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HARMONIC UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY USING INTEGRAL OPERATOR

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Abstract. In this paper we define and investigate a new class of harmonic functions by using an integral operator with varying arguments. We obtain coefficients inequalities, extreme points and distortion theorem.

1. Introduction

A continuous complex-valued function $f = u+iv$ which is defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D . In any simply-connected domain we can write

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$, $z \in D$ (see [4]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0)-1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as follows:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ \ g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1.2}$$

In 1984 Clunie and Shell-Small [[4] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses (see [[9], [11], [15] and [16]).
The integral operator I_λ^n is defined as follows (see [2], with $p = 1$, also see [7], with $\ell = 0$):
(i) $I^0 f(z) = f(z)$;
(ii) $I_\lambda^1 f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} f(t)dt$;
...
(iii) $I_\lambda^n f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} I_\lambda^{n-1} f(t)$ $(\lambda \geq 0; n \in \mathbb{N} = \{1, 2, 3, ...\})$
In this paper we now define the integral operator for harmonic univalent functions $f(z)$ such that $h(z)$ and $g(z)$ are given by (1.2) as follows:

$$I_\lambda^n f(z) = I_\lambda^n h(z) + (-1)^n \overline{I_\lambda^n g(z)}, \tag{1.3}$$

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where

$$I_\lambda^n h(z) = z + \sum_{k=2}^{\infty}[1+\lambda(k-1]^{-n} a_k z^k \text{ and } I_\lambda^n g(z) = \sum_{k=1}^{\infty}[1+\lambda(k-1]^{-n} b_k z^k \ (|b_1|<1).$$

With the help of the modified integral operator I_λ^n we let $G_H(m,n,\lambda;\gamma,\rho)$ be the family of harmonic functions $f = h + \overline{g}$, which satisfy the condition

$$Re\left\{\left(1+\rho e^{i\alpha}\right)\frac{I_\lambda^n f(z)}{I_\lambda^m f(z)} - \rho e^{i\alpha}\right\} \geq \gamma \tag{1.4}$$

$$\left(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; \lambda \geq 0; m \in \mathbb{N},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; m > n; z \in U\right),$$

where $I_\lambda^n f$ is defined by (1.3).

we note that:
(i) Taking $m = n+1$, $\rho = 0$, $\lambda = 1$, $G_H(n+1,n;1,\gamma,0) = H(n,\gamma)$ $(m \in \mathbb{N}; n \in \mathbb{N}_0; 0 \leq \gamma < 1)$ (see Cotirla [5]);
(ii) Taking $\lambda = 1$, $G_H(m,n,1;\gamma,\rho) = G_H(m,n;\gamma,\rho)$ (see El-Ashwah et al. [6]);
(iii) Taking $m = n+1$, $\rho = 0$, $G_H(m,n,\lambda;\gamma,0) = H_\lambda(n;\gamma)$ $(n \in \mathbb{N}_0; \lambda > 0; 0 \leq \gamma < 1)$ (see Seoudy [14], with $p = 1$);
(iv) Taking $\lambda = 1$, then $G_H(m,n,1,\gamma,\rho) = H(m,n;\gamma)$ (see Güney and Sakar [8]).
Also we note that, by the special choices of α, γ, ρ, m and n, we obtain:

(i) Taking $\rho = 0$, then $G_H(m,n,\lambda,\gamma,0) = H(m,n,\lambda;\gamma) = \Bigg\{ f \in S_H :$

$$Re\left\{\frac{I_\lambda^n f(z)}{I_\lambda^m f(z)}\right\} > \beta\,(\lambda > 0; 0 \leq \gamma < 1; m \in \mathbb{N}; n \in \mathbb{N}_0; m > n; z \in U)\Bigg\};$$

(ii) $G_H(n+1,n,\lambda;\gamma,\rho) = G_H(n,\lambda;\gamma,\rho) = \Bigg\{ f \in S_H : Re\left\{\left(1+\rho e^{i\alpha}\right)\frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)}\right.$

$$\left. - \rho e^{i\alpha}\right\} \geq \gamma\,(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; \lambda \geq 0; n \in \mathbb{N}_0; z \in U)\Bigg\};$$

(iii) $G_H(1,0,\lambda;\gamma,\rho) = G_H(\lambda,\gamma,\rho) = \Bigg\{ f \in S_H : Re\left\{\left(1+\rho e^{i\alpha}\right)\frac{f(z)}{I_\lambda^1 f(z)} - \rho e^{i\alpha}\right\} \geq \gamma$

$$(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; \lambda \geq 0; z \in U)\Bigg\}.$$

Definition 1 [10]. Let V_H denoted the class of functions $f = h + \overline{g}$ for which h and g are of the form (1.2) and there exist a real number ϕ so that, mod 2π,

$$\arg(a_k) + (k-1)\phi \equiv \pi \quad \text{and} \ \arg(b_k) + (k-1)\phi \equiv 0,\ k \geq 2.$$

We also let $V_{\overline{H}} = \overline{H} \cap V_H$.
Some subclasses of harmonic univalent functions with varying arguments were itroduced and studied by various authors (see [1] and [13]).
Also we now define the subclass $V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$ consists of harmonic functions $f_n = h + \overline{g}_n$ in $G_H(m,n,\lambda;\gamma,\rho)$ such that h and g_n are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad g_n(z) = \sum_{k=1}^{\infty} b_k z^k \ (|b_1| < 1). \tag{1.5}$$

and there exist a real number ϕ such that, mod 2π,

$$\arg(a_k) + (k-1)\phi \equiv \pi,\ k \geq 2 \text{ and } \arg(b_k) + (k+1)\phi \equiv (n-1)\pi,\ k \geq 1. \tag{1.6}$$

Also we note that, by the special choices of α, γ, m and n, we obtain:
(i) Taking $\rho = 0$, $V_{\overline{H}}(m,n,\lambda;\gamma,0) = V_{\overline{H}}(m,n,\lambda;\beta)$ $(m \in \mathbb{N}; n \in \mathbb{N}_0; \lambda \geq 0; 0 < \gamma \leq 1)$;
(ii) $V_{\overline{H}}(n+1,n,\lambda;\gamma,\rho) = V_{\overline{H}}(n,\lambda;\gamma,\rho)$ $(n \in \mathbb{N}_0; \lambda \geq 0)$;
(iii) $V_{\overline{H}}(1,0,\lambda;\gamma,\rho) = V_{\overline{H}}(\lambda,\gamma,\rho)$.

2. Geometric properties of the classes $G_H(m,n,\lambda;\gamma,\rho)$ and $V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$

Unless otherwise mentioned, we assume in the reminder of this paper that, $\alpha \in \mathbb{R}$, $0 \le \gamma < 1$, $\rho \ge 0$, $\lambda \ge 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$. We begin with a sufficient coefficients condition for functions in the class $G_H(m,n,\lambda;\gamma,\rho)$.

Theorem 1. *Let $f = h + \overline{g}$ be such that h and g are given by (1.2) and if*

$$\sum_{k=1}^{\infty}\left[\frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(\gamma+\rho)\,[1+\lambda(k-1)]^{-m}}{1-\gamma}\,|a_k| + \frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(-1)^{m-n}\,(\gamma+\rho)\,[1+\lambda(k-1)]^{-m}}{1-\gamma}\,|b_k|\right] \le 2, \quad (2.1)$$

where $a_1 = 1$. Then $f \in G_H(m,n,\lambda;\gamma,\rho)$.

Proof. To prove that $f \in G_H(m,n,\lambda;\gamma,\rho)$. We only need to show that if (2.1) holds, then the condition (1.4) is satisfied, then. Our aim is to show that

$$Re\left\{\frac{\left(1+\rho e^{i\alpha}\right) I_\lambda^n f(z) - \rho e^{i\alpha} I_\lambda^m f(z)}{I_\lambda^m f(z)}\right\} = Re\frac{A(z)}{B(z)} \ge \gamma. \quad (2.2)$$

Using the fact that $Re\{w\} > \gamma$ if and only if $|1-\gamma+w| > |1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \ge 0, \quad (2.3)$$

where $A(z) = \left(1+\rho e^{i\alpha}\right) I_\lambda^n f(z) - \rho e^{i\alpha} I_\lambda^m f(z)$ and $B(z) = I_\lambda^m f(z)$. Substituting for $A(z)$ and $B(z)$ in the left side of (2.3) we obtain

$$\begin{aligned}
&\left|(1+\rho e^{i\alpha}) I_\lambda^n f(z) - \rho e^{i\alpha} I_\lambda^m f(z) + (1-\gamma) I_\lambda^m f(z)\right| \\
&- \left|(1+\rho e^{i\alpha}) I_\lambda^n f(z) - \rho e^{i\alpha} I_\lambda^m f(z) - (1-\gamma) I_\lambda^m f(z)\right| \\
= &\Bigg|(2-\gamma)\,z + \sum_{k=2}^{\infty}[\left([1+\lambda(k-1)]^{-n} + (1-\gamma)\,[1+\lambda(k-1)]^{-m}\right) \\
&+\rho e^{i\alpha}\left([1+\lambda(k-1)]^{-n} - [1+\lambda(k-1)]^{-m}\right)]a_k z^k \\
&-(-1)^n \sum_{k=1}^{\infty}[\left((\gamma-1)\,[1+\lambda(k-1)]^{-m} - (-1)^{m-n}\,[1+\lambda(k-1)]^{-n}\right) \\
&+\rho e^{i\alpha}\left([1+\lambda(k-1)]^{-m} - (-1)^{m-n}\,[1+\lambda(k-1)]^{-n}\right)]\,\overline{b_k z^k}\Bigg| \\
&-\Bigg|\gamma z - \sum_{k=2}^{\infty}[\left([1+\lambda(k-1)]^{-n} - (1+\gamma)\,[1+\lambda(k-1)]^{-m}\right) \\
&+\rho e^{i\alpha}\left([1+\lambda(k-1)]^{-n} - [1+\lambda(k-1)]^{-m}\right)]a_k z^k \\
&+(-1)^n \sum_{k=1}^{\infty}[\left((1+\gamma)\,[1+\lambda(k-1)]^{-m} - (-1)^{m-n}\,[1+\lambda(k-1)]^{-n}\right) \\
&+\rho e^{i\alpha}\left([1+\lambda(k-1)]^{-m} - (-1)^{m-n}\,[1+\lambda(k-1)]^{-n}\right)]\overline{b_k z^k}\Bigg|
\end{aligned}$$

$$\begin{aligned}
\geq\ & (2-\gamma)|z| - \sum_{k=2}^{\infty}\left[(1+\rho)[1+\lambda(k-1)]^{-n} - (\gamma+\rho-1)[1+\lambda(k-1)]^{-m}\right]|a_k||z|^k \\
& - \sum_{k=1}^{\infty}\left|(1+\rho)[1+\lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma+\rho-1)[1+\lambda(k-1)]^{-m}\right||b_k||z|^k \\
& -\gamma|z| - \sum_{k=2}^{\infty}\left[(1+\rho)[1+\lambda(k-1)]^{-n} - (\gamma+\rho+1)[1+\lambda(k-1)]^{-m}\right]|a_k||z|^k \\
& - \sum_{k=1}^{\infty}\left|(1+\rho)[1+\lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma+\rho+1)[1+\lambda(k-1)]^{-m}\right||b_k||z|^k \\
& +\rho e^{i\alpha}\left([1+\lambda(k-1)]^{-m} - (-1)^{m-n}[1+\lambda(k-1)]^{-n}\right)]\overline{b_k z^k}\Big| \\
>\ & 2\Bigg\{(1-\gamma) - \Bigg[\sum_{k=2}^{\infty}\left[(1+\rho)[1+\lambda(k-1)]^{-n} - (\gamma+\rho)[1+\lambda(k-1)]^{-m}\right]|a_k| \\
& \qquad + \sum_{k=1}^{\infty}\left[(1+\rho)[1+\lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma+\rho)[1+\lambda(k-1)]^{-m}\right]|b_k|\Bigg]\Bigg\} \\
\geq\ & 0, \text{ this by using (2.1).}
\end{aligned}$$

The harmonic univalent functions

$$\begin{aligned}
f(z) &= z + \sum_{k=2}^{\infty} \frac{1-\gamma}{(1+\rho)[1+\lambda(k-1)]^{-n} - (\gamma+\rho)[1+\lambda(k-1)]^{-m}} x_k z^k \\
&\quad + \sum_{k=1}^{\infty} \frac{1-\gamma}{(1+\rho)[1+\lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma+\rho)[1+\lambda(k-1)]^{-m}} \overline{y_k z^k}, \qquad (2.4)
\end{aligned}$$

where $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$, shows that the coefficient bound given by (2.1) is sharp. This completes the proof of Theorem 1. □

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f_n = h + g_n$, where h and g_n are of the form (1.5).

Theorem 2. *Let $f_n = h+g_n$, where h and g_n are given by (1.5). Then $f_n \in V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$, if and only if the coefficient condition (2.1) holds.*

Proof. Since $V_{\overline{H}}(m,n,\lambda;\gamma,\rho) \subseteq G_H(m,n,\lambda;\gamma,\rho)$, we only need to prove the "only if" part of the theorem. For functions $f_n = h + g_n$, where h and g_n are given by (1.5), the inequality (1.4) with $f = f_n$ is equivalent to

$$Re\left\{\frac{(1+\rho e^{i\alpha})[z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^{-n}a_k z^k + (-1)^n\sum_{k=1}^{\infty}[1+\lambda(k-1)]^{-n}\overline{b}_k\overline{z}^k]}{z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^{-m}a_k z^k + (-1)^m\sum_{k=1}^{\infty}[1+\lambda(k-1)]^{-m}\overline{b}_k\overline{z}^k}\right\}$$

$$-Re\left\{\frac{(\gamma+\rho e^{i\alpha})[z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^{-m}a_k z^k + (-1)^n\sum_{k=1}^{\infty}[1+\lambda(k-1)]^{-m}\overline{b}_k\overline{z}^k]}{z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^{-m}a_k z^k + (-1)^m\sum_{k=1}^{\infty}[1+\lambda(k-1)]^{-m}\overline{b}_k\overline{z}^k}\right\} > 0.$$

The above condition holds for all values of $\alpha \in \mathbb{R}$ and $z \in U$. Upon choosing ϕ according (1.6) and substituting $\alpha = 0$ and $z = re^{i\phi}(0 < r < 1)$, we must have

$$\frac{E}{1 - \left[\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{-m}|a_k| - (-1)^{m+n-1}\sum_{k=1}^{\infty}[1+\lambda(k-1)]^{-m}|b_k|\right]r^{k-1}} > 0, \qquad (2.5)$$

where

$$\begin{aligned} E &= (1-\gamma) - \left(\sum_{k=2}^{\infty} \left[(1+\rho)[1+\lambda(k-1)]^{-n} - (\gamma+\rho)\,[1+\lambda(k-1)]^{-m} \right] |a_k| \right) r^{k-1} \\ &\quad - \left(\sum_{k=1}^{\infty} \left[(1+\rho)[1+\lambda(k-1)]^{-n} - (-1)^{m-n}\,(\gamma+\rho)\,[1+\lambda(k-1)]^{-m} \right] |b_k| \right) r^{k-1}. \end{aligned}$$

If the inequality (2.1) does not hold, then E is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. But this is a contradiction, then the proof of Theorem 2 is completed. □

We now obtain the distortion theorem for functions in the class $V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$.

Theorem 3. *Let $f_n = h + g_n$, where h and g_n are given by (ref1.5) and $f_n \in V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$. Then for $|z| = r < 1$, we have*

$$\begin{aligned} |f_n(z)| &\leq (1+|b_1|)\,r + \left[\frac{1-\gamma}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}} \right. \\ &\quad \left. - \frac{(1+\rho) - (-1)^{m-n}\,(\gamma+\rho)}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}}\,|b_1| \right] r^2, \qquad (2.6) \end{aligned}$$

and

$$\begin{aligned} |f_n(z)| &\geq (1-|b_1|)\,r - \left[\frac{1-\gamma}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}} \right. \\ &\quad \left. - \frac{(1+\rho) - (-1)^{m-n}\,(\gamma+\rho)}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}}\,|b_1| \right] r^2. \qquad (2.7) \end{aligned}$$

The equalities in (2.6) and (2.7) are attained for the functions f given by

$$\begin{aligned} f_{1,\phi}(z) &= (1+|b_1|)\,\overline{z} + \left[\frac{1-\gamma}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}} \right. \\ &\quad \left. - \frac{(1+\rho) - (-1)^{m-n}\,(\gamma+\rho)}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}}\,|b_1| \right] \overline{z}^2 e^{-i\phi}, \qquad (2.8) \end{aligned}$$

and

$$\begin{aligned} f_{2,\phi}(z) &= (1-|b_1|)\,\overline{z} - \left[\frac{1-\gamma}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}} \right. \\ &\quad \left. - \frac{(1+\rho) - (-1)^{m-n}\,(\gamma+\rho)}{(1+\rho)(1+\lambda)^{-n} - (\gamma+\rho)\,(1+\lambda)^{-m}}\,|b_1| \right] \overline{z}^2 e^{-i\phi}, \qquad (2.9) \end{aligned}$$

where $|b_1| \leq \dfrac{1-\gamma}{(1+\rho) - (-1)^{m-n}\,(\gamma+\rho)}$.

Proof. We prove the first inequality.

Let $f_n \in V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$, we have

$$\begin{aligned}
|f_n(z)| &\leq (1+|b_1|)\, r + \sum_{k=2}^{\infty} (|a_k|+|b_k|)\, r^k \leq (1+|b_1|)\, r + r^2 \sum_{k=2}^{\infty} (|a_k|+|b_k|) \\
&\leq (1+|b_1|)\, r + r^2 \left[\tfrac{1-\gamma}{(1+\rho)(1+\lambda)^{-n}-(\gamma+\rho)(1+\lambda)^{-m}} \sum_{k=2}^{\infty} \tfrac{(1+\rho)(1+\lambda)^{-n}-(\gamma+\rho)(1+\lambda)^{-m}}{1-\gamma} (|a_k|+|b_k|) \right] \\
&\leq (1+|b_1|)\, r + \frac{1-\gamma}{(1+\rho)(1+\lambda)^{-n}-(\gamma+\rho)(1+\lambda)^{-m}} \\
&\quad \cdot r^2 \sum_{k=2}^{\infty} \left[\frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(\gamma+\rho)[1+\lambda(k-1)]^{-m}}{1-\gamma} |a_k| \right. \\
&\qquad \left. + \frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(-1)^{m-n}(\gamma+\rho)[1+\lambda(k-1)]^{-m}}{1-\gamma} |b_k| \right] \\
&\leq (1+|b_1|)\, r + \frac{1-\gamma}{(1+\rho)(1+\lambda)^{-n}-(\gamma+\rho)(1+\lambda)^{-m}} \left[1 - \frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma} |b_1| \right] r^2 \\
&\leq (1+|b_1|)\, r + \left[\frac{1-\gamma}{(1+\rho)(1+\lambda)^{-n}-(\gamma+\rho)(1+\lambda)^{-m}} - \frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)(1+\lambda)^{-n}-(\gamma+\rho)(1+\lambda)^{-m}} |b_1| \right] r^2.
\end{aligned}$$

The proof of the second inequality is similar, thus it is left.

□

The bounds given in Theorem 3 for functions $f_n = h + \overline{g}_n$ such that h and g_n are given by (1.5) also hold for functions $f = h + \overline{g}$ such that h and g are given by (1.2) if the coefficient condition (2.1) is satisfied.
Next using the same technique used earlier by Aghalary [1] we obtain the extreme points of the class $V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$.

Theorem 4. *The closed convex hull of the class $V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$ denoted by $clcoV_{\overline{H}}(m,n;\gamma,\rho)$ is*

$$\left\{ f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \in G_H(m,n,\lambda;\gamma,\rho) : \right.$$

$$\sum_{k=1}^{\infty} \left[\frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(\gamma+\rho)[1+\lambda(k-1)]^{-m}}{1-\gamma} |a_k| \right.$$

$$\left. \left. + \frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(-1)^{m-n}(\gamma+\rho)[1+\lambda(k-1)]^{-m}}{1-\gamma} |b_k| \right] \leq 2 \right\},$$

where a_1=1. Set $\lambda_k = \frac{1-\gamma}{(1+\rho)[1+\lambda(k-1)]^{-n}-(\gamma+\rho)[1+\lambda(k-1)]^{-m}}$ and $\mu_k = \frac{1-\gamma}{(1+\rho)[1+\lambda(k-1)]^{-n}-(-1)^{m-n}(\gamma+\rho)[1+\lambda(k-1)]^{-m}}$. For b_1 fixed, $|b_1| \leq \frac{1-\gamma}{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}$, the extreme points of the class $V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$ are

$$\left\{ z + \lambda_k x z^k + \overline{b_1 z} \right\} \cup \left\{ \overline{z + \mu_k x z^k + b_1 z} \right\}, \tag{2.10}$$

where $k \geq 2$ and $|x| = 1 - \dfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}$.

Proof. Any function $f \in V_{\overline{H}}(m,n,\lambda;\gamma,\rho)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| e^{i\beta_k} z^k + \overline{b_1 z} + \overline{\sum_{k=2}^{\infty} |b_k| e^{i\delta_k} z^k},$$

where the coefficients satisfy the inequality (2.1). Set

$$h_1(z)=z,\ g_1(z)=b_1z,\ h_k(z)=z+\lambda_k e^{i\beta_k}z^k,\ g_k(z)=b_1z+\mu_k e^{i\delta_k}z^k, k=2,3,...,$$

writing $X_k = \dfrac{|a_k|}{\lambda_k}$, $Y_k = \dfrac{|b_k|}{\mu_k}$, $k = 2, 3, ...$ and $X_1 = 1 - \sum_{k=2}^{\infty} X_k$,
$Y_1 = 1 - \sum_{k=2}^{\infty} Y_k$, we have

$$f(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + \overline{Y_k g_k(z)} \right).$$

In particular, setting

$$f_1(z) = z + \overline{b_1 z},$$

and

$$f_k(z) = z + \lambda_k x z^k + \overline{b_1 z} + \overline{\mu_k y z^k},$$

$$\left(k \geq 2, |x| + |y| = 1 - \frac{(1+\rho) - (-1)^{m-n}(\gamma+\rho)}{1-\gamma} |b_1| \right),$$

we see that extreme points of the class $V_{\overline{H}}(m, n, \lambda; \gamma, \rho)$ are contained in $\{f_k(z)\}$. To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$\begin{aligned} f_1(z) &= \frac{1}{2}\left\{ f_1(z) + \lambda \left(1 - \frac{(1+\rho)[1+\lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma+\rho)[1+\lambda(k-1)]^{-m}}{1-\gamma} |b_1| \right) z^2 \right\} \\ &\quad + \frac{1}{2}\left\{ f_1(z) - \lambda \left(1 - \frac{(1+\rho)[1+\lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma+\rho)[1+\lambda(k-1)]^{-m}}{1-\gamma} |b_1| \right) z^2 \right\}, \end{aligned}$$

a convex linear combination of functions in the class $V_{\overline{H}}(m, n, \lambda; \gamma, \rho)$. Next we will show if both $|x| \neq 0$ and $|y| \neq 0$, then f_k is not an extreme point. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1 + \epsilon$ and $B = 1 - \left| \frac{\epsilon x}{y} \right|$, we then see that both

$$t_1(z) = z + \lambda_k x A z^k + \overline{b_1 z + \mu_k y B z^k}$$

and

$$t_2(z) = z + \lambda_k x (2 - A) z^k + \overline{b_1 z + \mu_k y (2 - B) z^k},$$

are in the class $V_{\overline{H}}(m, n, \lambda; \gamma, \rho)$ and note that

$$f_k(z) = \frac{1}{2}(t_1(z) + t_2(z)).$$

The extremal coefficient bounds shows that functions of the form (2.10) are the extreme points for the class$V_{\overline{H}}(m, n, \lambda; \gamma, \rho)$, then the proof of Theorem 4 is completed. □

Finally we will examine the closure properties of the class $V_{\overline{H}}(m, n, \lambda; \gamma, \rho)$ under the generalized Bernardi-Libera-Livingston integral operator (see [3, 12]) $L_c(f)$ which is defined by

$$L_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \quad (c > -1). \tag{2.11}$$

Theorem 5. *Let $f_n = h + g_n \in V_{\overline{H}}(m, n, \lambda; \gamma, \rho)$, where h and g_n are given by (1.5). Then $L_c(f_n(z))$ belongs to the class $V_{\overline{H}}(m, n, \lambda; \gamma, \rho)$.*

Proof. From the representation of $L_c(f_n(z))$, it follows that

$$\begin{aligned} L_c(f_n(z)) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left(h(t)+\overline{g}_n(t)\right)dt \\ &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left\{t+\sum_{k=2}^{\infty}a_k t^k+\overline{\sum_{k=1}^{\infty}b_k t^k}\right\}dt \\ &= z+\sum_{k=2}^{\infty}A_k z^k+\overline{\sum_{k=1}^{\infty}B_k z^k}, \end{aligned}$$

where $A_k = \frac{c+1}{c+k}a_k$, $B_k = \frac{c+1}{c+k}b_k$. Therefore, we have,

$$\sum_{k=1}^{\infty}\left[\frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(\gamma+\rho)\left[1+\lambda(k-1)\right]^{-m}}{1-\gamma}\frac{c+1}{c+k}|a_k|\right.$$
$$\left.+\frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(-1)^{m-n}(\gamma+\rho)\left[1+\lambda(k-1)\right]^{-m}}{1-\gamma}\frac{c+1}{c+k}|b_k|\right]$$
$$\leq \sum_{k=1}^{\infty}\left[\frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(\gamma+\rho)\left[1+\lambda(k-1)\right]^{-m}}{1-\gamma}|a_k|\right.$$
$$\left.+\frac{(1+\rho)[1+\lambda(k-1)]^{-n}-(-1)^{m-n}(\gamma+\rho)\left[1+\lambda(k-1)\right]^{-m}}{1-\gamma}|b_k|\right]\leq 2,$$

and the proof of Theorem 5 is completed. □

Remark 1. *(i) Putting* $m = n + 1$ $(n \in \mathbb{N}_0)$, $\rho = 0$ *and* $\lambda = 1$ *in the above results, we obtain the corresponding results obtained by Cotirla* [5]*);*
(ii) Putting $\lambda = 1$ *in the above results, we obtain the corresponding results obtained by El-Ashwah et al.* [6]*;*
(iii) Putting $m = n+1$ $(n \in \mathbb{N}_0)$ *and* $\rho = 0$ *in the above results, we obtain the corresponding results obtained by Seoudy [*[14]*, with* $p = 1$*];*
(iv) Putting $\lambda = 1$ *in the above results, we obtain the corresponding results obtained by Güney and Sakar* [8].

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