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BAXTER PERMUTATION AND INVERSION MATRIX

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Abstract. In this paper we give some examples of Baxter permutation as inversiom matrix . Hence we identify the Baxter permutation directly through the inversion matrix .

1. Introduction

The Artin's braid group B_n and the symmetric group S_n, have respectively the presentations ([1]) :

$$B_n = \left\{ \begin{array}{c} \sigma_i,\ i=1,2,...,n-1:\ \sigma_i\sigma_j \ = \ \sigma_j\sigma_i \ \text{ if } |i-j|>1, \\ \sigma_i\sigma_{i+1}\sigma_i \ = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } i=1,2,...,n-2 \end{array} \right\} \tag{1}$$

A positive braid in B_n is the braid which can be written as a word in positive powers of generators σ_i, and without use of the inverse elements σ_i^{-1}. The set of all positive braids form a monoid of positive braids denoted by B_n^+. The positive permutation braids, PPBs S_n^+, were first defined by Elrifai [2]) , where a braid is a positive permutation braid if it is positive and each pair of its strings cross at most once. PPBs represent a geometric analogue of permutations, and $S_n^+ \subseteq B_n^+ \subseteq B_n$.

In ([3]) Elrifai and Anis constructed an isomorphic group of matrices to a finite symmetric group, which is based on the inversion of permutations.

They construct a group of binary matrices which is isomorphic to a symmetric group. Starting with a permutation $\alpha \in S_n$ and from its inversion set , and define a unique binary matrix $M(\alpha)$, called inversion matrix of α. Then construct a group $M_n(F) = \{M(\alpha) : \alpha \in S_n\} \cong S_n$, over the field $F = \{0,1\}$ with addition mod 2 .

2. Existence and uniqueness

2.1. Inversion matrix.

A permutation matrix is a square binary matrix that has exactly one entry 1 in each row and each column and $0s$ elsewhere. In the i $\underline{th}$ row, the entry $\alpha(i)$ equals 1, for a permutation α.

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For a permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix},$$

the permutation matrix P_α equals

$$P_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(For $\alpha = (\alpha(1)\alpha(2)...\alpha(n)) \in S_n$, $\alpha(i)$ is the image of i under α, define the following:

- An inversion of α is the pair $(\alpha(i), \alpha(j))$ where $i < j$ and $\alpha(i) > \alpha(j)$.
- The inversion set of α is $Inv(\alpha) = \{(\alpha(i), \alpha(j)) : i < j, \alpha(i) > \alpha(j)\}$.
- Let $l_i = |\{\alpha(j) : i > j, \alpha(i) < \alpha(j)\}|$.
- The inversion vector or " the Lehmer code" of α is the $n-$tuple $L(\alpha) = (l_1, l_2, ..., l_n)$.
- The inversion number of α is $I(\alpha) = l_1 + l_2 + ... + l_n$.
- The inversion family of order n is $L_n = \{L(\alpha) : \alpha \in S_n\}$.

$I(\alpha)$ is the length of α, i.e. the smallest word of generators τ_i that needed to represent α, and there is a $1-1$ correspondence between S_n and the inversion family L_n . [3].

(An inversion matrix of $\alpha \in S_n$, is the matrix

$$M(\alpha) = (m_{ij})_{n\times n} = \left\{ \begin{array}{ll} 1 & \text{if } i < j \text{ and } \alpha(i) > \alpha(j) \\ 0 & \text{otherwise} \end{array} \right\} \quad (2)$$

2.2. **Baxter permutations.** (Let S_n be the set of all permutations of $\{1, ..., n\}$. A permutation $\pi \in S_n$ is called a Baxter permutation if it satisfies the following conditions for all $1 \leq a \quad b \quad c \quad d \leq n$,

- If π_a $+1$=π_d and $\pi_b \quad \pi_d$ then $\pi_c \quad \pi_d$.
- If π_d+1=π_a and $\pi_c \quad \pi_a$ then $\pi_b \quad \pi_a$. [4].

For example (25314) is a Baxter permutation, but (5327146) is not. It is clear from the definition that the inverse of a Baxter permutation is also Baxter.

2.2.1. *321-avoiding Baxter permutations with further restriction.*
In ([4]) consider the permutations in $B_n(321)$ with the entry 1 preceding the entry 2. Let

$$R_n = \left\{\pi \in \mathbb{B}_n(321) : \pi^{-1}(1) \quad \pi^{-1}(2)\right\} \quad (3)$$

For example,

$$R_3 = \{123, 132, 312\}$$

and

$$R_4 = \{1234, 1243, 1324, 1342, 1423, 3124, 3412, 4123\}.$$

For $n > 3$, we classify the permutations $\pi = \pi_1...\pi_n \in R_n$ into the following four classes.

- If $\pi_n = n$ then we label π by (2_1).

- If $\pi_{n-1} = n$ then we label π by (3_1).
- If $\pi = (3, 4, ..., n, 1, 2)$ then we label π by (3_2).
- Otherwise, we label π by (2_2).

2.3. **Representation 321-avoiding Baxter permutations as Inversion matrix.**

In ([4]) they classify the permutations $\pi = \pi_1 ... \pi_n \in R_n$ into the following four classes, under the condition

$$R_n = \left\{ \pi \in \mathbb{B}_n(321) : \pi^{-1}(1) \quad \pi^{-1}(2) \right\}$$

We begin by studing the inversion matrix of some Baxter permutation.

In $R_4 = \{1234, 1243, 1324, 1342, 1423, 3124, 3412, 4123\}$, we have the inversion matrix as follows:

$$M(e) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M(\sigma_1\sigma_2) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M(\sigma_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M(\sigma_3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M(\sigma_3\sigma_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M(\sigma_2\sigma_1\sigma_3\sigma_2) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M(\sigma_1\sigma_2\sigma_3) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M(\sigma_2\sigma_3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

We identify the Baxter permutation directly through the inversion matrix through the study of one of these properties:

- If all the elements of column n equal to zero in the inversion matrix represents class (2_1).
- In the inversion matrix if the item corresponding to $n-1$ row and n column equal to 1, the inversion matrix represents class (3_1).
- In the inversion matrix if the item corresponding to i row and $n-1$ column , i row and n column equal to 1, $i = 1, 2, ..., n-1$, the item corresponding to n row and $n-1$ row equal to zero , the inversion matrix represents class (3_2).
- In the rest of the case the inversion matrices under the condition mentioned represents class (2_2).

For α in R_n, and let $M_\alpha = (m_{ij})$, then define the matrix,

$$M_\alpha = \left\{ \begin{array}{llll} CLASS & (2_1) & \text{if} & (m_{in}) = 0 \qquad\qquad i = 1, 2, ..., n \\ \\ CLASS & (3_1) & \text{if} & (m_{n-1n}) = 1 \\ \\ CLASS & (3_2) & \text{if} & (m_{in}) = (m_{in-1}) = 1, \quad i = 1, 2, ..., n-2 \\ & & & (m_{n-1j}) = (m_{nj}) = 0 \quad j = 1, 2, ..., n \\ \\ CLASS & (2_2) & & Otherwise \end{array} \right\}$$

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