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WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS FOR SEMILINEAR BOUNDARY DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with the study of weighted pseudo almost periodic mild solutions for semilinear boundary differential equations. Namely, some sufficient conditions for the existence and uniqueness of weighted pseudo almost periodic mild solutions of semilinear boundary differential equations are obtain.

1. Introduction

The notation of pseudo almost periodicity, introduced by Zhang [1] is related to and more general than almost periodicity. Since then, this notion was utilized to investigated various type of functional differential equations and partial differential equations [2–4]. Recently, a new generalization of pseudo almost periodicity was introduced by Diagana [5]. Such a new concept is called weighted pseudo almost periodicity. As applications, some existence and uniqueness theorems of weighted pseudo almost periodic solutions were obtained [6–8].

In this paper, we consider the semilinear boundary differential equation

$$\begin{cases} u'(t) = A_m u(t) + f(t, u(t)),\ t \in \mathbb{R}, \\ Lu(t) = g(t, u(t)),\ t \in \mathbb{R}. \end{cases}$$

The first equation stands in a Banach space $(X, \|\cdot\|)$ and the second one is in the boundary space ∂X, $(A_m, D(A_m))$ is a densely defined linear operator on X, $L : D(A_m) \to \partial X$ is a bounded linear operator, and $f \in C(\mathbb{R} \times X, X)$, $g \in C(\mathbb{R} \times X, \partial X)$. This kind of boundary differential equation is motivated by retarded differential equations, by population dynamics equations and by boundary control problems. Boundary differential equations are widely used to model the scientific problems in physical, biology and other subjects and for this reason, this type equations have received much attention in recent years. Some properties of the solutions have been studied in several contexts, see [9–12] for more details. However, to the best our knowledge, the weighted pseudo almost periodic solutions

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to semilinear boundary differential equations has not been treated in the literature yet. This is one of the key motivations of this study.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. Section 3 is devoted to the existence and uniqueness of weighted pseudo almost periodic solutions to semilinear boundary differential equations. In the last section of this paper, we apply the abstract results to the retarded differential equations.

2. Preliminaries and Basic Results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two Banach spaces and $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. For A being a linear operator on X, $D(A), \rho(A), R(\lambda, A)$ stand for the domain, the resolvent set and the resolvent of A. In order to facilitate the discussion below, we further introduce the following notations:

- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R}\times Y, X)$: the Banach space of bounded continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times Y$ to X) with the supremum norm.
- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times Y, X)$): the set of continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times Y$ to X).
- $B(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology.

2.1. **Extrapolation Banach Space.** Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Define on X a new norm by

$$\|x\|_{-1} = \|(\lambda - A)^{-1}u\|,\ x \in X, \lambda \in \rho(A).$$

The completion of $(X, \|\cdot\|_{-1})$ is called the extrapolation space of X associated to A and will be denoted by X_{-1}. By the resolvent equation, the space X_{-1} does not depend on λ.

Since $T(t)$ commutes with the operator resolvent $R(\lambda, A) := (\lambda I - A)^{-1}$, the extension of $T(t)$ to X_{-1} exists and defines a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ which is generated by A_{-1} with $D(A_{-1}) = X$.

We recall that the Favard class associated to a generator A (or $T(\cdot)$) is the Banach space

$$F_A := \left\{ x \in X, \sup_{t>0} \frac{1}{t}\|e^{-\omega t}T(t)x - x\| < \infty \right\}$$

endowed with the norm

$$\|x\|_{F_A} := \sup_{t>0} \frac{1}{t}\|e^{-\omega t}T(t)x - x\|,$$

here $\omega > \omega_0(T(\cdot))$, the growth bound of $T(\cdot)$. It is clear that F_A is independent of the choice of ω, contains the domain of A, $F_A \hookrightarrow X \hookrightarrow F_{A_{-1}} \hookrightarrow X_{-1}$, and

$$(\lambda - A_{-1}) : F_A \to F_{A_{-1}} \tag{2.1}$$

is an isomorphism for every $\lambda \in \rho(A)$ (see [13] for more details).

Definition 2.1. [13] A C_0-semigroup $(T(t))_{t\geq 0}$ is said to be hyperbolic if it satisfies the following properties:

(i) there exist two subspace X_S (the stable space) and X_U (the unstable space) of X such that $X = X_S \oplus X_U$;

(ii) $T(t)$ is defined on X_U, $T(t)X_U \subset X_U$, and $T(t)X_S \subset X_S$ for all $t \geq 0$.

(iii) there exist constants $M, \delta > 0$ such that

$$\|T(t)P_S\| \leq Me^{-\delta t},\ t \geq 0,\ \|T(t)P_U\| \leq Me^{\delta t},\ t \leq 0,$$

where P_S and P_U are the projection onto X_S and X_U respectively.

In the sequel we need the following fundamental lemma, see [14] .

Lemma 2.1. *Let $f : \mathbb{R} \to F_{A_{-1}}$ be a bounded function, then the following assertions hold:*
(i) $\int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}f(s)ds \in X$ *for all* $t \in \mathbb{R}$.
(ii) $\int_{t}^{\infty} T_{-1}(t-s)P_{U,-1}f(s)ds \in X$ *for all* $t \in \mathbb{R}$.
(iii) $\left\|\int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}f(s)ds\right\| \leq Ce^{-\delta t}\int_{-\infty}^{t} e^{\delta s}\|f(s)\|_{F_{A_{-1}}}ds$ *for all* $t \in \mathbb{R}$.
(iv) $\left\|\int_{t}^{\infty} T_{-1}(t-s)P_{U,-1}f(s)ds\right\| \leq Ce^{\delta t}\int_{t}^{\infty} e^{-\delta s}\|f(s)\|_{F_{A_{-1}}}ds$ *for all* $t \in \mathbb{R}$.

2.2. **Weighted pseudo almost periodicity.** First, let us recall some definitions of almost periodic function and weighted pseudo almost periodic function.

Definition 2.2. A function $f \in C(\mathbb{R}, X)$ is called almost periodic if for each $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ such that $\|f(t+\tau) - f(t)\| < \varepsilon$ for each $t \in \mathbb{R}$. The number τ is called an ε-translation number of f and the collection of all functions will be denoted by $AP(\mathbb{R}, X)$.

Let U be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $T > 0$ and each $\rho \in U$, set

$$\mu(T, \rho) := \int_{-T}^{T} \rho(t)\mathrm{d}t.$$

Define

$$U_\infty := \{\rho \in U : \lim_{T\to\infty} \mu(T, \rho) = \infty\},\ U_B := \{\rho \in U_\infty : \rho \text{ is bounded and } \inf_{x\in\mathbb{R}} \rho(x) > 0\}.$$

It is clear that $U_B \subset U_\infty \subset U$.

For $\rho \in U_\infty$, define

$$PAP_0(\mathbb{R}, X, \rho) := \left\{ f \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(s)\|f(s)\|ds = 0 \right\}.$$

$$PAP_0(\mathbb{R}\times X, X, \rho) := \left\{ f \in BC(\mathbb{R}\times X, X) : \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(s)\|f(s,u)\|ds = 0 \right.$$
$$\left. \text{uniformly in } u \in X \right\}.$$

Definition 2.3. [5] Let $\rho \in U_\infty$. A function $f \in C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times X, X)$) is called weighted pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$ (resp. $AP(\mathbb{R}\times X, X)$) and $\varphi \in PAP_0(\mathbb{R}, X, \rho)$ (resp. $PAP_0(\mathbb{R}\times X, X, \rho)$). Denote by $WPAP(\mathbb{R}, X, \rho)$ (resp. $WPAP(\mathbb{R}\times X, X, \rho)$) the set of such functions.

Definition 2.4. Let $\rho_1, \rho_2 \in U_\infty$. ρ_1 is said to be equivalent to ρ_2 (i.e., $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$

It is trivial to show that " $\sim$ " is a binary equivalence relation on U_∞. The equivalence class of a given weight $\rho \in U_\infty$ which is denoted by $cl(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho \in U_\infty} cl(\rho)$.

Let $\rho \in U_\infty, s \in \mathbb{R}$, defined ρ_s by $\rho_s(t) = \rho(t+s)$ for $t \in \mathbb{R}$ and

$$U_T = \{\rho \in U_\infty : \rho \sim \rho_s \text{ for each } s \in \mathbb{R}\}.$$

It is trivial to see that U_T contains various kinds of weights such as $1, (1+t^2)/(2+t^2), e^t$, and $1+|t|^n$ with $n \in \mathbb{N}$ et al.

It is obvious that $WPAP(\mathbb{R}, X, \rho)$ (resp. $WPAP(\mathbb{R} \times X, X, \rho)$), $\rho \in U_T$ is a Banach space when endowed with the supremum norm $\|\cdot\|$.

Lemma 2.2. *[8] $PAP_0(\mathbb{R}, X, \rho)$ with $\rho \in U_T$ is translation invariant, that is, $\varphi \in PAP_0(\mathbb{R}, X, \rho)$ and $s \in \mathbb{R}$ implies that $\varphi(\cdot - s) \in PAP_0(\mathbb{R}, X, \rho)$.*

3. Existence and uniqueness of $WPAP$ solutions to (3.1)

Consider the semilinear boundary differential equation

$$\begin{cases} u'(t) = A_m u(t) + f(t, u(t)), \ t \in \mathbb{R}, \\ Lu(t) = g(t, u(t)), \ t \in \mathbb{R}. \end{cases} \tag{3.1}$$

The first equation stands in a Banach space $(X, \|\cdot\|)$ and the second one is in the boundary space ∂X, $(A_m, D(A_m))$ is a densely defined linear operator on X, $L : D(A_m) \to \partial X$ is a bounded linear operator, and $f \in C(\mathbb{R} \times X, X)$, $g \in C(\mathbb{R} \times X, \partial X)$.

In this section, we make the following assumptions.

(H_1) There exists a new norm $|\cdot|$ which makes the domain $D(A_m)$ complete and then denoted by X_m. The space is continuous embedded in X and $A_m \in B(X_m, X)$.

(H_2) The restriction $A := A_m|_{ker(L)}$ generates a C_0-semigroup $T(\cdot)$ on X.

(H_3) $L \in B(X_m, \partial X)$ is surjective.

(H_4) There exist positive constants γ, λ_0 such that

$$\|Lu\| \geq \gamma(\lambda - \lambda_0), \ u \in ker(\lambda - A_m), \lambda \in \rho(A), \lambda > \lambda_0.$$

(H_5) The semigroup $T(\cdot)$ is hyperbolic on X.

It is not difficult to see that for some $\lambda \in \rho(A)$, X_m can be decomposed as

$$X_m = kerL \oplus ker(\lambda - A_m),$$

thus, the restriction $L : ker(\lambda - A_m) \to \partial X$ is then a bijection and its inverse $L_\lambda \in B(\partial X, X)$ and $L_\lambda L$ is a projection onto $ker(\lambda - A_m)$, one can see [15] for more details. It is shown in [15] that

$$R(\mu, A)L_\lambda = R(\lambda, A)L_\mu \text{ for all } \lambda, \mu \in \rho(A). \tag{3.2}$$

From [16], (H_4) is equivalent to the fact that the operator

$$L_\lambda : \partial X \to F_A \text{ is bounded for all } \lambda > \lambda_0. \tag{3.3}$$

Definition 3.1. A mild of (3.1) is a continuous function $u : \mathbb{R} \to X$ satisfying

$$(i) \ \int_s^t u(\tau)d\tau \in X_m, \ (ii) \ u(t) - u(s) = A_m \int_s^t u(\tau)d\tau + \int_s^t f(\tau, u(\tau))d\tau,$$

$$(iii) \ L\int_s^t u(\tau)d\tau = \int_s^t g(\tau, u(\tau))d\tau,$$

for all $t \geq s, t, s \in \mathbb{R}$.

As in [10], we transform (3.1) to the equivalent semilinear differential equation

$$u'(t) = A_{-1}u(t) + f(t, u(t)) - A_{-1}L_0 g(t, u(t)),\ t \in \mathbb{R}, \tag{3.4}$$

where $L_0 := (L|_{ker(A_m)})^{-1}$.

Lemma 3.1. *[10] Assume that (H_1)-(H_3) are satisfied. A function u is a mild solution of (3.1) if and only if u is a mild solution of (3.4).*

In this subsection, we deal with the case that the nonlinear perturbation in (3.1) is weighted pseudo almost periodic, i.e., $f \in WPAP(\mathbb{R} \times X, X, \rho)$, $g \in WPAP(\mathbb{R} \times X, \partial X, \rho)$, $\rho \in U_T$.

First, we study the existence and uniqueness of weighted pseudo almost periodic solutions for the linear inhomogeneous differential equation

$$u'(t) = A_{-1}u(t) + h(t),\ t \in \mathbb{R}. \tag{3.5}$$

Definition 3.2. A mild of (3.5) is a continuous function $u : \mathbb{R} \to X$ satisfying

$$u(t) = T(t-s)u(s) + \int_s^t T_{-1}(t-\tau)h(\tau)d\tau \tag{3.6}$$

for all $t \geq s, t, s \in \mathbb{R}$.

Lemma 3.2. *Let $h \in WPAP(\mathbb{R}, F_{A_{-1}}, \rho)$, then (3.5) has a unique mild solution $u \in WPAP(\mathbb{R}, X, \rho)$ given by*

$$u(t) = \int_{-\infty}^t T_{-1}(t-\tau)P_{S,-1}h(\tau)d\tau - \int_t^{\infty} T_{-1}(t-\tau)P_{U,-1}h(\tau)d\tau,\ t \in \mathbb{R}.$$

Proof. It is clear that $u(\cdot)$ is well defined in X and $u(\cdot)$ satisfies (3.6), hence u is a mild solution of (3.5). Since $h \in WPAP(\mathbb{R}, F_{A_{-1}}, \rho)$, let $h = h_1 + h_2$, where $h_1 \in AP(\mathbb{R}, F_{A_{-1}})$, $h_2 \in PAP_0(\mathbb{R}, F_{A_{-1}}, \rho)$, then $u(t) := u_1(t) + u_2(t)$, where

$$u_1(t) = \int_{-\infty}^t T_{-1}(t-\tau)P_{S,-1}h_1(\tau)d\tau - \int_t^{\infty} T_{-1}(t-\tau)P_{U,-1}h_1(\tau)d\tau,\ t \in \mathbb{R}.$$

$$u_2(t) = \int_{-\infty}^t T_{-1}(t-\tau)P_{S,-1}h_2(\tau)d\tau - \int_t^{\infty} T_{-1}(t-\tau)P_{U,-1}h_2(\tau)d\tau,\ t \in \mathbb{R}.$$

First, we show that $u_1 \in AP(\mathbb{R}, X)$. Since $h_1 \in AP(\mathbb{R}, F_{A_{-1}})$, then for each $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that for every $a \in \mathbb{R}$, there exists a number $\sigma \in [a, a + l(\varepsilon)]$ satisfy $\|h_1(t+\tau) - h_1(t)\|_{F_{A_{-1}}} < \frac{\delta\varepsilon}{2C}$ for all $t \in \mathbb{R}$.

Then

$$\begin{aligned}\|u_1(t+\sigma)-u_1(t)\| &= \|\int_{-\infty}^{t+\sigma} T_{-1}(t+\sigma-\tau)P_{S,-1}h_1(\tau)d\tau \\ &\quad - \int_{t+\sigma}^{\infty} T_{-1}(t+\sigma-\tau)P_{U,-1}h_1(\tau)d\tau \\ &\quad - \int_{-\infty}^{t} T_{-1}(t-\tau)P_{S,-1}h_1(\tau)d\tau + \int_{t}^{\infty} T_{-1}(t-\tau)P_{U,-1}h_1(\tau)d\tau\| \\ &\le \|\int_{-\infty}^{t+\sigma} T_{-1}(t+\sigma-\tau)P_{S,-1}h_1(\tau)d\tau - \int_{-\infty}^{t} T_{-1}(t-\tau)P_{S,-1}h_1(\tau)d\tau\| \\ &\quad + \|\int_{t+\sigma}^{\infty} T_{-1}(t+\sigma-\tau)P_{U,-1}h_1(\tau)d\tau - \int_{t}^{\infty} T_{-1}(t-\tau)P_{U,-1}h_1(\tau)d\tau\| \\ &\le \|\int_{-\infty}^{t} T_{-1}(t-\tau)P_{S,-1}(h_1(\tau+\sigma)-h_1(\tau))d\tau\| \\ &\quad + \|\int_{t}^{\infty} T_{-1}(t-\tau)P_{U,-1}(h_1(\tau+\sigma)-h_1(\tau))d\tau\| \\ &\le Ce^{-\delta t}\int_{-\infty}^{t} e^{\delta\tau}\|h_1(\tau+\sigma)-h_1(\tau)\|_{F_{A_{-1}}}d\tau \\ &\quad + Ce^{\delta t}\int_{t}^{\infty} e^{-\delta\tau}\|h_1(\tau+\sigma)-h_1(\tau)\|_{F_{A_{-1}}}d\tau \\ &< \varepsilon,\end{aligned}$$

hence, $u_1 \in AP(\mathbb{R}, X)$.

To complete the proof, we show that $u_2 \in PAP_0(\mathbb{R}, X, \rho)$. In fact, for $T > 0$, one has

$$\begin{aligned}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|u_2(t)\|\rho(t)dt &\le \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|\int_{-\infty}^{t} T_{-1}(t-\tau)P_{S,-1}h_2(\tau)d\tau\|\rho(t)dt \\ &\quad + \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|\int_{t}^{\infty} T_{-1}(t-\tau)P_{U,-1}h_2(\tau)d\tau\|\rho(t)dt \\ &\le \frac{C}{\mu(T,\rho)}\int_{-T}^{T}\int_{-\infty}^{t} e^{-\delta(t-\tau)}\|h_2(\tau)\|_{F_{A_{-1}}}\rho(t)d\tau dt \\ &\quad + \frac{C}{\mu(T,\rho)}\int_{-T}^{T}\int_{t}^{\infty} e^{-\delta(\tau-t)}\|h_2(\tau)\|_{F_{A_{-1}}}\rho(t)d\tau dt \\ &= \frac{C}{\mu(T,\rho)}\int_{-T}^{T}\int_{0}^{\infty} e^{-\delta s}\|h_2(t-s)\|_{F_{A_{-1}}}\rho(t)ds dt \\ &\quad + \frac{C}{\mu(T,\rho)}\int_{-T}^{T}\int_{0}^{\infty} e^{-\delta s}\|h_2(t+s)\|_{F_{A_{-1}}}\rho(t)ds dt \\ &= C\int_{0}^{\infty} e^{-\delta s}\left(\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|h_2(t-s)\|_{F_{A_{-1}}}\rho(t)dt\right)ds \\ &\quad + C\int_{0}^{\infty} e^{-\delta s}\left(\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|h_2(t+s)\|_{F_{A_{-1}}}\rho(t)dt\right)ds,\end{aligned}$$

Since $\rho \in U_T$, from Lemma 2.2, it follows that $h_2(\cdot - s), h_2(\cdot + s) \in PAP_0(\mathbb{R}, F_{A_{-1}}, \rho)$, then

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|h_2(t-s)\|_{F_{A_{-1}}} \rho(t)dt = 0,$$

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|h_2(t+s)\|_{F_{A_{-1}}} \rho(t)dt = 0,$$

so by Lebesgue dominated convergence theorem,

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|u_2(t)\|\rho(t)dt = 0,$$

then $u_2 \in PAP_0(\mathbb{R}, F_{A_{-1}}, \rho)$, hence $u \in WPAP(\mathbb{R}, X, \rho)$. □

Next, for the semilinear differential equation

$$u'(t) = A_{-1}u(t) + h(t, u(t)),\ t \in \mathbb{R}. \tag{3.7}$$

By the fixed point theorem, one has the following conclusion.

Lemma 3.3. *Assume that* $h \in WPAP(\mathbb{R} \times X, F_{A_{-1}}, \rho), \rho \in U_T$,

$$\|h(t, u(t)) - h(t, v(t))\|_{F_{A_{-1}}} \le k\|u - v\|,\ u, v \in X, t \in \mathbb{R}.$$

If $kC < \frac{\delta}{2}$, *then* (3.7) *has a unique mild solution* $u \in WPAP(\mathbb{R}, X, \rho)$, *which satisfies*

$$u(t) = \int_{-\infty}^{t} T_{-1}(t-\tau)P_{S,-1}h(\tau, u(\tau))d\tau - \int_{t}^{\infty} T_{-1}(t-\tau)P_{U,-1}h(\tau, u(\tau))d\tau,\ t \in \mathbb{R}.$$

Proof. Define the operator $\Gamma : WPAP(\mathbb{R}, X, \rho) \to WPAP(\mathbb{R}, X, \rho)$ by

$$(\Gamma u)(t) = \int_{-\infty}^{t} T_{-1}(t-\tau)P_{S,-1}h(\tau, u(\tau))d\tau - \int_{t}^{\infty} T_{-1}(t-\tau)P_{U,-1}h(\tau, u(\tau))d\tau,\ t \in \mathbb{R}.$$

By Lemma 3.2, Γ is well defined.

For $u, v \in WPAP(\mathbb{R}, X, \rho)$,

$$\begin{aligned}\|(\Gamma u)(t) - (\Gamma v)(t)\| &\le Ce^{-\delta t} \int_{-\infty}^{t} e^{\delta\tau} \|h(\tau, u(\tau)) - h(\tau, v(\tau))\|_{F_{A_{-1}}} d\tau \\ &\quad + Ce^{\delta t} \int_{t}^{\infty} e^{-\delta\tau} \|h(\tau, u(\tau)) - h(\tau, v(\tau))\|_{F_{A_{-1}}} d\tau \\ &\le \frac{2kC}{\delta}\|u - v\|.\end{aligned}$$

By the Banach contraction mapping principle, one has Γ has a unique fixed point in $WPAP(\mathbb{R}, X, \rho)$, which is the unique $WPAP$ mild solution to (3.7). The proof is complete. □

Theorem 3.1. *Assume that* (H_1)-(H_5) *are satisfied, the functions* $f \in WPAP(\mathbb{R} \times X, X, \rho)$, $g \in WPAP(\mathbb{R} \times X, \partial X, \rho)$ *are globally Lipschitzian with small constants. Then* (3.1) *has a unique mild* $u \in WPAP(\mathbb{R}, X, \rho)$.

Proof. From (2.1), (3.2), (3.3), $A_{-1}L_0$ is a bounded operator from ∂X to $F_{A_{-1}}$. Hence by $f \in WPAP(\mathbb{R} \times X, X, \rho)$, $g \in WPAP(\mathbb{R} \times X, \partial X, \rho)$ and the injection $X \hookrightarrow F_{A_{-1}}$, it follows that $h(t,u) := f(t,u) - A_{-1}L_0 g(t,u) \in WPAP(\mathbb{R} \times X, F_{A_{-1}}, \rho)$ and $h(t,u)$ is globally Lipschitzian with a small constant. Hence by Lemma 3.1, Lemma 3.3, there exists a unique mild solution $u \in WPAP(\mathbb{R}, X, \rho)$ of (3.1). □

4. Examples

Consider the following retarded partial differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + \alpha u(t,x) + g(t, u(t-r,x)), \ t \in \mathbb{R}, x \in [0,\pi] \\ u(t,0) = u(t,\pi) = 0, \ t \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $r > 0$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$, $g(t, u(t-r,\cdot)) \in L^2(0,\pi)$ and $3 < \alpha < 4$.

Let $E = L^2(0,\pi)$, then (4.1) can be rewrite as

$$\frac{d}{dt}v(t) = Bv(t) + g(t, v_t), \ t \in \mathbb{R} \tag{4.2}$$

with $v, g : \mathbb{R} \to E$ such that $v(t) = u(t,\cdot)$ and $g(t,\varphi) = g(t, \varphi(-r,\cdot))$, $v_t \in C([-r,0], E)$, and B is the operator defined in E by

$$By = y'' + \alpha y, \ y \in D(B) = \{y \in W^{2,2}([0,\pi], E) : y(0) = y(\pi) = 0\}.$$

It is well know that B generates an immediately compact semigroup in E, and $\lambda \in \sigma(B)$ if and only if there exists $n \in \mathbb{N}$ such that $\lambda = \alpha - n^2$, see [17].

As in [10], we transform (4.2) to the boundary differential equation, by setting

$$X = C([-r,0], E), \ \partial X = E, \ A_m = \frac{d}{d\sigma},$$

$$D(A_m) = \{f \in C^1([-r,0], E) : f(0) \in D(B)\},$$

$$X_m = (D(A_m), |\cdot|), \ |f| = \|f\|_\infty + \|f'\|_\infty + \|Bf(0)\|, \ f \in D(A_m),$$

and the boundary operator L is defined on X_m by $Lf = f'(0) - Bf(0), \ f \in X_m$. The operator $A = A_m|_{kerL}$ is given by

$$A = \frac{d}{d\sigma}, \ D(A) = \{f \in C^1([-r,0], E) : f(0) \in D(B), f'(0) = Bf(0)\},$$

and it generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. By [10], (H_1)-(H_4) are satisfied. From [13], $\sigma(A) = \sigma(B)$ and $3 < \alpha < 4$, so $\sigma(B) \cap i\mathbb{R} = \emptyset$, then the semigroup $(T(t))_{t\geq 0}$ is hyperbolic.

Theorem 4.1. *If $g \in WPAP(\mathbb{R} \times C([-r,0], E), E)$ and globally Lipschitzian with a small constant, then (4.1) admits a unique mild solution u such that $u_t \in WPAP(\mathbb{R}, C([-r,0], E))$.*

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