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SOME CLASSES OF MULTIVALENT HARMONIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper, new classes of multivalent harmonic functions defined by convolution are considered. Coefficient bounds, representation theorem and distortion bounds for functions of these classes are obtained.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic in a complex domain D if both u and v are harmonic in D. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small [1]).

Denote by H the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

For $m \in \mathbb{N} = \{1, 2, ...\}$, h and g analytic in U, denote by $H(m)$ the set of all multivalent harmonic functions $f = h + \overline{g}$ defined in U, where h and g defined by

$$h(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=m}^{\infty} b_n z^n, \quad |b_m| < 1. \tag{1.1}$$

Denote by $H = H(1)$.

Let F be a fixed multivalent harmonic function given by

$$F(z) = H(z) + \overline{G(z)} = z^m + \sum_{n=m+1}^{\infty} |A_n| z^n + \overline{\sum_{n=m}^{\infty} |B_n| z^n}, \quad |b_m| < 1. \tag{1.2}$$

Recall the Hadamard product (or convolution) of f and F by:

$$(f * F)(z) = z^m + \sum_{n=m+1}^{\infty} a_n |A_n| z^n + \overline{\sum_{n=m}^{\infty} b_n |B_n| z^n}. \tag{1.3}$$

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Using the convolution (1.3) and for $0 \leq \gamma < 1, k \geq 0, \theta \in R, 0 \leq \lambda \leq 1, z' = \frac{\partial}{\partial \theta}(z = re^{i\theta}), m \geq 1$ and $f \in H(m)$, we define the subclass $S_m H(F, \lambda, \gamma, k)$ by

$$\text{Re}\left\{(1 + ke^{i\theta})\frac{z(f * F)'(z)}{z'\left[(1-\lambda)z^m + \lambda(f * F)(z)\right]} - kme^{i\theta}\right\} \geq m\gamma. \tag{1.4}$$

Since $f'(z) = \frac{\partial}{\partial \theta}(f(re^{i\theta})) = i(zh'(z) - \overline{zg'(z)})$, (1.4) is equivalent to:

$$\text{Re}\left\{(1 + ke^{i\theta})\frac{\left[z(h * H)'(z) - z\overline{(g * G)'(z)}\right]}{(1-\lambda)z^m + \lambda[(h * H)(z) + \overline{(g * G)(z)}]} - kme^{i\theta}\right\} \geq m\gamma. \tag{1.5}$$

For special choices of the fixed function F, we obtain the following new classes:

(*i*) For $A_n = B_n = \Gamma_n(\alpha_1) = \dfrac{(\alpha_1)_{n-m}...(\alpha_q)_{n-m}}{(\beta_1)_{n-m}...(\beta_s)_{n-m}(1)_{n-m}}, \alpha_1, ...\alpha_q, \beta_1, ...\beta_s$ are postive real numbers, the class $S_m H(F, \lambda, \gamma, k)$ reduces to

$$S_m H(\alpha_1, \lambda, \gamma, k) = \text{Re}\left\{(1 + ke^{i\theta})\frac{z(H_{m,q,s}(\alpha_1)(f)(z))'}{z'[(1-\lambda)z^m + \lambda H_{m,q,s}(\alpha_1)(f)(z)]} - kme^{i\theta}\right\} \geq m\gamma, \tag{1.6}$$

where, $H_{m,q,s}(\alpha_1)$ is the modified Dziok-Srivastava operator (see [2] and [3]) which contains many other operators considered earlier for special values of the parameters α_i , β_j, q, s;

(*ii*) For $A_n = B_n = \left[\frac{m+l+\delta(n-m)}{m+l}\right]^s, \delta, l, s \geq 0$, the class $S_m H(F, \lambda, \gamma, k)$ reduces to

$$S_m H(\delta, l, s, \lambda, \gamma, k) = \text{Re}\left\{(1 + ke^{i\theta})\frac{z(I_m^s(\delta,l)(f)'(z))'}{z'[(1-\lambda)z^m + \lambda I_m^s(\delta,l)(f))(z)]} - kme^{i\theta}\right\} \geq m\gamma, \tag{1.7}$$

where $I_m^s(\delta, l)$ is the modified Cătăs operator (see [4]) which contains many other operators considered earlier for special values of the parameters s, l, δ;

(*iii*) For $A_n = B_n = \left[\frac{m+l}{m+l+\delta(n-m)}\right]^s, \delta, l, s \geq 0$, the class $S_m H(F, \lambda, \gamma, k)$ reduces to

$$S_m H(\delta, l, s, \lambda, \gamma, k) = \text{Re}\left\{(1 + ke^{i\theta})\frac{z(J_m^s(\delta,l)(f)(z))'}{z'[(1-\lambda)z^m + \lambda(J_m^s(\delta,l)(f))(z)]} - kme^{i\theta}\right\} \geq m\gamma, \tag{1.8}$$

where $J_m^s(\delta, l)$ is the modified operator for the operator $J_m^s(\delta, l)$ introduced and studied by El-Ashwah and Aouf [5] and Aouf et al. [6], which contains in turn other operators considered earlier for special values of the parameters s, l, δ.

Also, for $F(z) = f(z)$, the class $S_m H(F, \lambda, \gamma, k)$ reduces to the class $G_H(k, m, \gamma, \lambda)$ introduced and studied by Ahuja et al. [7], which for $\lambda = k = 1$, reduces to the class $R(m, \gamma)$ introduced and studied by Jahangiri et al. [8]. For $m = k = 1$, $S_1 H(F, \lambda, \gamma, 1) = R_H(F, \lambda, \gamma)$ which was introduced and studied by Murugusundaramoorthy and Vijaya [9] and for $\lambda = 1$, m replaced by p, $S_p H(F, 1, \gamma, k) = H_F(p, \gamma, k)$ with $t = 1$ which was introduced and studied by Ahuja et al. [10].

Denote by $TH(m)$ the subclass of $H(m)$ consisting of functions $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^m - \sum_{n=m+1}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=m}^{\infty} |b_n| z^n \ , |b_m| < 1. \tag{1.9}$$

Finally, we define the class $TS_m H(F, \lambda, \gamma, k) = S_m H(F, \lambda, \gamma, k) \cap TH(m)$.

In this paper we obtain necessary and sufficient coefficient bounds for functions in the class $TS_mH(F,\lambda,\gamma,k)$. A represntaion theorem, inclusion properties, and distortion bounds for functions of this class are also obtained.

2. Main results

Unless otherwise mentioned, we assume that $0\leq\gamma<1, k\geq 0, \theta\in R, 0\leq\lambda\leq 1$ and $m\in\mathbb{N}$.

We begin with a sufficient condition for functions in the class $S_mH(F,\lambda,\gamma,k)$.

Theorem 1. *Let $f=h+\overline{g}$, where h and g be given by (1.1). Then $f\in S_mH(F,\lambda,\gamma,k)$ if*

$$\sum_{n=m+1}^{\infty}\left[n(k+1)-m\lambda(k+\gamma)\right]|a_nA_n|+\sum_{n=m}^{\infty}\left[n(k+1)-+m\lambda(k+\gamma)\right]|b_nB_n|$$

$$\leq\frac{1}{2}\left[m(1-\gamma)+1-|m(1-\gamma)-1|\right]. \tag{2.1}$$

Proof. In view of (1.5) , we need to prove that $Re\{\zeta\}>m\gamma$, where

$$\zeta=\frac{(1+ke^{i\theta})\left[z(h*H)'(z)-z\overline{(g*G)'(z)}\right]-mke^{i\theta}\left\{(1-\lambda)z^m+\lambda[(h*H)(z)+\overline{(g*G)(z)}]\right\}}{(1-\lambda)z^m+\lambda[(h*H)(z)+\overline{(g*G)(z)}]}=\frac{\Phi(z)}{\Psi(z)}, \tag{2.2}$$

Using the fact that $\mathrm{Re}\{\zeta\}\geqslant m\gamma$ if and only if $|1-m\gamma+\zeta|\geqslant|1+m\gamma-\zeta|$ in U, it suffices to show that

$$|\Phi(z)+\Psi(z)(1-m\gamma)|-|\Phi(z)-\Psi(z)(1+m\gamma)|\geqslant 0. \tag{2.3}$$

Substituting for $\Phi(z)$ and $\Psi(z)$, we have

$$\begin{aligned}
&|\Phi(z)+\Psi(z)(1-m\gamma)|-|\Phi(z)-\Psi(z)(1+m\gamma)|\\
=&\left|[1+m(1-\gamma)]z^m+\sum_{n=m+1}^{\infty}[n+(n-\lambda m)ke^{i\theta}+\lambda(1-m\gamma)]a_nA_nz^n\right.\\
&\left.-\sum_{n=m}^{\infty}[n+(n+\lambda m)ke^{i\theta}-\lambda(1-m\gamma)]\overline{b_nB_nz^n}\right|\\
&-\left|[m(1-\gamma)-1]z^m-\sum_{n=m+1}^{\infty}[n+(n-\lambda m)ke^{i\theta}-\lambda(1+m\gamma)]a_nA_nz^n\right.\\
&\left.-\sum_{n=m}^{\infty}[n+(n+\lambda m)ke^{i\theta}+\lambda(1+m\gamma)]\overline{b_nB_nz^n}\right|
\end{aligned}$$

$$\begin{aligned}
\geq\ & [1+m(1-\gamma)-|m(1-\gamma)-1|]\,|z|^m \\
& -2\sum_{n=m+1}^{\infty}[n+(n-\lambda m)k-\lambda m\gamma]\,|a_nA_n|\,|z|^n \\
& -2\sum_{n=m}^{\infty}[n+(n+\lambda m)k+\lambda m\gamma]\,|b_nB_n|\,|z|^n \\
\geq\ & [1+m(1-\gamma)-|m(1-\gamma)-1|]. \\
& .\left\{1-\sum_{n=m+1}^{\infty}\frac{2[n+(n-\lambda m)k-\lambda m\gamma]}{[1+m(1-\gamma)-|m(1-\gamma)-1|]}\,|a_nA_n|\right. \\
& \left.-\frac{[n+(n+\lambda m)k+\lambda m\gamma]}{[1+m(1-\gamma)-|m(1-\gamma)-1|]}\,|b_nB_n|\right\}.
\end{aligned}$$

By hypothesis (2.1), last expression is nonnegative. Thus the proof is completed. The coeficient bounds (2.1) is sharp for the function

$$f(z)=z^m+\sum_{n=m+1}^{\infty}\frac{[1+m(1-\gamma)-|m(1-\gamma)-1|]}{2[n+(n-\lambda m)k-\lambda m\gamma]A_n}X_nz^n + \sum_{n=m}^{\infty}\frac{[1+m(1-\gamma)-|m(1-\gamma)-1|]}{2[n+(n+\lambda m)k+\lambda m\gamma]B_n}\overline{y}_n\overline{z}^n, \tag{2.4}$$

where $\sum\limits_{n=m+1}^{\infty}|x_n|+\sum\limits_{n=m}^{\infty}|y_n|=1$, shows that the coefficient bound given by (2.1) is sharp.

Corollary 1. For $m\geq 1/(1-\gamma)$, *then* $f\in S_mH(F,\lambda,\gamma,k)$ *if*

$$\sum_{n=m+1}^{\infty}[n(k+1)-m\lambda(k+\gamma)]\,|a_nA_n|+\sum_{n=m}^{\infty}[n(k+1)+m\lambda(k+\gamma)]\,|b_nB_n|\leq 1.$$

Corollary 2. For $1\leq m\leq 1/(1-\gamma)$, *then* $f\in S_mH(F,\lambda,\gamma,k)$ *if*

$$\sum_{n=m+1}^{\infty}[n(k+1)-m\lambda(k+\gamma)]\,|a_nA_n|+\sum_{n=m}^{\infty}[n(k+1)+m\lambda(k+\gamma)]\,|b_nB_n| \leq m(1-\gamma).$$

Theorem 2. *Let* $f=h+\overline{g}$ *be given by* (1.9). *Then*

(*i*) for $1\leq m\leq 1/(1-\gamma)$, $f\in TS_mH(F,\lambda,\gamma,k)$ *if and only if*

$$\sum_{n=m+1}^{\infty}[n(k+1)-m\lambda(k+\gamma)]\,|a_nA_n|+\sum_{n=m}^{\infty}[n(k+1)+m\lambda(k+\gamma)]\,|b_nB_n| \leq m(1-\gamma); \tag{2.5}$$

(*ii*) for $m\geq 1/(1-\gamma)$, $f\in TS_mH(F,\lambda,\gamma,k)$ *if and only if*

$$\sum_{n=m+1}^{\infty}[n(k+1)-m\lambda(k+\gamma)]\,|a_nA_n|+\sum_{n=m}^{\infty}[n(k+1)+m\lambda(k+\gamma)]\,|b_nB_n| \leq 1. \tag{2.6}$$

Proof. Since $S_mH(F,\lambda,\gamma,k)\subset TS_mH(F,\lambda,\gamma,k)$, we only need to prove the "only if " part of the theorem. For f of the form (1.9), then

$$\operatorname{Re}\left\{\frac{(1+ke^{i\theta})\left[z(h*H)'(z)-z\overline{(g*G)'(z)}\right]-m(ke^{i\theta}+\gamma)\left\{(1-\lambda)z^m+\lambda[(h*H)(z)+\overline{(g*G)(z)}]\right\}}{(1-\lambda)z^m+\lambda[(h*H)(z)+\overline{(g*G)(z)}]}\right\}\geq 0,$$

that is

$$\operatorname{Re}\left\{\frac{m(1-\gamma)z^m-\sum\limits_{n=m+1}^{\infty}[n(1+ke^{i\theta})-\lambda m(ke^{i\theta}+\gamma)]a_nA_nz^n-\sum\limits_{n=m}^{\infty}[n(1+ke^{i\theta})+\lambda m(ke^{i\theta}+\gamma)]\overline{b_nB_nz^n}}{z^z-\sum\limits_{n=m+1}^{\infty}\lambda a_nA_nz^n+\overline{\sum\limits_{n=m}^{\infty}\lambda b_nB_nz^n}}\right\}\geq 0. \tag{2.7}$$

The condition (2.7) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, and noting that $\operatorname{Re}\{-e^{i\theta}\}\geq -\left|e^{i\theta}\right|=-1$, we have

$$\frac{m(1-\gamma)-\sum\limits_{n=m+1}^{\infty}[n(1+ke^{i\theta})-\lambda m(ke^{i\theta}+\gamma)]a_nA_nr^{n-m}-\sum\limits_{n=m}^{\infty}[n(1+ke^{i\theta})+\lambda m(ke^{i\theta}+\gamma)]b_nB_nr^{n-m}}{1-\sum\limits_{n=m+1}^{\infty}\lambda a_nA_nr^{n-m}+\sum\limits_{n=m}^{\infty}\lambda b_nB_nr^{n-m}}\geq 0. \tag{2.8}$$

If the condition (2.7) does not hold, then the numerator in (2.8) is negative for r sufficiently close to 1. Hence, there exist $z_0=r_0$ in $(0,1)$ for which the quotient of (2.8) is negative. This contradicts the required condition for $f\in TS_mH(F,\lambda,\gamma,k)$. This completes the proof of Theorem 2.

Theorem 3. *If* $f\in TS_mH(F,\lambda,\gamma,k)$, *then for* $|z|=r<1, |A_{m+1}|\leq|A_n|\leq|B_n|\ (n\geq m+1)$ *and* $A_{m+1}\neq 0$, *we have*

$$|f(z)|\leq\begin{cases}(1+|b_m|)r^m+\left[\frac{m(1-\gamma)}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\right.\\ \left.-\frac{m[1+k+\lambda(k+\gamma)]}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\left|b_mB_m\right|\right]r^{m+1},\\ [1+k+\lambda(k+\gamma)]\left|b_mB_m\right|<(1-\gamma)\leq\frac{1}{m};\\ (1+|b_m|)r^m+\left[\frac{1}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\right.\\ \left.-\frac{[(m+1)(k+1)+m\lambda(k+\gamma)]}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\left|b_mB_m\right|\right]r^{m+1},\\ [(m+1)(k+1)+m\lambda(k+\gamma)]\left|b_mB_m\right|<1 \text{ and } m(1-\gamma)\geq 1\end{cases} \tag{2.9}$$

and

$$|f(z)|\geq\begin{cases}(1-|b_m|)r^m-\left[\frac{m(1-\gamma)}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\right.\\ \left.-\frac{m[1+k+\lambda(k+\gamma)]}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\left|b_mB_m\right|\right]r^{m+1},\\ [1+k+\lambda(k+\gamma)]\left|b_mB_m\right|<(1-\gamma)\leq\frac{1}{m};\\ (1-|b_m|)r^m-\left[\frac{1}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\right.\\ \left.-\frac{[(m+1)(k+1)+m\lambda(k+\gamma)]}{[(m+1)(k+1)-m\lambda(k+\gamma)]|A_{m+1}|}\left|b_mB_m\right|\right]r^{m+1},\\ [(m+1)(k+1)+m\lambda(k+\gamma)]\left|b_mB_m\right|<1 \text{ and } m(1-\gamma)\geq 1\end{cases} \tag{2.10}$$

The results are sharp.

Proof. For $m(1-\gamma) \leq 1, f \in TS_m H(F, \lambda, \gamma, k)$ and $|A_{m+1}| \leq |A_n| \leq |B_n|$ $(n \geq m+1)$. From (1.9), we have

$$\begin{aligned} |f(z)| &= \left| z^m - \sum_{n=m+1}^{\infty} |a_n| z^n + \sum_{n=m}^{\infty} |b_n| \overline{z^n} \right| \\ &= \left| z^m + |b_m| \overline{z^m} - \sum_{n=m+1}^{\infty} [|a_n| - |b_n|] \overline{z^n} \right| \\ &\leq (1 + |b_m|) r^m + \sum_{n=m+1}^{\infty} [|a_n| + |b_n|] r^{m+1} \\ &\leq (1 + |b_m|) r^m + \frac{m(1-\gamma)}{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|} \cdot \\ &\quad \cdot \sum_{n=m+1}^{\infty} \frac{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|}{m(1-\gamma)} [|a_n| + |b_n|] r^{m+1} \end{aligned}$$

$$\begin{aligned} &\leq (1 + |b_m|) r^m + \frac{m(1-\gamma)}{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|} \cdot \\ &\quad \cdot \left(\sum_{n=m+1}^{\infty} \tfrac{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|}{m(1-\gamma)} |a_n| + \tfrac{[(m+1)(k+1) + m\lambda(k+\gamma)] |A_{m+1}|}{m(1-\gamma)} |b_n| \right) r^{m+1} \\ &\leq (1 + |b_m|) r^m + \frac{m(1-\gamma)}{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|} \cdot \\ &\quad \cdot \left(\sum_{n=m+1}^{\infty} \tfrac{[(m+1)(k+1) - m\lambda(k+\gamma)]}{m(1-\gamma)} |a_n A_n| + \tfrac{[(m+1)(k+1) + m\lambda(k+\gamma)]}{m(1-\gamma)} |b_n B_n| \right) r^{m+1}. \end{aligned}$$

From (2.5), we have

$$\begin{aligned} |f(z)| &\leq (1 + |b_m|) r^m + \tfrac{m(1-\gamma)}{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|} \left(1 - \tfrac{[(m+1)(k+1) + m\lambda(k+\gamma)]}{m(1-\gamma)} |b_m B_m| \right) r^{m+1} \\ &\leq (1 + |b_m|) r^m + \left(\tfrac{m(1-\gamma)}{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|} - \tfrac{m[k+1+\lambda(k+\gamma)] |b_m B_m|}{[(m+1)(k+1) - m\lambda(k+\gamma)] |A_{m+1}|} \right) r^{m+1}. \end{aligned}$$

This completes the first inequality of the theorem. The proof of the others are similar and so, we omit them.

Theorem 4. *Let $A_n \neq 0, n \geq m+1, B_n \neq 0, n \geq m$. Then $f \in clcoTS_m H(F, \lambda, \gamma, k)$ if and only if f can be expressed as*

$$f(z) = X_m h_m(z) + \sum_{n=m+1}^{\infty} \left(X_n h_n(z) + Y_n g_n(z) \right), \qquad (2.11)$$

where $h_m(z) = z^m$,

$$h_n(z) = \begin{cases} z^m - \dfrac{m(1-\gamma)}{[n(1+k) - m(k+\gamma)] |A_n|} z^n & (n \geq m+1, m(1-\gamma) \leq 1), \\ z^m - \dfrac{1}{[n(1+k) - m(k+\gamma)] |A_n|} z^n & (n \geq m+1, m(1-\gamma) \geq 1), \end{cases} \qquad (2.12)$$

$$g_n(z)=\begin{cases} z^m+\dfrac{m(1-\gamma)}{[n(1+k)+m(k+\gamma)]\,|B_n|}\overline{z}^{\;n}\ (n\geq m, m(1-\gamma)\leq 1),\\ z^m+\dfrac{1}{[n(1+k)+m(k+\gamma)]\,|B_n|}\overline{z}^{\;n}\ (n\geq m, m(1-\gamma)\geq 1),\end{cases} \quad (2.13)$$

$$X_m+\sum_{n=m+1}^{\infty}X_n+\sum_{n=m}^{\infty}Y_n=1 \ \textit{and}\ X_n,Y_n\geqslant 0.$$

Proof. Let $m(1-\gamma)\leq 1$. For f of the form (2.11), we have

$$f(z)=z^m-\sum_{n=m+1}^{\infty}\frac{m(1-\gamma)}{[n(1+k)-m(k+\gamma)]\,|A_n|}X_nz^n+\sum_{n=m}^{\infty}\frac{m(1-\gamma)}{[n(1+k)+m(k+\gamma)]\,|B_n|}Y_n\overline{z}^n. \quad (2.14)$$

Since, $0\leq X_m\leq 1$, we have

$$\begin{aligned} &\sum_{n=m+1}^{\infty}\frac{[n(1+k)-m(k+\gamma)]\,|A_n|}{m(1-\gamma)}\frac{m(1-\gamma)}{[n(1+k)-m(k+\gamma)]\,|A_n|}X_n\\ &+\sum_{n=m}^{\infty}\frac{[n(1+k)+m(k+\gamma)]\,|B_n|}{m(1-\gamma)}\frac{m(1-\gamma)}{[n(1+k)+m(k+\gamma)]\,|B_n|}Y_n\\ =\ &\sum_{n=m+1}^{\infty}X_n+\sum_{n=m}^{\infty}Y_n=1-X_m\leq 1.\end{aligned}$$

Consequently, $f\in TS_mH(F,\lambda,\gamma,k)$.

Conversely, let $f\in TS_mH(F,\lambda,\gamma,k)$.Then

$$|a_n|\leq\frac{m(1-\gamma)}{[n(1+k)-m(k+\gamma)]\,|A_n|},\quad |b_n|\leq\frac{m(1-\gamma)}{[n(1+k)+m(k+\gamma)]\,|B_n|}. \quad (2.15)$$

Putting

$$X_n=\frac{[n(1+k)-m(k+\gamma)]\,|a_nA_n|}{m(1-\gamma)},\quad Y_n=\frac{[n(1+k)+m(k+\gamma)]\,|b_nB_n|}{m(1-\gamma)}, \quad (2.16)$$

and

$$X_m=1-\left(\sum_{n=m+1}^{\infty}X_n+\sum_{n=m}^{\infty}Y_n\right)\geq 0,$$

we have

$$\begin{aligned} f(z) &= z^m - \sum_{n=m+1}^{\infty} |a_n| z^n + \sum_{n=m}^{\infty} |b_n| \overline{z}^n \\ &= z^m - \sum_{n=m+1}^{\infty} \tfrac{m(1-\gamma)}{[n(1+k)-m(k+\gamma)]|A_n|} X_n z^n + \sum_{n=m}^{\infty} \tfrac{m(1-\gamma)}{[n(1+k)+m(k+\gamma)]|B_n|} Y^n \overline{z}^n \\ &= z^m - \sum_{n=m+1}^{\infty} (z^m - h_n(z)) X_n - \sum_{n=m}^{\infty} (z^m - g_n(z)) Y_n \\ &= \left[1 - \left(\sum_{n=m+1}^{\infty} X_n + \sum_{n=m}^{\infty} Y_n\right)\right] z^m + \sum_{n=m+1}^{\infty} h_n(z) X_n + \sum_{n=m}^{\infty} g_n(z) Y_n \\ &= X_m z^m + \sum_{n=m+1}^{\infty} h_n(z) X_n + \sum_{n=m}^{\infty} g_n(z) Y_n. \end{aligned}$$

Thus f can be expressed in the form (2.11). The case for $m(1-\gamma) \geq 1$ can be proved in the same manner and hence we omit it.

Theorem 5. *The class $TS_m H(F, \lambda, \gamma, k)$ is closed under convex combinations.*

Proof. For $i = 1, 2, ...,$ let the functions f_i given by

$$f_i(z) = z^m - \sum_{n=m+1}^{\infty} |a_{in}| z^n + \sum_{n=m}^{\infty} |b_{in}| \overline{z}^n \tag{2.17}$$

are in the class $TS_m H(F, \lambda, \gamma, k)$ and suppose that the fixed functions F_i are given by

$$F_i(z) = z^m + \sum_{n=m+1}^{\infty} |A_{in}| z^n + \sum_{n=m}^{\infty} |B_{in}| \overline{z}^n. \tag{2.18}$$

For $0 \leq \mu_i \leq 1, \sum_{i=1}^{\infty} \mu_i = 1$, the convex combinations can be expressed in the form

$$\sum_{i=1}^{\infty} \mu_i f_i = z^m - \sum_{n=m+1}^{\infty} \left(\sum_{i=1}^{\infty} \mu_i |a_{in}|\right) z^n + \sum_{n=m}^{\infty} \left(\sum_{i=1}^{\infty} \mu_i |b_{in}|\right) \overline{z}^n. \tag{2.19}$$

From (2.5) and (2.6), we have

$$\begin{aligned}
&\sum_{n=m+1}^{\infty}[n+(n-m\lambda)k-\lambda m\gamma]\sum_{i=1}^{\infty}\mu_i\left|a_{in}A_{in}\right| \\
&+\sum_{n=m}^{\infty}[n+(n+m\lambda)k+\lambda m\gamma]\sum_{i=1}^{\infty}\mu_i\left|b_{in}B_{in}\right| \\
=\ &\sum_{i=1}^{\infty}\mu_i\left\{\sum_{n=m+1}^{\infty}[n+(n-m\lambda)k-\lambda m\gamma]\left|a_{in}A_{in}\right|\right. \\
&\left.+\sum_{n=m}^{\infty}[n+(n+m\lambda)k+\lambda m\gamma]\left|b_{in}B_{in}\right|\right\} \\
\leq\ &\begin{cases} m(1-\gamma)\sum\limits_{i=1}^{\infty}\mu_i=m(1-\gamma) & if\ m(1-\gamma)\leq 1 \\ \sum\limits_{i=1}^{\infty}\mu_i=1 & if\ m(1-\gamma)\geq 1. \end{cases}
\end{aligned}$$

That is, that,

$$\sum_{i=1}^{\infty}\mu_i f_i(z)\in TS_mH(F,\lambda,\gamma,k),$$

which complets the proof of Theorem 5.

Remark. (**i**) *Putting* $\lambda=1$ and replacing m by p *in Theorems 1, 2, 3, 4 and 5 and Corollaries 1, 2 and 3, respectively , we obtain the results obtained by Ahuja et al. [10 , Theorems 2.1, 2.4, 2.5, 2.6 and 2.8 and Corollaries 2.2, 2.3 and 2.7, respectively , with* $t=1$*];*

(*ii*) *Putting* $k=m=1$ *and* $A_n=B_n$ *in Theorems 1, 2, 3, 4 and 5, respectively , we obtain the results obtained by Murugusundaramoorthy and Vijaya [8 , Theorems 1, 2, 3, 4 and 5, respectively];*

(*iii*) For different choices of the function F, as stated in (*i*), (*ii*) and (*iii*) in the introduction, we obtain new results corresponding to the corresponding classes.

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