Electronic Journal of Mathematical Analysis and Applications,
Vol. 2(2) July 2014, pp. 46-55.
ISSN: 2090-792X (online)
http://fcag-egypt.com/Journals/EJMAA/

GROWTH OF SOLUTIONS WITH $L^{2(\rho+2)}$-NORM TO SYSTEM OF DAMPED WAVE EQUATIONS WITH STRONG SOURCES

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Abstract. In the present paper we will prove that the solutions of system of damping wave equations with source terms supplemented with the initial and boundary conditions grow exponentially in a bounded domain, with positive initial energy.

1. Introduction

We consider the following system:

$$\begin{cases} (|u_t|^{m-2}u_t)' - \Delta u + \left(a_1 |u|^k + a_2 |v|^l\right) u_t &= f_1(u,v), \\ (|v_t|^{m-2}v_t)' - \Delta v + \left(a_3 |v|^\theta + a_4 |u|^\varrho\right) v_t &= f_2(u,v), \end{cases} \tag{1.1}$$

where $m \geq 2, k, l, \theta, \varrho \geq 1$ and the two functions $f_1(u,v)$ and $f_2(u,v)$ given by

$$\begin{aligned} f_1(u,v) &= a_5|u+v|^{2(\rho+1)}(u+v) + a_6|u|^\rho u|v|^{(\rho+2)} \\ f_2(u,v) &= a_5|u+v|^{2(\rho+1)}(u+v) + a_6|u|^{(\rho+2)}|v|^\rho v, \rho > -1 \end{aligned} \tag{1.2}$$

In (1.1), $u = u(t,x), v = v(t,x)$ where $x \in \Omega$ is a bounded domain of R^n with a smooth boundary $\partial\Omega$ and $t > 0$, $a_i > 0, i = 1, 2, ...$ Our system is supplemented with the following initial conditions

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), x \in \Omega \tag{1.3}$$

and boundary conditions

$$u(x) = v(x) = 0, x \in \partial\Omega. \tag{1.4}$$

This type of problems are not only important from the theoretical point of view, but also arise in many physical applications and describe a great deal of models in applied science, many questions in physics and engineering give rise to problems that deal with system of nonlinear wave equations.

2000 *Mathematics Subject Classification.* 35L05, 35L20, 58G16.
Key words and phrases. Exponential growth, Strong nonlinear source, Positive initial energy.
Submitted Jan. 11, 2014.

Authors in [17] considered the following system

$$\begin{cases} u_{tt} - \Delta u + u - |v|^{\rho+2}\,|u|^{\rho}\,u &= f_1(x), \\ v_{tt} - \Delta v + v - |u|^{\rho+2}\,|v|^{\rho}\,v &= f_2(x), \end{cases} \tag{1.5}$$

in $\Omega \times (0, T)$. Using the method of potential well, the authors determined the existence of weak solutions of system (1.5).

In [2] Agre and Rammaha studied the following system :

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t &= f_1(u, v), \\ v_{tt} - \Delta v + |v_t|^{r-1}v_t &= f_2(u, v), \end{cases} \tag{1.6}$$

in $\Omega \times (0, T)$ with initial and boundary conditions and the nonlinear functions f_1 and f_2 satisfying appropriate conditions. They proved under some restrictions on the parameters and the initial data many results on the existence of a weak solution. They also showed that any weak solution with negative initial energy blows up in finite time using the same techniques as in [9].

In [21], author considered the same problem treated in [2], and he improved the blow up result for a large class of initial data in which the initial energy can take positive values.

In the work [16], authors considered the nonlinear viscoelastic system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1}\,u_t = f_1(u, v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1}\,v_t = f_2(u, v), \end{cases} \quad x \in \Omega, t > 0 \tag{1.7}$$

where

$$\begin{aligned} f_1(u, v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u, v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{1.8}$$

and they prove a global nonexistence theorem for certain solutions with positive initial energy, the main tool of the proof is a method used in [21].

Recently, in [19] M. A. Rammaha and Sawanya Sakuntasathien focus on the global well-posedness of the system of nonlinear wave equations

$$\begin{cases} u_{tt} - \Delta u + \left(d\,|u|^{k} + e\,|v|^{l}\right)u_t &= f_1(u, v), \\ v_{tt} - \Delta v + \left(d'\,|v|^{\theta} + e'\,|u|^{\varrho}\right)v_t &= f_2(u, v), \end{cases} \tag{1.9}$$

in a bounded domain $\Omega \subset R^n$, $n = 1, 2, 3$, with Dirichlet boundary conditions. The nonlinearities $f_1(u, v)$ and $f_2(u, v)$ act as a strong source in the system. Under some restriction on the parameters in the system, they obtain several results on the existence and uniqueness of solutions.

We will prove that, under some restrictions on the initial data and (with positive initial energy) for some conditions on the functions f_1 and f_2, the solution of problem (1.1)-(1.4) grows exponentially *i.e*

$$\lim_{t\to\infty}\left[\|u\|_{2(\rho+2)} + \|v\|_{2(\rho+2)}\right] \to \infty.$$

2. Assumptions

The constants $c_i, i = 0, 1, 2, ..$ used throughout this paper are positive generic constants, which may be different in various occurrences.

We introduce the following definition of weak solution to (1.1)-(1.4)

Definition 2.1. *A pair of functions (u, v) is said to be a weak solution of (1.1)-(1.4) on $[0, T]$ if $u, v \in C_w([0, T], H_0^1(\Omega)), u_t, v_t \in C_w([0, T], L^m(\Omega)), (u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega), (u_1, v_1) \in L^m(\Omega) \times L^m(\Omega)$ and (u, v) satisfies,*

$$\begin{aligned} \int_0^t \int_\Omega (|u_s|^{m-2} u_s)' \phi dx ds + \int_0^t \int_\Omega \nabla u(s) \nabla \phi dx ds \\ + \int_0^t \int_\Omega \left((a_1 |u|^k + a_2 |v|^l \right) u_s \phi dx ds = \int_0^t \int_\Omega f_1(u, v) \phi dx ds \\ \int_0^t \int_\Omega (|v_s|^{m-2} v_s)' \psi dx ds + \int_0^t \int_\Omega \nabla v(s) \nabla \psi dx ds \\ + \int_0^t \int_\Omega \left((a_3 |v|^\theta + a_4 |u|^\varrho \right) v_s \psi dx ds = \int_0^t \int_\Omega f_2(u, v) \psi dx ds \end{aligned} \tag{2.1}$$

for all test functions $\phi, \psi \in H_0^1(\Omega) \cap L^m(\Omega)$, for almost all $t \in [0, T]$. Where $C_w([0, T], X)$ denotes the space of weakly continuous functions from $[0, T]$ into Banach space X

We introduce the "modified" energy functional $E(t)$ associated to our system:

$$2E(t) = \frac{2(m-1)}{m} (\|u_t\|_m^m + \|v_t\|_m^m) + J(u, v) - 2 \int_\Omega F(u, v) dx. \tag{2.2}$$

where

$$J(u, v) = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \tag{2.3}$$

We make use Sobolev imbedding $H_0^1(\Omega) \subset L^{2(\rho+2)}(\Omega)$, for

$$\begin{cases} -1 < \rho & \textbf{if}\, n = 1, 2 \\ -1 < \rho \leq \frac{4-n}{n-2} & \textbf{if}\, n \geq 3. \end{cases} \tag{2.4}$$

There exists a function $F(u, v)$ such that

$$\begin{aligned} F(u, v) &= \frac{1}{2(\rho+2)} [u f_1(u, v) + v f_2(u, v)] \\ &= \frac{1}{2(\rho+2)} \left[a_5 |u + v|^{2(\rho+2)} + 2a_6 |uv|^{\rho+2} \right] \geq 0, \end{aligned} \tag{2.5}$$

where

$$\frac{\partial F}{\partial u} = f_1(u, v), \frac{\partial F}{\partial v} = f_2(u, v). \tag{2.6}$$

The following technical lemmas will play an important role in the sequel.

Lemma 2.2. [21] *There exist a positive constant c_1 such that*

$$F(u, v) \leq \frac{c_1}{2(\rho+2)} \left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)} \right). \tag{2.7}$$

It is not hard to see this lemma.

Lemma 2.3. *Suppose that (2.4) holds. Then there exists $\eta > 0$ such that for any $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ the inequality*

$$2(\rho+2)\int_{\Omega} F(u,v)dx \leq \eta\,(J(u,v))^{\rho+2} \tag{2.8}$$

holds.

3. Proof of result

We take $a_i = 1, i = 1, 2...$ for convenience, and let us introduce the following:

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \alpha_1 = B^{-\frac{(\rho+2)}{(\rho+1)}}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \tag{3.1}$$

where η given in (2.8).

We first state (without proof, it is similar to that in [19]) a local existence theorem for $n = 1, 2, 3$. Unfortunately, due to the strong nonlinearities on f_1, f_2 the well known techniques of constructing approximations by the Faedo-Galerkin allowed us to prove the local existence result only for $n \leq 3$, where the local existence result in the case of $n > 3$ is still an open problem.

Theorem 3.1. *Let $n = 1, 2, 3$. Suppose that (2.4) holds. Then, there exists a local weak solution in the sense of Definition (2.1) of problem (1.1)-(1.4) defined on $[0, T]$ for some $T > 0$, and (u, v) satisfies the energy inequality*

$$\begin{aligned} E(t) \quad &+ \quad \int_s^t \left(\int_{\Omega}\left(|u(\tau)|^k + |v(\tau)|^l\right)u_\tau^2 dx + \int_{\Omega}\left(|v(\tau)|^\theta + |u(\tau)|^\varrho\right)v_\tau^2 dx\right)d\tau \\ &\leq \quad E(s) \end{aligned} \tag{3.2}$$

for all $T \geq t \geq s \geq 0$, where $E(t)$ is given in (2.2).

The following theorem asserts that the weak solution furnished by Theorem 3.1 grows exponentially with positive initial energy under condition (3.3).

Theorem 3.2. *Suppose that (2.4) holds. Assume further that*

$$2(\rho+2) > \max(m, k+2, l+2, \theta+2, \varrho+2) \tag{3.3}$$

Then any solution of problem (1.1)-(1.4) with initial data satisfying

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 > \alpha_1^2, \qquad \text{and} \qquad E(0) < E_1 \tag{3.4}$$

grows exponentially, where the constants α_1 and E_1 are defined in (3.1).

The following lemma is very useful to prove our main result for positive initial energy $E(0) > 0$. It is similar to that in [21].

Lemma 3.3. *Let the assumption (2.4) be fulfilled. Let (u, v) be a solution of $(1.1) - (1.4)$. Assume further that $E(0) < E_1$ and*

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 > \alpha_1^2. \tag{3.5}$$

Then there exists a constant $\alpha_2 > \alpha_1$ such that

$$J(u,v) > \alpha_2^2, \tag{3.6}$$

and

$$2(\rho+2)\int_{\Omega} F(u,v)dx \geq (B\alpha_2)^{2(\rho+2)}, \forall t \geq 0. \tag{3.7}$$

Proof. (of theorem 3.2)
We set

$$H(t) = E_1 - E(t). \tag{3.8}$$

By using the definition of $H(t)$, we get

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) u_t^2 dx \\ &\quad + \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) v_t^2 dx. \\ &\geq 0, \forall t \geq 0. \end{aligned}$$

Consequently, since $E'(t)$ is absolutely continuous,

$$H(0) = E_1 - E(0) > 0.$$

Then,

$$\begin{aligned} 0 &< H(0) \leq H(t) = E_1 - \frac{m-1}{m}\left(\|u_t\|_m^m + \|v_t\|_m^m\right) - \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &\quad + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned} \tag{3.9}$$

From (2.2) and (3.6), we obtain, for all $t \geq 0$,

$$\begin{aligned} & E_1 - \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ < \; & E_1 - \frac{1}{2}\alpha_1^2 + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ = \; & -\frac{1}{2(\rho+2)}\alpha_1^2 + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ < \; & \frac{c_0}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned}$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right], \forall t > 0. \tag{3.10}$$

Then we define the functional

$$L(t) = H(t) + \varepsilon \int_\Omega u|u_t|^{m-2}u_t + v|v_t|^{m-2}v_t dx, \tag{3.11}$$

for ε small to be chosen later to get small perturbation of $E(t)$ and we will show that $L(t)$ grows exponentially.

By taking a derivative of (3.11) and by equations (1.1), we obtain

$$
\begin{aligned}
L'(t) &= H'(t)+\varepsilon\left(\|u_t\|_m^m+\|v_t\|_m^m\right)-\varepsilon\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\\
&\quad -\varepsilon\int_\Omega u\left(|u(t)|^k+|v(t)|^l\right)u_t dx\\
&\quad -\varepsilon\int_\Omega v\left(|v(t)|^\theta+|u(t)|^\varrho\right)v_t dx\\
&\quad +\varepsilon\int_\Omega \left(uf_1(u,v)+vf_2(u,v)\right)dx.
\end{aligned}
\tag{3.12}
$$

The definition of $H(t)$ leads to

$$
\begin{aligned}
L'(t) &= H'(t)+(1+\frac{2(m-1)}{m})\varepsilon\left(\|u_t\|_m^m+\|v_t\|_m^m\right)\\
&\quad -\varepsilon\int_\Omega u\left(|u(t)|^k+|v(t)|^l\right)u_t dx\\
&\quad -\varepsilon\int_\Omega v\left(|v(t)|^\theta+|u(t)|^\varrho\right)v_t dx\\
&\quad +\varepsilon\left(1-\frac{1}{(\rho+2)}\right)\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right)+2\varepsilon H(t)-2\varepsilon E_1.
\end{aligned}
\tag{3.13}
$$

Then using (3.7) we obtain, for $c_3>0$

$$
\begin{aligned}
L'(t) &\geq H'(t)+(1+\frac{2(m-1)}{m})\varepsilon\left(\|u_t\|_m^m+\|v_t\|_m^m\right)\\
&\quad +\varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right)+2\varepsilon H(t),\\
&\quad -\varepsilon\int_\Omega u\left(|u(t)|^k+|v(t)|^l\right)u_t dx\\
&\quad -\varepsilon\int_\Omega v\left(|v(t)|^\theta+|u(t)|^\varrho\right)v_t dx
\end{aligned}
\tag{3.14}
$$

In order to estimate the last two terms in (3.14) we have:

$$
\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|uu_t|dx\leq\lambda_1\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)u^2dx \tag{3.15}
$$
$$
+\frac{1}{4\lambda_1}\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)u_t^2dx.
$$

and

$$
\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|vv_t|dx\leq\lambda_2\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)v^2dx
$$
$$
+\frac{1}{4\lambda_2}\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)v_t^2dx. \tag{3.16}
$$

Inserting the estimates (3.15), (3.16) into (3.14), we obtain

$$\begin{aligned} L'(t) \quad \geq \quad & H'(t) + (1+\frac{2(m-1)}{m})\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & +\varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + \|uv\|_{\rho+2}^{\rho+2}\right) + 2\varepsilon H(t) \\ & -\varepsilon\lambda_1\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)u^2dx \\ & -\varepsilon\frac{1}{4\lambda_1}\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)u_t^2dx \\ & -\varepsilon\lambda_2\int_\Omega\left(|v(t)|^\theta + |u(t)|^\varrho\right)v^2dx \\ & -\varepsilon\frac{1}{4\lambda_2}\int_\Omega\left(|v(t)|^\theta + |u(t)|^\varrho\right)v_t^2dx \end{aligned} \tag{3.17}$$

Consequently, by using Young's inequality for some $\delta, \delta_1 > 0$, we have

$$\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)u^2dx = \|u\|_{k+2}^{k+2} + \int_\Omega |v|^l u^2dx$$
$$\leq \|u\|_{k+2}^{k+2} + \frac{l}{l+2}\delta^{(l+2)/l}\|v\|_{l+2}^{l+2} + \frac{2}{l+2}\delta^{-(l+2)/(2)}\|u\|_{l+2}^{l+2},$$

and

$$\int_\Omega\left(|v(t)|^\theta + |u(t)|^\varrho\right)v^2dx = \|v\|_{\theta+2}^{\theta+2} + \int_\Omega |u|^\varrho v^2dx$$
$$\leq \|v\|_{\theta+2}^{\theta+2} + \frac{\varrho}{\varrho+2}\delta_1^{(\varrho+2)/\varrho}\|u\|_{\varrho+2}^{\varrho+2} + \frac{2}{\varrho+2}\delta_1^{-(\varrho+2)/(2)}\|v\|_{\varrho+2}^{\varrho+2}.$$

By using lemma 2.2, (3.17) becomes

$$\begin{aligned} L'(t) \quad \geq \quad & H'(t) + (1+\frac{2(m-1)}{m})\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & +\varepsilon c_4\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) + 2\varepsilon H(t) \\ & -\varepsilon\frac{1}{4\lambda_1}\int_\Omega\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^2dx - \varepsilon\frac{1}{4\lambda_2}\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)|u_t|^2dx \\ & -\varepsilon\lambda_2\left(\|v\|_{\theta+2}^{\theta+2} + \frac{\varrho}{\varrho+2}\delta_1^{(\varrho+2)/\varrho}\|u\|_{\varrho+2}^{\varrho+2} + \frac{2}{\varrho+2}\delta_1^{-(\varrho+2)/(2)}\|v\|_{\varrho+2}^{\varrho+2}\right) \\ & -\varepsilon\lambda_1\left(\|u\|_{k+2}^{k+2} + \frac{l}{l+2}\delta^{(l+2)/l}\|v\|_{l+2}^{l+2} + \frac{2}{l+2}\delta^{-(l+2)/(2)}\|u\|_{l+2}^{l+2}\right) \end{aligned}$$

Since (3.3) holds, there exists $M_1 > 0, M_2, M_3$, for λ_1, λ_2 fixed

$$\begin{aligned} L'(t) \quad \geq \quad & (1-\varepsilon M_1)H'(t) + (1+\frac{2(m-1)}{m})\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + 2\varepsilon H(t) \\ & +\varepsilon M_2\|u\|_{2(\rho+2)}^{2(\rho+2)} + \varepsilon M_3\|v\|_{2(\rho+2)}^{2(\rho+2)} \end{aligned} \tag{3.18}$$

where $M_2 = \left(c_4 - \lambda_1(\frac{2}{l+2}\delta^{-(l+2)/(2)} + 1) - \lambda_2\frac{\varrho}{\varrho+2}\delta_1^{(\varrho+2)/\varrho}\right)$,
$M_3 = \left(c_4 - \lambda_1\frac{l}{l+2}\delta^{(l+2)/l} - \lambda_2(1 + \frac{2}{\varrho+2}\delta_1^{-(\varrho+2)/(2)})\right)$.

Choosing δ, δ_1 such that $M_2, M_3 > 0$, then we pich ε small enough so that $(1-\varepsilon M_1) \geq 0$ and $L(0 > 0$. Consequently, there exists $\Gamma > 0$ such that (3.18) becomes

$$L'(t) \geq \varepsilon\Gamma\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \tag{3.19}$$

Thus, the functional $L(t)$ is strictly positive and increasing for all $t \geq 0$.

By Holder's and Young's inegualities, since (3.3), we estimate

$$\begin{aligned} & \left|\int_\Omega u|u_t|^{m-1}dx\right| \\ \leq \ & \|u\|_m\|u_t\|_m^{(m-1)} \\ \leq \ & C|\Omega|^{\frac{1}{m}-\frac{1}{2(\rho+2)}}\left(\|u\|_{2(\rho+2)}^m + \|u_t\|_m^m\right), \\ \leq \ & C|\Omega|^{\frac{1}{m}-\frac{1}{2(\rho+2)}}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_m^m\right), \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} & \left|\int_\Omega v|v_t|^{m-1}dx\right| \\ \leq \ & \|v\|_m\|v_t\|_m^{(m-1)} \\ \leq \ & C|\Omega|^{\frac{1}{m}-\frac{1}{2(\rho+2)}}\left(\|v\|_{2(\rho+2)}^m + \|v_t\|_m^m\right), \\ \leq \ & C|\Omega|^{\frac{1}{m}-\frac{1}{2(\rho+2)}}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v_t\|_m^m\right), \end{aligned} \tag{3.21}$$

Also, by noting that

$$\begin{aligned} L(t) &= H(t) + \varepsilon\int_\Omega uu_t|u_t|^{m-2} + vv_t|v_t|^{m-2}dx \\ &\leq c_5\left(H(t) + \left|\int_\Omega u|u_t|^{m-1} + v|v_t|^{m-1}dx\right|\right) \qquad (3.22) \\ &\leq c_6\left[H(t) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_m^m + \|v_t\|_m^m\right], \forall t \geq 0, \end{aligned}$$

and combining with (3.22) and (3.19), we arrive at

$$\frac{dL(t)}{dt} \geq \xi L(t), \xi > 0, \forall t \geq 0. \tag{3.23}$$

Integration of (3.23) between 0 and t gives us $L(t) \geq L(0)\exp(\xi t)$ and for ε small enough, we have

$$L(t) \leq H(t) \leq \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right], \forall t > 0.$$

then,

$$L(0)\exp(\xi t) \leq \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right], \forall t > 0.$$

This completes the proof. □

Question: One can consider the same problem and may ask questions on asymptotic behavior of the solutions (If it existes): as time goes to infinity, what is the asymptotic behavior of solutions? More generally, what is the long time behavior of solutions when initial data vary in any bounded set in a Sobolev space associated with the problem.

References

[1] M. Abdelli, Abbes Benaissa, *Energy decay of solutions of a degenerate Kirchhoff equation with a weak nonlinear dissipation*, Nonlinear Analysis 69 (2008) 1999-2008

[2] K. Agre and M. A. Rammaha, *Systems of nonlinear wave equations with damping and source terms*, Diff .Integral Equations, 19(2007), 1235-1270.

[3] D. D. Ang and A. P. N. Dinh. *Strong solutions of a quasilinear wave equation with non linear damping,* SIAM Journal on Mathematical Analysis, vol. 19, no. 2, pp. 337-347, 1988.

[4] A. Benaissa, D. Ouchenane and Kh. zennir, *Blow up of positive initial-energy solutions to systems of nonlinear wave equations with degenerate damping and source terms*, Nonlinear Studies, Vol. 19, No. 4, pp. 523-535, 2012.

[5] A. Benaissa, S. Mokeddem, *Global existence and energy decay of solutions to the Cauchy problem for a wave equation with a weakly nonlinear dissipation*, Abstr. Appl. Anal, 11(2004) 935-955.

[6] A. Benaissa, S. A. Messaoudi, *Blow up of solutions of a nonlinear wave equation*, j. Appl. Math, 2(2), 105-108, (2002).

[7] A. Benaissa, S. A. Messaoudi, *Blow up of solutions for Kirchhoff equation of q-Laplacien type with nonlinear dissipation*, Colloq. Math. 94(1)(2002)103-109.

[8] A. Benaissa, S. A. Messaoudi, *Blow up of solutions of a quasilinear wave equation with nonlinear dissipation*, j. Partial Differential Equations 15(3)(2002)61-67.

[9] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source term*, J. Diff .Eq., 109(1994), 295-308.

[10] M. Kafini, S. Messaoudi, *A blow-up result in a Cauchy viscoelastic problem*, Applied Mathematics Letters 21(2008)549–553.

[11] Khaled zennir, *Exponential growth of solutions with L_p-norm of a nonlinear viscoelastic hyperbolic equation*, J. Nonlinear Sci. Appl. 6 (2013), 252-262

[12] H. A. Levine and J. Serrin. *Global nonexistence theorems for quasilinear evolution equations with dissipation,* Archive for Rational Mechanics and Analysis, vo l37, no. 4, pp. 341-361, 1997.

[13] H. A. Levine, S. R. Park, and J. Serrin. *Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation,*Journal of Mathematical Analysis and Applications, vo 228, no. 1, pp. 181-205, 1998.

[14] S. A. Messaoudi, *Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation*, J. Math .Anal. Appl. 320(2006)902-915.

[15] S. A. Messaoudi and B. Said-Houari, *Global non-existence of solutions of a class of wave equations with non-linear damping and source terms*, Math. Meth. Appl. Sci. 2004, 27:1687–1696.

[16] S. A. Messaoudi and B. Said-Houari, *Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms*, J. Math. Anal. Appl. 365 (2010) 277-287.

[17] Milla Miranda, M. Medeiros, L. A.: *On the existence of global solutions of a coupled nonlinear klein-gordon equations,* Funkcial. Ekvac. 30(1), 147-161(1987).

[18] M. A. Rammaha and Sawanya Sakuntasathien, *Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms*, Nonlinear Analysis 72(2010)2658-2683.

[19] M. A. Rammaha and Sawanya Sakuntasathien, *Critically and degenerately damped systems of nonlinear wave equations with source terms*, Applicable Analysis 72 (2010) 1201-1227.

[20] M. Reed, *Abstract Non Linear Wave Equations*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1976.

[21] B. Said-Houari, *Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms*, Differential and Integral Equations . Volume23, Numbers1-2(2010),79-92.

[22] B. Said-Houari, *Exponential growth of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms*, Z. Angew. Math. Phys. 62(2011),115–133.

[23] B. Said-Houari, S. A. Messaoudi and A. Guesmia, *General decay of solutions of a nonlinear system of viscoelastic wave equations*, Nonlinear Difffer. Equ. Appl. 2011 Springer Basel AG.

[24] G. Todorova. *Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms,* Comptes Rendus de l'académie des Sciences. Série 1, vo 326, no. 2, pp. 191-196, 1998.

[25] E. Vitillaro.*Some New Results on Global Nonexistence and Blow-up for Evolution Problems with Positive Initial Energy*, Rend. Istit. Mat. Trieste, Vol. XXXI Suppl. 2, 245-275 (2000).

[26] E. Vitillaro. *Global nonexistence theorems for a class of evolution equations with dissipative,*Archive for Rational Mechanics and Analysis,*v*o 149, no. 2, pp. 155-182, 1999.

[27] Z. Yang. *Blow up of solutions for a class of nonlinear evolution equations with nonlinear damping and source term,* Mathematical Methods in the Applied Sciences,*v*ol. 25, no.10, pp. 825-833, 2002.

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