New analytical internal long wave solutions described by the Benjamin Ono equation in deep water

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Abstract

Herein, we investigate new and different types of internal wave solutions in deep water. These solutions are described by the Benjamin Ono equation. We use the unified method to obtain these solutions in polynomial function type which are classified into three categories, namely solitary, soliton and elliptic wave solutions.

Keywords: Benjamin Ono equation; traveling wave solutions; Unified Method

Introduction

In literature, different physical phenomena in the real life and the nature are described by nonlinear evolution equations (NLEEs). Under using symbolic computation systems, the studying of the exact solutions of the NLEEs has attracted the attention of research community. A variety of approaches have been investigated and applied to the NLEEs, including the unified method (UM) [1-5] and its generalized form [6-10], the extended Jacobi elliptic function expansion method [11, 12], the Bernoulli sub-equation function method [13, 14], the sine-Gordon expansion method [15, 16] and the Ricatti equation expansion [17, 18].

Our main object in this paper to apply the UM [1-5] to investigate new types of wave solutions for the Benjamin Ono equation (BOE) [19, 20].

The BOE is given by

$$(1)\, u_{tt} + \lambda\,(u^2)_{xx} + \mu\, u_{xxxx} = 0,$$

where λ and μ are arbitrary constants. The BOE has a good application for describing internal long waves in deep water. In [21-23], different types of solution for Eq. (1) are obtained by using the extended homoclinic test. Among these solutions doubly periodic, rogue waves, breather solitary waves and lump wave solutions.

The article is arranged as follows: In section 2, the mathematical formulation of the UM is introduced. The application of the UM to the BOE is given in section 3. Conclusions are provided in section 4.

2- Mathematical formulation for the unified method (UM)

Consider the nonlinear evolution equation (NLEE) of the type

$$F\left(u, u_t, u_{x1}, \dots, u_{x_q}, u_{x1\,x2}, u_{x1\,x3}, \dots\right) = 0, \quad (2)$$

where F is a polynomial in its arguments and $u = u\left(t, x_1, \dots,\ x_q\right)$.

When $x_1, x_2, \dots, x_q$ and t are missing in Eq. (2), then each physical observable u possess $(q+1)$ basic traveling wave solutions that satisfy the equation

$$H(U, U', U'', \dots) = 0, \qquad \xi = \alpha_0 t + \sum_{s=1}^{q} \alpha_s x_s \quad (3)$$

where $U = U(\xi), U' = \frac{dU}{d\xi}, \alpha_0$ and α_s are arbitrary constants.

In this section, we find the traveling wave solutions (in polynomial function or rational function forms) for the NLEE given by (3) via the UM [1-5]. The outline of this method are presented as follow.

(I) Polynomial solutions.

To obtain the solutions of Eq. (3) in polynomial function forms, we assume that

$$U = U(\xi) = \sum_{i=0}^{n} p_i \Gamma^i(\xi),$$
$$\left(\Gamma'(\xi)\right)^{\tau} = \sum_{j=0}^{\tau\, m} b_j \Gamma^j(\xi), \ \ \xi = \alpha_0 t + \sum_{s=1}^{q} \alpha_s x_s \ , \tau = 1,2, \tag{4}$$

where p_i, b_j, α_0 and α_s are constants. The UM provides the balance principle technique to evaluate the relation between the two parameters n and m and satisfies the consistency condition between the arbitrary functions in the solutions given by Eq. (4) (for details see [1-5]).

It worth mentioning that, the UM solves (4) to elementary solutions or elliptic solutions when $\tau = 1$ or $\tau = 2$ respectively.

(II) Rational solutions.

To get these solutions, we suppose that

$$U = U(\xi) = \sum_{i=0}^{n} p_i \Gamma^i(\xi) \Big/ \sum_{i=0}^{k} q_i \Gamma^i(\xi) \ \ , n \geq k$$
$$\left(\Gamma'(\xi)\right)^{\tau} = \sum_{j=0}^{\tau\, m} b_j \Gamma^j(\xi), \ \ \xi = \alpha_0 t + \sum_{s=1}^{q} \alpha_s x_s \ , \tau = \ , \tag{5}$$

where p_i, q_i, b_j, α_0 and α_s are constants. Similarly, The UM provides the balance principle technique to evaluate the relation between the two parameters n, k and m and satisfies the consistency condition between the arbitrary functions in the solutions given by Eq. (5) (for details see [1-5]). Furthermore, the values of τ give different types for these solutions by the same criteria described in (I).

Here, we find only the solutions in the polynomial function type.

3 Exact solutions of BOE using the UM

Applying the transformation $u(x,t) = P(\xi),\ \xi = \alpha_1 x + \alpha_2 t$ on Eq. (1), it generates the following ordinary differential equation

$$\alpha_1{}^4\,\mu\,P^4(\xi) + \left(\alpha_2{}^2 + 2\alpha_1{}^2\,\lambda\,P(\xi)\right)P''(\xi) + 2\alpha_1{}^2\,\lambda\,P'^2(\xi) = 0 \tag{6}$$

where $P' = \frac{dP}{d\xi}$, and α_1 , α_2 are constants.

Integrating Eqs. (6) twice while taking the constants of integration as zero, it gives

$$\alpha_1{}^4\,\mu\,P''(\xi) + \alpha_1{}^2\,\lambda\,P^2(\xi) + \alpha_2{}^2 P(\xi) = 0 \tag{7}$$

In the next sub-section, we use the UM technique to find the traveling wave solutions of Eq. (7) in the polynomial function type.

3.1 Polynomial function solutions of the BOE

To find the polynomial function solutions of the BOE, we assume that

$$P(\xi) = \sum_{i=0}^{n} p_i \Gamma^i(\xi),$$

$$\left(\Gamma'(\xi)\right)^\tau = \sum_{j=0}^{\tau m} b_j \Gamma^j(\xi), \quad \tau = 1,2, \tag{8}$$

where p_i and b_j are constants. By considering the homogeneous balance relation between P'' and P^2 in Eq. (7), we get $n = 2(m-1)$ and $m = 2, 3,...$ Here, we con ne ourselves to find these solutions when $m = 2$ and $\tau = 1$ or $\tau = 2$.

3.1.1 Solitary wave solution

To obtain this solution, we consider $\tau = 1$ in the auxiliary equation given by (8). From Eq. (8), we have

$$P(\xi) = p_0 + p_1\Gamma(\xi) + p_2\Gamma^2(\xi)\,,$$

$$\Gamma'(\xi) = b_2\Gamma^2(\xi) + b_1\Gamma(\xi) + b_0 \qquad (9)$$

Substituting Eq. (9) into Eq. (7) and equating the coefficients of $\Gamma(\xi)$ to zero, we obtain a set of algebraic equations. By means of a symbolic computations package, we get the following two sets of algebraic equations

$$p_0 = -\frac{(b_1^2 + 2b_2b_0)\alpha_2^2}{R^2\,\alpha_1^2\,\lambda}\,, p_1 = -\frac{6b_1b_2\alpha_2^2}{R^2\,\alpha_1^2\,\lambda}\,, p_2 = -\frac{6b_2^2\alpha_2^2}{R^2\,\alpha_1^2\,\lambda},$$

$$\mu = \frac{\alpha_2^2}{R^2\,\alpha_1^2} \qquad (10)$$

where $R = \sqrt{b_1^2 - 4b_2b_0}$.

Solving the auxiliary equation $\Gamma'(\xi) = b_2\Gamma^2(\xi) + b_1\Gamma(\xi) + b_0$ and substituting together with (10) into Eq. (9), we get the solution of Eq. (1), namely

$$u_1(x,t) = \frac{\alpha_2^2}{2\alpha_1^2\,\lambda}\left(1 - 3\tanh^2\left(\frac{1}{2}\,R\,\xi\right)\right) \qquad (11)$$

where $\xi = \alpha_1\,x + \alpha_2\,t$.

3.1.2 Soliton wave solution

Here we find the soliton wave solution. We consider $\tau = 2$ in the auxiliary equation given by (8). From Eq. (8), we have

$$P(\xi) = p_0 + p_1\Gamma(\xi) + p_2\Gamma^2(\xi)\,,$$

$$\Gamma'(\xi) = \Gamma(\xi)\sqrt{b_2\Gamma^2(\xi) + b_1\Gamma(\xi) + b_0} \tag{12}$$

Substituting (12) into Eq. (7), we get a set of algebraic equations that yield

$$p_0 = -\frac{\alpha_2^2}{\alpha_1^2\,\lambda}\,, p_1 = -\frac{12b_2\alpha_2^2}{b_1\alpha_1^2\,\lambda}\,, p_2 = -\frac{24\,b_2^2\alpha_2^2}{b_1^2\,\alpha_1^2\,\lambda},$$

$$\mu = \frac{4\,b_2\alpha_2^2}{b_1^2\,\alpha_1^4}, b_0 = \frac{b_1^2}{4b_2} \tag{13}$$

Solving the auxiliary equation $\Gamma'(\xi) = b_2\Gamma^2(\xi) + \; b_1\Gamma(\xi) + b_0$ and substituting together with (13) into Eq. (12), we get the solution of Eq. (1)

$$u_2(x,t) = -\frac{(1 + 8\,b_1\,\exp(\frac{b_1\xi}{2\sqrt{b_2}}) + 4\,b_1^2\exp(\frac{b_1\xi}{\sqrt{b_2}}))\,\alpha_2^2}{(1 - 2\,b_1\,\exp(\frac{b_1\xi}{2\sqrt{b_2}}))^2\alpha_1^2\,\lambda} \tag{14}$$

where $\xi \;=\; \alpha_1\,x \;+\; \alpha_2\,t\,, b_2 > 0$.

3.1.3 Elliptic wave solution

Here we find the elliptic wave solution. We consider $\tau \;=\; 2$ in the auxiliary equation given by (8). From Eq. (8), we have

$$P(\xi) = p_0 + p_1\Gamma(\xi) + p_2\Gamma^2(\xi)\,,$$

$$\Gamma'(\xi) = \Gamma(\xi)\sqrt{b_4\Gamma^4(\xi) + b_2\Gamma^2(\xi) + b_0} \tag{15}$$

Substituting from (15) into Eq. (7), we get

$$p_0 = -\frac{\alpha_2^2}{2\,\alpha_1^2\,\lambda}(1 + \frac{b_2}{H})\,, p_1 = 0\,, p_2 = -\frac{3b_4\alpha_2^2}{2H\,\alpha_1^2\,\lambda}\,, \mu = \frac{\alpha_2^2}{4H\,\alpha_1^4}, \tag{16}$$

$$H=\sqrt{b_2^2-3b_4b_0}.$$

Substituting from Eq. (15) into Eq. (7), we obtain the solution of Eq. (1)

$$u_3(x,t)=-\frac{\left(b_2+H+3b_4\Gamma^2(\xi)\right)}{2H\alpha_1^4\lambda} \tag{17}$$

where $\Gamma(\xi)$ is obtained by solving the auxiliary equation given by Eq. (15), $\xi = \alpha_1 x + \alpha_2 t$.

It must be noted that $b_i, i = 0, 2, 4,$ in Eq. (17) are arbitrary constants and that for particular values of bi, we get different Jacobi elliptic functions solutions [24].

According to the classification in [24], namely

$$b_4=\frac{4}{k}, b_2=-(k^2+6k+1), b_0=k^4+2k^3+k^2, \qquad 0<k<1, \tag{18}$$

the auxiliary function takes the form

$$\Gamma(\xi) = \frac{k\ dn\,(\xi,k)\ cn\,(\xi,k)}{k\ (sn^2(\xi,k)-1)}$$

and the solution given by Eq.(17) will be in the form

$$u_3^*(x,t)=-\frac{1}{2H\alpha_1^2\lambda}\left(b_2+H+3b_4\left(\frac{k\ dn\,(\xi,k)\ cn\,(\xi,k)}{k\ (sn^2(\xi,k)-1)}\right)^2\right) \tag{19}$$

We mention that $0 < k < 1$ is called the modulus of the Jacobi elliptic functions. When $k \to 0$, $sn(\xi)$, $cn(\xi)$ and $dn(\xi)$ degenerate to $sin(\xi)$, $cos(\xi)$ and 1 respectively. While when $k \to 1$, $sn(\xi)$, $cn(\xi)$ and $dn(\xi)$ degenerate to $\tanh(\xi)$, $\operatorname{sech}(\xi)$ and $\operatorname{sech}(\xi)$ respectively.

4 Conclusion

In this paper, we found different types of solutions of the BOE. Among these solutions: solitary wave, soliton wave and elliptic wave solutions. The solutions of the BOE are obtained by using the UM. This method can not only give a unified formulation to uniformly construct polynomial solutions, but also can provide us a guideline to classify the types of these solutions according to the given parameters. The method which we have proposed in this work is also a standard, direct and computerized method, which allow us to do complicated and tedious algebraic calculation.

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حلول تحليلية جديدة للموجات الطويلة الداخلية الموصوفة بمعادلة بنيامين- أونو في المياه العميقة

جمال المطوع
الهيئة العامة للتعليم التطبيقي والتدريب - المعهد العالي للخدمات الإدارية – الكويت

الملخص

قمنا في هذا البحث بدراسة أنواع مختلفة وجديدة من حلول الموجات الداخلية في المياه العميقة.

يتم وصف هذه الحلول باستخدام معادلة بنيامين- أونو في المياه العميقة. استخدمنا الطريقة الموحدة لايجاد هذه الحلول في شكل دالة كثيرة حدود والتي يتم تصنيفها لثلاث فئات وهي حلول ذات موجة منعزلة ، حلول ذات موجة سوليتونية ،و حلول ذات موجه إهليجية.

الكلمات المفتاحية: معادلة بنيامين أونو - حلول ذات موجة متنقلة - الطريقة الموحدة.