$ given by (1.1) belongs to the class $N_b^{\gamma}(\lambda, \ell, m, q, s, \alpha_1, \beta_1; \Phi)$ and $b > 0$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{bB_1}{2(1+2\gamma)\Gamma_3(\alpha_1)}\left[\frac{1+\ell}{1+\ell+2\lambda}\right]^m\left[\frac{B_2}{B_1} + bB_1 - \frac{2\mu bB_1(1+2\gamma)}{(1+\gamma)^2 b_2^2}\frac{(1+\ell+2\lambda)^m(1+\ell)^m}{(1+\ell+\lambda)^{2m}}\frac{\Gamma_3(\alpha_1)}{(\Gamma_2(\alpha_1))^2}\right] \\ \qquad \text{if } \mu \le \chi_1, \\ \frac{bB_1}{2(1+2\gamma)\Gamma_3(\alpha_1)}\left[\frac{1+\ell}{1+\ell+2\lambda}\right]^m \qquad \text{if } \chi_1 \le \mu \le \chi_2, \\ -\frac{bB_1}{2(1+2\gamma)\Gamma_3(\alpha_1)}\left[\frac{1+\ell}{1+\ell+2\lambda}\right]^m\left[\frac{B_2}{B_1} + bB_1 - \frac{2\mu bB_1(1+2\gamma)}{(1+\gamma)^2 b_2^2}\frac{(1+\ell+2\lambda)^m(1+\ell)^m}{(1+\ell+\lambda)^{2m}}\frac{\Gamma_3(\alpha_1)}{(\Gamma_2(\alpha_1))^2}\right] \\ \qquad \text{if } \mu \ge \chi_2, \end{cases}$$

where

$$\chi_1 = \frac{(1+\gamma)^2}{2bB_1(1+2\gamma)}\frac{(1+\ell+\lambda)^{2m}}{(1+\ell+2\lambda)^m(1+\ell)^m}\frac{(\Gamma_2(\alpha_1))^2}{\Gamma_3(\alpha_1)}\left[-1 + \frac{B_2}{B_1} + bB_1\right],$$

and

$$\chi_2 = \frac{(1+\gamma)^2}{2bB_1(1+2\gamma)}\frac{(1+\ell+\lambda)^{2m}}{(1+\ell+2\lambda)^m(1+\ell)^m}\frac{(\Gamma_2(\alpha_1))^2}{\Gamma_3(\alpha_1)}\left[1 + \frac{B_2}{B_1} + bB_1\right].$$

The result is sharp.

The proof of Theorem 3 is similar to the proof Theorem 2, so the details are omitted.

Theorem 3. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + ...\ (B_1 > 0)$. Let $f(z)$ given by (1.1) belongs to the class $N^{\gamma}(g, b; \Phi)$ and $b > 0$. Let σ_3 be given by

$$\sigma_3 = \frac{(1+\gamma)^2 b_2^2}{2bB_1(1+2\gamma) b_3}\left[\frac{B_2}{B_1} + bB_1\right]. \tag{2.20}$$

If $\sigma_1 < \mu \leq \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{(1+\gamma)^2 b_2^2}{2bB_1(1+2\gamma) b_3}\left[1 - \frac{B_2}{B_1} - bB_1 + \frac{2\mu b B_1(1+2\gamma) b_3}{(1+\gamma)^2 b_2^2}\right]|a_2|^2 \leq \frac{bB_1}{2(1+2\gamma) b_3}. \tag{2.21}$$

If $\sigma_3 < \mu \leq \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{(1+\gamma)^2 b_2^2}{2bB_1(1+2\gamma) b_3}\left[1 + \frac{B_2}{B_1} + bB_1 - \frac{2\mu b B_1(1+2\gamma) b_3}{(1+\gamma)^2 b_2^2}\right]|a_2|^2 \leq \frac{bB_1}{2(1+2\gamma) b_3}, \tag{2.22}$$

where σ_1 and σ_2 are given in Theorem 2. The result is sharp.

Remark 2. Putting $g(z) = z + \sum_{k=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(k-1)+1\right]^m z^k$ $(\alpha, \beta, \lambda, \delta \geq 0,\ \lambda > \delta,\ \beta > \alpha,\ m \in \mathbb{N}_0)$, $b = 1$ and $\gamma = 0$ in Theorem 3, we obtain the result obtained by Ramadan and Darus [13,Remark 2].

Putting $g(z) = \frac{z}{1-z}$ and $b = 1$ in Theorem 3, we obtain the following corollary improving the result obtained by Shanmugam and Sivasubramanian [17, Remark 1].

Corollary 13. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots .. \ (B_1 > 0)$. Let $f(z)$ given by (1.1) belongs to the class $M_\gamma(\Phi)$. Let σ_3 be given by

$$\eta_3 = \frac{(1+\gamma)^2}{2B_1(1+2\gamma)}\left[\frac{B_2}{B_1} + B_1\right]. \tag{2.20}$$

If $\eta_1 < \mu \leq \eta_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{(1+\gamma)^2}{2B_1(1+2\gamma)}\left[1 - \frac{B_2}{B_1} - B_1 + \frac{2\mu B_1(1+2\gamma)}{(1+\gamma)^2}\right]|a_2|^2 \leq \frac{B_1}{2(1+2\gamma)}. \tag{2.21}$$

If $\eta_3 < \mu \leq \eta_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{(1+\gamma)^2}{2B_1(1+2\gamma)}\left[1 + \frac{B_2}{B_1} + B_1 - \frac{2\mu B_1(1+2\gamma)}{(1+\gamma)^2}\right]|a_2|^2 \leq \frac{B_1}{2(1+2\gamma)}, \tag{2.22}$$

where η_1 and η_2 are given in Corollary 9. The result is sharp.

Putting $g(z) = \frac{z}{1-z}$ and $\gamma = 0$ in Theorem 3, we obtain the following corollary:

Corollary 14. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots .. \ (B_1 > 0)$. Let $f(z)$ given by (1.1) belongs to the class $S_b^*(\Phi)$ and $b > 0$. Let σ_3^* be given by

$$\sigma_3^* = \frac{1}{2bB_1}\left[\frac{B_2}{B_1} + bB_1\right].$$

If $\sigma_1^* < \mu \leq \sigma_3^*$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{2bB_1}\left[1 - \frac{B_2}{B_1} - (1-2\mu) bB_1\right]|a_2|^2 \leq \frac{bB_1}{2}.$$

If $\sigma_3^* < \mu \leq \sigma_2^*$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{2bB_1}\left[1 + \frac{B_2}{B_1} + (1-2\mu) bB_1\right]|a_2|^2 \leq \frac{bB_1}{2},$$

where σ_1^* and σ_2^* are given in Corollary 10. The result is sharp.

Putting $g(z)=\frac{z}{1-z}$ and $\gamma=1$ in Theorem 3, we obtain the following corollary:
Corollary 15. Let $\Phi(z)=1+B_1z+B_2z^2+.....\ (B_1>0)$. Let $f(z)$ given by (1.1) belongs to the class $C_b(\Phi)$ and $b>0$. Let σ_3^{**} be given by

$$\sigma_3^{**}=\frac{(1+\gamma)^2 b_2^2}{2bB_1(1+2\gamma)b_3}\left[\frac{B_2}{B_1}+bB_1\right].$$

If $\sigma_1^{**}<\mu\le\sigma_3^{**}$, then

$$\left|a_3-\mu a_2^2\right|+\frac{2}{3bB_1}\left[1-\frac{B_2}{B_1}-\left(1-\frac{3\mu}{2}\right)bB_1\right]|a_2|^2\le\frac{bB_1}{6}.$$

If $\sigma_3^{**}<\mu\le\sigma_2^{**}$, then

$$\left|a_3-\mu a_2^2\right|+\frac{2}{3bB_1}\left[1+\frac{B_2}{B_1}+\left(1-\frac{3\mu}{2}\right)bB_1\right]|a_2|^2\le\frac{bB_1}{6},$$

where σ_1^{**} and σ_2^{**} are given in Corollary 11. The result is sharp.

Putting $g(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)z^k$ $(m\in\mathbb{Z};\ \ell\ge0;\ \lambda\ge0;$ $q\le s+1;\ q,s\in\mathbb{N}_0)$ and $\Gamma_k(\alpha_1)$ be given by (1.6) in Theorem 3, we obtain the following corollary:
Corollary 16. Let $\Phi(z)=1+B_1z+B_2z^2+.....\ (B_1>0)$. Let $f(z)$ given by (1.1) belongs to the class $N_b^{\gamma}(\lambda,\ell,m,q,s,\alpha_1,\beta_1;\Phi)$ and $b>0$. Let χ_3 be given by

$$\chi_3=\frac{(1+\gamma)^2 b_2^2}{2bB_1(1+2\gamma)b_3}\left[\frac{B_2}{B_1}+bB_1\right].$$

If $\chi_1<\mu\le\chi_3$, then

$$\left|a_3-\mu a_2^2\right|+\frac{(1+\gamma)^2b_2^2}{2bB_1(1+2\gamma)b_3}\frac{(1+\ell+\lambda)^{2m}}{(1+\ell)^m(1+\ell+2\lambda)^m}\frac{(\Gamma_2(\alpha_1))^2}{\Gamma_3(\alpha_1)}\left[1-\frac{B_2}{B_1}-bB_1+\frac{2\mu bB_1(1+2\gamma)}{(1+\gamma)^2}\frac{(1+\ell+2\lambda)^m(1+\ell)^m}{(1+\ell+\lambda)^{2m}}\frac{\Gamma_3(\alpha_1)}{(\Gamma_2(\alpha_1))^2}\right]|a_2|^2$$
$$\le\frac{bB_1(1+\ell)^m}{2(1+2\gamma)(1+\ell+2\lambda)^m\Gamma_3(\alpha_1)}.$$

If $\chi_3<\mu\le\chi_2$, then

$$\left|a_3-\mu a_2^2\right|+\frac{(1+\gamma)^2b_2^2}{2bB_1(1+2\gamma)b_3}\frac{(1+\ell+\lambda)^{2m}}{(1+\ell)^m(1+\ell+2\lambda)^m}\frac{(\Gamma_2(\alpha_1))^2}{\Gamma_3(\alpha_1)}\left[1+\frac{B_2}{B_1}+bB_1-\frac{2\mu bB_1(1+2\gamma)}{(1+\gamma)^2}\frac{(1+\ell+2\lambda)^m(1+\ell)^m}{(1+\ell+\lambda)^{2m}}\frac{\Gamma_3(\alpha_1)}{(\Gamma_2(\alpha_1))^2}\right]|a_2|^2$$
$$\le\frac{bB_1(1+\ell)^m}{2(1+2\gamma)(1+\ell+2\lambda)^m\Gamma_3(\alpha_1)},$$

where χ_1 and χ_2 are given in Corollary 12. The result is sharp.

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