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CONCENTRATION DISTRIBUTION AROUND A GROWING GAS BUBBLE IN A BIOTISSUES UNDER THE EFFECT OF SUCTION AND INJECTION

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Abstract. The paper presents the concentration distribution around a growing nitrogen gas bubble in the blood and other bio tissues of divers who surface too quickly under the effect of suction and injection. The modification of Mohammadein and Mohamed model (2010) for ambient pressure through the decompression process is considered as variant and constant. The mathematical model is solved analytically to find the growth rate of a gas bubble in bio tissues after decompression in the ambient pressure. The growth process is affected by ascent rate , tissue diffusivity , initial concentration difference , surface tension and void fraction .The gas bubbles grows slowly in bio tissues of divers under the effect of suction than injection and Mohammadein and Mohamed model (2010) respectively. Results are compared with Mohammadein and Mohamed model.

1. Introduction

Decompression sickness (DCS) is a dangerous disease caused by nitrogen bubbles; which appears in the blood and other tissues for divers who surface too quickly or people who flight for long distances from the earth (Fig.1). The growth problem is discussed for unsteady flow in tissue by Mohammadein and Mohamed [6]; in case of three-region model [8]. Moreover, the concentration distribution around a stationary growing gas bubble in tissue is obtained analytically for two main growth stages.

Srinivasan, R. S. et al. [8] have solved the problem in the case of quasi-static pressure. Mohammadein and Mohamed [6] solved the problem when the effect of changing in concentration with the time takes place. The growth stages can be repeated sequentially, while the diver ascents quickly to a lower-pressure sea level and dives horizontally, and so on until he reaches the sea level pressure (1 atm.)

The same decompression effect may be occurred when aviators or astronauts are exposed to low-pressure environments, in this case $P_0 = P_{atm} = 101.325 N.m^{-2}$(the

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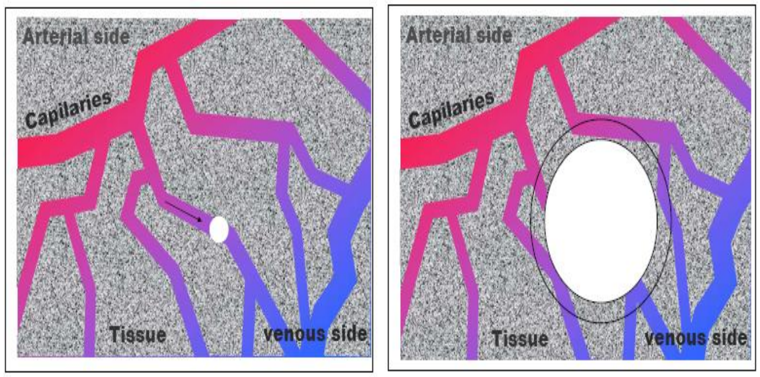

FIGURE 1. On the left, in the initial phase of the decompression, an arterial bubble enters a tissue capillary net. It exchanges gas with the surrounding tissues and starts growing. If it reaches a critical radius, it might block the blood supply and cause ischemia. On the right, in the last phase of the decompression, a bubble has grown to a large volume using dissolved gas available in the surrounding tissue. Its mechanical action might cause pain. [4]

sea level pressure)

In this work, the problem in more general case, at which the effect of suction and injection in tissues of divers is considered. The normal and critical gas bubbles in a bio tissues is illustrated in Fig.1.

2. Analysis

A single gas bubble as in Fig.2 is considered to grow inside a tissue between two finite boundaries R_0 and R_m under the effect of suction and injection in biotissues. The growth is affected by some parameters such as the pressure difference between the bubble pressure $P_g(R(t),t)$ and the ambient pressure $P_{amb}(t)$, surface tension of the mixture inside the bio tissue at the bubble boundary, concentration difference between the two phases and other physical parameters.
The growth of the gas bubble has been studied based on the following assumptions:

- Gases are considered to be ideal.
- The bubble is assumed to have a spherical geometry.
- Pressure inside the bubble is assumed to be uniform.
- Gas density distribution inside the bubble is assumed to be uniform.
- The viscosity of the fluid is omitted.
- The growth performed under the effect of suction and injection processes.

The mathematical model describing the current problem consists of four main equations ($mass, diffusion, Fick's$ and concentration equations).

Mass Balance equation: Assuming equilibration of tissue gas with venous blood gas. The rate of gas uptake by the biotissue is the amount carried by

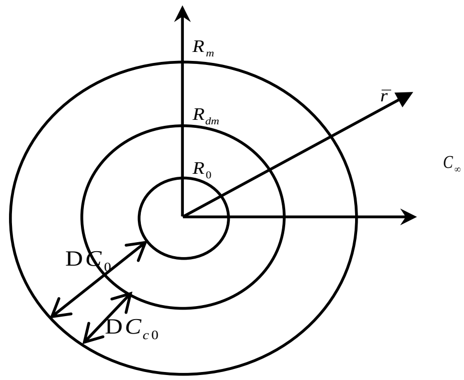

FIGURE 2. The growth stages.

the blood per unit time less than flux into the gas bubble. Thus, the mass equation has the form

$$\alpha_T V_T \frac{dP_T}{dt} = \alpha_T V_T \dot{Q}(P_a - P_T) - \frac{1}{\Re T}\frac{d}{dt}(P_g V_g) \tag{1}$$

Diffusion equation: The gas diffusion through the tissue without suction or injection and convection due to bubble movement [6] is described by

$$\frac{\partial C(r,t)}{\partial t} = D_T \nabla^2 C(r,t) \tag{2}$$

Fick's equation: The molar flux of gas through the bubble surface equals the rate of change of molar concentration of gas in the bubble. Thus

$$\frac{1}{\Re T}\frac{d}{dt}(\frac{4}{3}\pi R^3 P_g) = 4\pi R^2 D_T (\frac{\partial C}{\partial r})_{r=R} \tag{3}$$

Pressure Balance equation: Effects of surface tension at the gas-liquid interface of the bubble through the Laplace equation, neglecting tissue viscoelastic effects, is

$$P_g = P_{atm} + \frac{2\sigma}{R} \tag{4}$$

2.1. **Analytical Solution.** Assuming spherical symmetry, the above equation (2) with suction and injection and neglecting convection term is described in the form

$$\frac{\partial C(r,t)}{\partial t} = D_T(\frac{\partial^2 C}{\partial r^2} + \frac{2}{r}\frac{\partial C}{\partial r}) + \frac{bD_T}{r}\frac{\partial C}{\partial r}, \tag{5}$$

where

$$b = \begin{cases} -1 & \text{suction effect} \\ 0 & \text{Mohammadein and Mohamed model [6]} \\ 1 & \text{injection effect} \end{cases}$$

For solving the diffusion Eq. (5), the method of combined variables is used to solve the diffusion Eq.(5) [6, 10], then

$$C(r,t) = C(s)$$

where

$$s = \frac{\beta r}{f(t)} \tag{6}$$

At $r = R$, then $s = \beta$ and

$$R = f(t). \tag{7}$$

Based on the above assumptions, Eq. (5) becomes, after separating of variables,

$$f(t)\dot{f}(t) = \frac{-\beta^2 D_T}{s}(\frac{1}{\frac{dC}{ds}})\frac{d^2C}{ds^2} - (2+b)\frac{\beta^2 D_T}{s^2} = D_T^2\mu. \tag{8}$$

The separation constant in the form $D_T^2\mu$ divides Eq. (8) into two differential equations

$$f(t)\dot{f}(t) = D_T^2\mu. \tag{9}$$

and

$$\frac{d}{ds}\ln(\frac{dC}{ds}) = -(\frac{\mu D_T s}{\beta^2} + \frac{(2+b)}{s}). \tag{10}$$

Apply integration to Eq. (9), and by using of the boundary condition at $t = t_0$, $R = R_0$, therefore

$$R(t) = \sqrt{2\mu D_T^2(t-t_0) + R_0^2}. \tag{11}$$

Now, integrating Eq. (10), we obtain

$$\frac{dC}{ds} = k_1 s^{-(2+b)}\exp(-\frac{\mu D_T s^2}{2\beta^2}). \tag{12}$$

For getting an expression for the constant k_1 . The initial condition (3), by using of Eq. (4), is modified to be

$$(\frac{\partial C}{\partial r})_{r=R} = \frac{1}{3\Re T D_t}(R\dot{P_{atm}} + \frac{4\sigma\dot{R}}{R} + 3P_{amb}\dot{R}). \tag{13}$$

and

$$(\frac{dC}{ds})_{s=\beta} = (\frac{r}{s}\frac{\partial C}{\partial r})_{r=R,s=\beta}. \tag{14}$$

Substituting from Eq. (14) into Eq. (12), then

$$k_1 = \frac{\beta^{(b+1)}(R^2\dot{P_{amb}} + 4\sigma\dot{R} + 3R\dot{R}P_{amb})}{3\Re T D_T}\exp(-\frac{\mu D_T s^2}{2\beta^2}). \tag{15}$$

From Eq. (12) into Eq. (14), we have

$$\frac{\partial C}{\partial r} = \frac{k_1}{R^{-(b+1)}}\beta^{-(b+1)}r^{-(b+2)}\exp(-\frac{\mu D_T s^2}{2\beta^2}) = \frac{k}{R^{-(b+1)}}r^{-(b+2)}\exp(-\frac{\mu D_T r^2}{2R^2}). \tag{16}$$

Integrating the previous equation through the interval from any instant t to at which the bubble reaches its maximum radius R_m , at this instant $C(R_m, t_m) = C_\infty$, then

$$C(r,t) - C_\infty = -kR^{(b+1)}\int_r^{R_m}\frac{1}{x^{(b+2)}}\exp(-\frac{\mu D_T x^2}{2R^2})dx. \tag{17}$$

At the bubble wall $r = R(t)$ the previous equation becomes

$$C(R(t),t) - C_\infty = -kR^{(b+1)}\int_{R(t)}^{R_m}\frac{1}{x^{(b+2)}}\exp(-\frac{\mu D_T x^2}{2R^2})dx. \tag{18}$$

Putting $y = \frac{x}{R}$, thus the previous equation can be written in the form

$$C(R(t),t) - C_\infty = -k\int_t^{\frac{R_m}{R(t)}} \frac{1}{y^{(b+2)}}\exp(-\frac{\mu D_T}{2}y^2)dy. \tag{19}$$

To find the relation between the bubble radius $R(t)$ and the time t; we assume that at $t = t_0 \Rightarrow R(t_0) = R_0$, then Eq. (21) becomes

$$\mu = \sqrt{\frac{8(3-b)\triangle C_0}{D_T^2 k(\varphi_0^{\frac{-1}{3}(3-b)} - 1)}} \tag{20}$$

since $D_T = 1$ and $0 < \varphi_0 < 1$, then $k_0 = \dfrac{R_0^2 \dot{P_{amb}} + 4\sigma\dot{R}_0 + 3R_0\dot{R}_0 P_{amb0}}{3\Re T D_T}$ (21)

Substituting for μ into Eq. (11) we get the relation of the bubble radius as a function of time.

$$R(t) = \sqrt{R_0^2 + 2D_T\sqrt{\frac{8(3-b)\triangle C_0}{k(\varphi_0^{\frac{-1}{3}(3-b)} - 1)}}(t-t_0)} \tag{22}$$

The initial growth velocity can get the following approximated value

$$\dot{R}_0 \approx \frac{\varphi_0^{\frac{1}{3}}}{R_0}D_T \tag{23}$$

In the follows, the growth process is divided into two main stages:

(1) The first stage takes place during the decompression of the ambient pressure $P_{amb}(t)$ as a function of time.
(2) The second stage takes place at the end of the decompression at which $\frac{d}{dt}P_{amb}(t) = 0$ i.e. $P_{amb}(t) = const = P_\infty$.

For the decompression stage, suppose the ambient pressure linearly decreases with time, i.e. $P_{amb}(t) = P_0 - \dot{\alpha}t$, where $\dot{\alpha}$ is the ascent rate [10], then

$$R_d(t) = \sqrt{R_0^2 + 2D_T\sqrt{\frac{8(3-b)\triangle C_0}{k_d(\varphi_0^{\frac{-1}{3}(3-b)} - 1)}}(t-t_0)}, \text{for} t_0 \le t \le t_{dm} \tag{24}$$

where

$$k_d = \frac{-\alpha R_0^2 + 4\sigma\dot{R}_0 + 3R_0\dot{R}_0(P_0 - \dot{\alpha}t_0)}{3\Re T D_T} \tag{25}$$

$$\triangle C_{d0} = C_\infty - C(R_0,t_0) \quad \text{and} \quad \varphi_{0d} = (\frac{R_0}{R_{dm}})^3 \tag{26}$$

At the end of the decompression stage, at which the ambient pressure becomes constant i.e. $P_{amb}(t) = const = P_\infty$, then

$$R_c(t) = \sqrt{R_{dm}^2 + 2D_T\sqrt{\frac{8(3-b)\triangle C_{d0}}{k_c(\varphi_0^{\frac{-1}{3}(3-b)} - 1)}}(t-t_{dm})}, \text{for} t_{dm} \le t \le t_m \tag{27}$$

where

$$k_c = \frac{4\sigma \dot{R_{c0}} + 3R_{dm}\dot{R_{c0}}P_\infty}{3\Re T D_T} \quad (28)$$

2.2. **Concentration distribution around a growing gas bubble under the effect of suction and injection in a biotissues.** The growth and concentration distribution in tissues during a growing nitrogen bubbles are studied in two intervals of time and bubble radius; which called decompression and after decompression stages respectively. For the decompression stage $R_0 \le r(t) \le R_{dm}$ and $t_0 \le t \le t_{dm}$. After decompression stage $R_{dm} \le r(t) \le R_m$ and $t_{dm} \le t \le t_m$.

The concentration distribution in tissues during a growing nitrogen bubbles is given by

Putting $y = \frac{x}{R}$ into Eq. (17), it becomes

$$C(r,t) - C_\infty = -k_d(t) \int_{\frac{r}{R_d}}^{\frac{R_m}{R_d}} \frac{1}{y^{(b+2)}} \exp(-\frac{\mu D_T}{2} y^2) dy. \quad (29)$$

$$C(r,t) - C_\infty = -\frac{k_d A^2}{2(3-b)} \left((\frac{R_m}{R})^{3-b} - (\frac{r}{R})^{3-b} \right). \quad (30)$$

Where

$$k_d(t) = \frac{-\alpha R_d^2(t) + 4\sigma \dot{R_d}(t) + 3R_d(t)\dot{R_d}(t)(P_0 - \dot{\alpha} t)}{3\Re T D_T} \quad (31)$$

and

$$\dot{R_d}(t) = \frac{D_T^2 \mu_d}{R_d(t)} \quad (32)$$

After decompression stage $R_{dm} \le r(t) \le R_m$ and $t_{dm} \le t \le t_m$, the gas distribution around the growing bubble is given by

$$C(r,t) - C_\infty = -k_c(t) \int_{\frac{r}{R_c}}^{\frac{R_m}{R_c}} \frac{1}{y^{(b+2)}} \exp(-\frac{\mu_c D_T}{2} y^2) dy. \quad (33)$$

where

$$k_c(t) = \frac{4\sigma \dot{R_c}(t) + 3R_c(t)\dot{R_c}(t)P_\infty}{3\Re T D_T} \quad (34)$$

and

$$\dot{R_c}(t) = \frac{D_T^2 \mu_c}{R_c(t)}. \quad (35)$$

3. Implementation

The following table shows the data which used to simulate the problem for decompression stage.

By using Mathematica program (Version 6.0) we get the following graphs that demonstrate the effect of the physical parameters on the growth of the gas bubble.

Parameter	Value	Unit	Parameter	Value	Unit
T	310 ($37°C$)	K	$R_\circ$	1.0×10^{-6}	m
p_0	200,000 [10]	N/m^2	α	3066.67 [10]	$N/m^2 s$
$\triangle C_0$	0.7	mol/m^3	$\Re$	8.314472 [5]	$Nm/molK$
σ	0.03 [8]	N/m	$t_\circ$	0.0	s
D_r	2.210^{-12}[8]	m^2/s			

4. Discussion of the Results

The growth of a gas bubble in tissue is obtained as a function of time and other physical parameters. On contrary, the previous problems are presented a numerical or an implicit solution as given by authors [1, 8, 10].The diffusion equation (5) for a stationary growing gas bubble in a bio tissue is solved under the effect of suction and injection processes. The problem of growth and concentration is solved by the method of combined variables under the effect of decompression in ambient pressure.

The growth of gas bubble in terms of time for three different values of parameter "b" when ambient pressure is constant or variant is given by Figs.3 and 4. It is observed that growth is proportional with the values of parameter "b". The growth of gas bubble in terms of time for three different values of gas diffusion coefficient in tissue when ambient pressure is variant is given by Fig.5. It is observed that growth is proportional with the values of gas diffusion coefficient in tissue. The Concentration distribution around a growing nitrogen bubble in the bio-tissues for diver when $b = -1, 0, 1$ for suction, zero suction and injection effects respectively are given by Figs. 6, 7, and 8. It is observed that the Concentration distribution is proportional inversely with the values of parameter "b".

5. Conclusion

The growth problem is discussed for unsteady flow in the bio tissues of divers. Based on the three-region model [8], the concentration distribution around a stationary growing gas bubble in the bio tissues is obtained analytically for the two main growth stages as given by Eqs. (29) and (33) respectively. The discussion of results and figures concluded the following remarks:

(1) The concentration gradient decreases while the growth process is taking place until it vanishes at complete growth of the bubble.
(2) The growth of bubble radius is proportional with the ascent rate $\dot{\alpha}$, the initial difference in concentration $\triangle C_0$, the diffusivity of the tissue D_T , the initial void fraction φ_0 and inversely proportional to the surface tension σ of the tissue.
(3) The growth of gas bubbles proportional with all values of parameter "b".
(4) The suction effect ($b = -1$) performs lower values of growth than that obtained in case of Mohammadein and Mohamed model ($b = 0.0$) [6] and injection effect ($b = 1$) respectively.
(5) The current problem can be used to avoid the divers from many bad side effects of the decompression sickness (DCS) under suction effects.

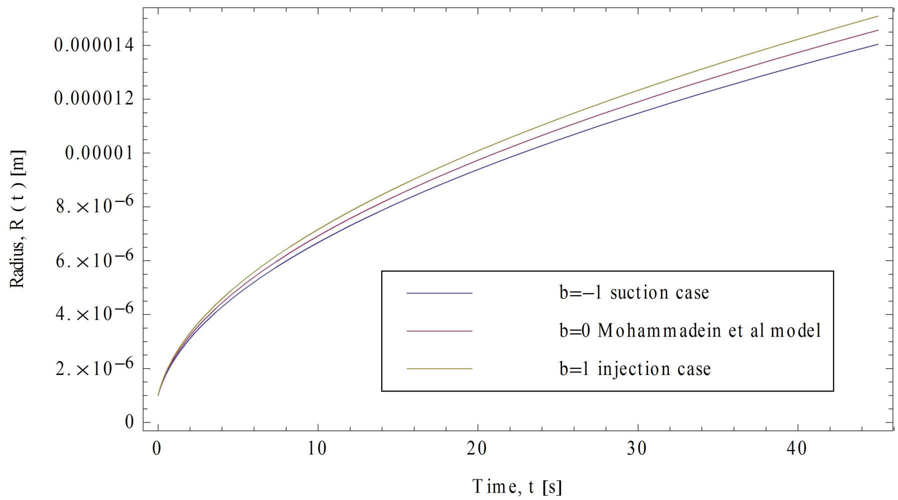

FIGURE 3. The growth of gas bubble in terms of time for three different values of parameter "b" when ambient pressure is constant.

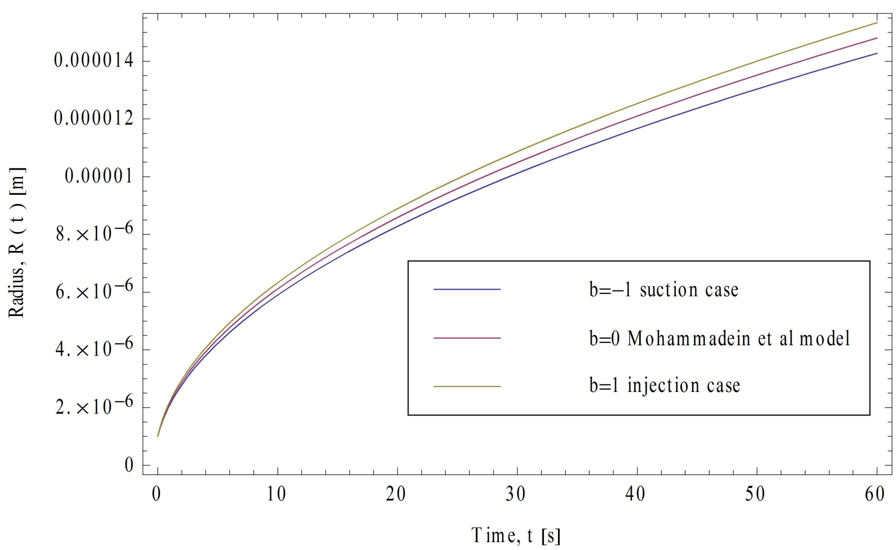

FIGURE 4. The growth of gas bubble in terms of time for three different values of parameter "b" when ambient pressure is variant.

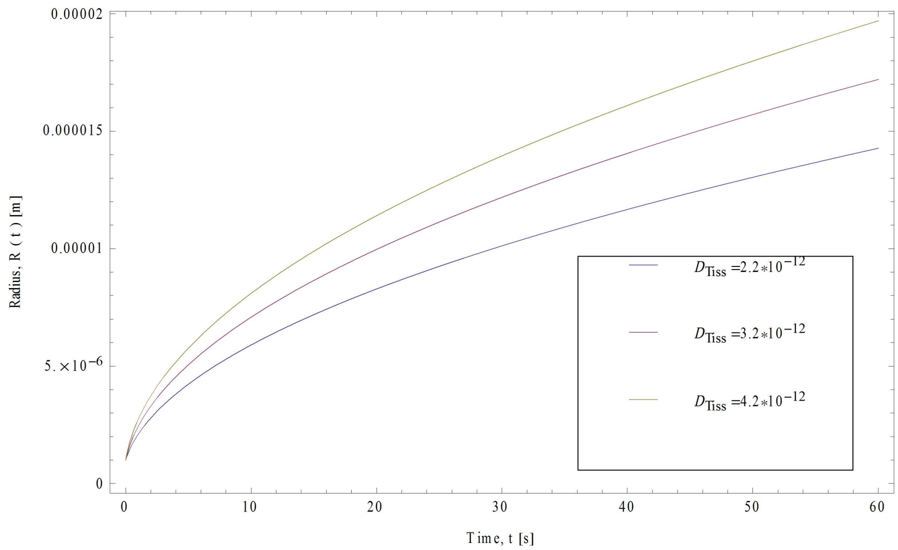

FIGURE 5. The growth of gas bubble in terms of time for three different values of gas diffusion coefficient in tissue when ambient pressure is variant.

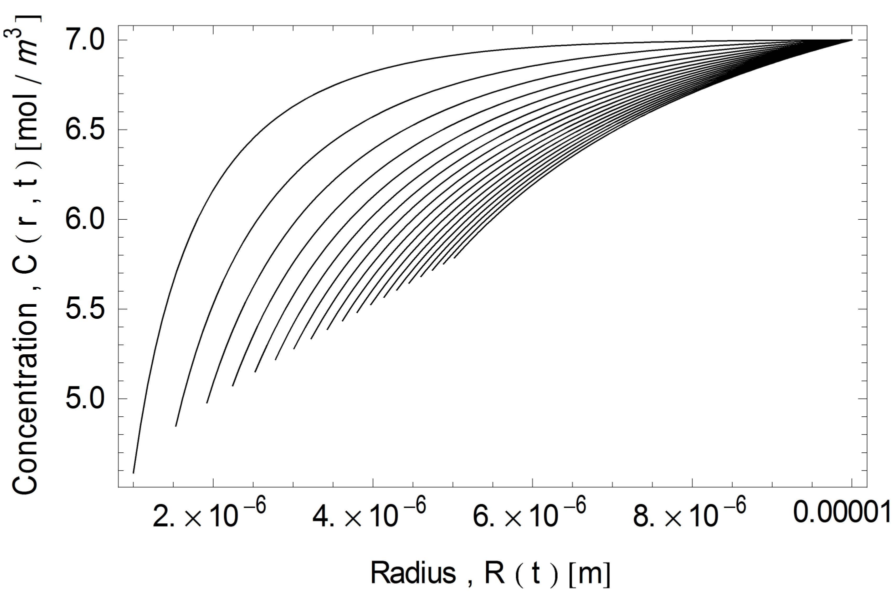

FIGURE 6. Concentration distribution around a growing nitrogen bubble in a bio-tissue for diver when b= -1 (suction effect).

REFERENCES

[1] Divinis, N. et al. Bubbles Growing in Supersaturated Solutions at Reduced Gravity. AIChE Journal, 50 (10), 2369-2382, (2004).

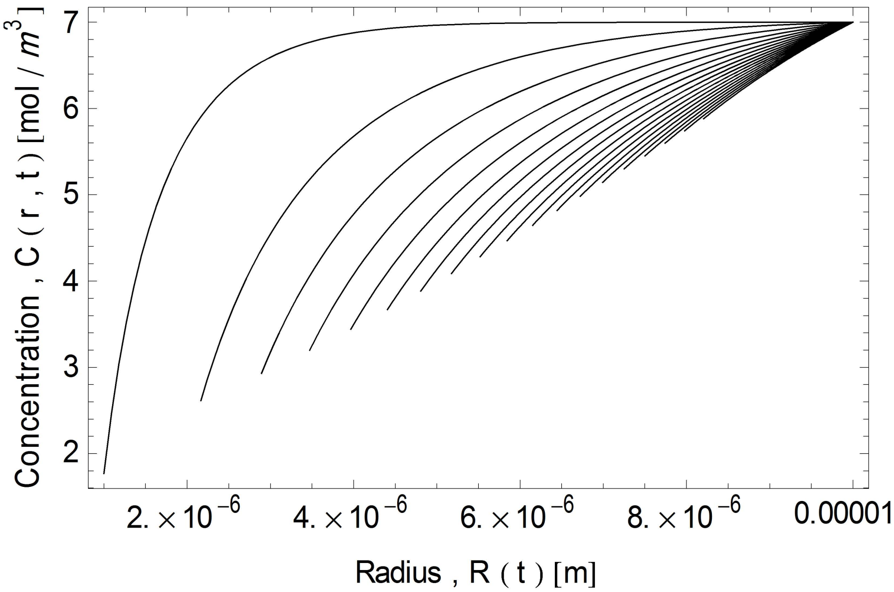

FIGURE 7. Concentration distribution around a growing nitrogen bubble in a bio-tissue for diver when b=0.0 (Mohammadein and Mohamed model [6]).

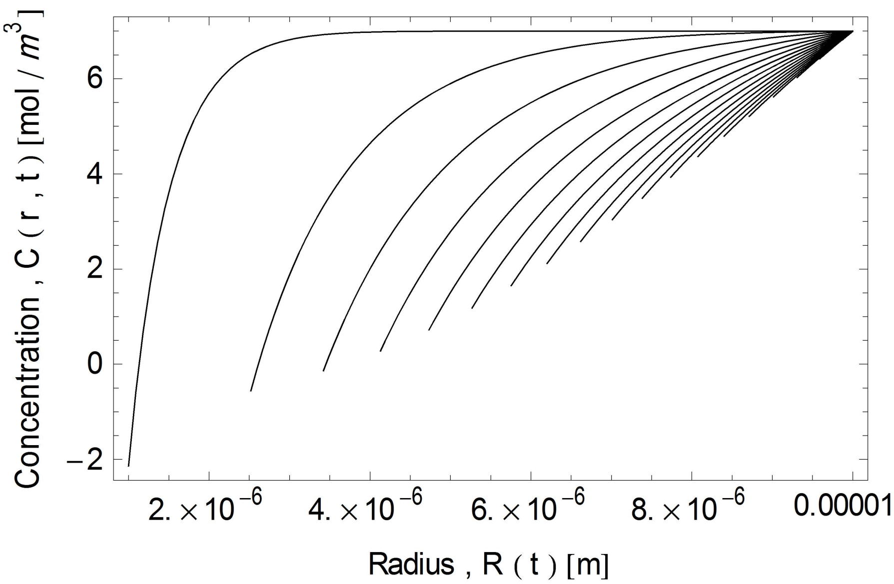

FIGURE 8. Concentration distribution around a growing nitrogen bubble in a bio-tissue for diver when b=1 (injection effect).

[2] Epstein, P. S. and Plesset, M. S. On the Stability of Gas Bubbles in Liquid-Gas Solutions. J. Chem. Phys. 18, 1505-1509, (1950).
[3] Hashemi, S. J. and Abedi, J. Advances in Modeling of New Phase Growth. J. Energy Fuels, 21 (4), 2147-2155, (2007).
[4] Lang, M.A. and Smith, N.E. (eds.). Proceedings of the Advanced Scientific Diving Workshop. February 23-24, 2006. Smithsonian Institution, Washington, DC. 277 p. (2006).
[5] Lide, David R. CRC Handbook of Chemistry and Physics. CRC Press, Boca Raton, FL, (2005).
[6] Mohammadein, S. A. and Mohamed, K.G. Concentration distribution around a growing gas bubble in tissue. J. Math. Bioscinces.225, 11-17(2010).
[7] Srinivasan, R. S., Gerth, W. A. and Powell, M. R. A Mathematical Model of Diffusion Limited Gas Bubble Dynamics in Tissue with Varying Diffusion Region Thickness. Respiration Physiology 123, 153-164, (2000).
[8] Srinivasan, R. S., Gerth, W. A. and Powell, M. R. Mathematical Models of Diffusion-Limited Gas Bubble Dynamics in Tissue. J. Appl. Physiol. 86, 732-741, (1999).
[9] Tikuisis, P., Gault, K. A. and Nishi, R. Y. Prediction of Decompression Illness Using Bubble Models. Undersea Hyperb. Med. 21, 129-143, (1994).
[10] Van Liew, H. D. and Burkard, M. E. Density of Decompression Bubbles and Competition for Gas among Bubbles, tissue and blood. J. Appl. Physiol. 75, 2293-2301, (1993).

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