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ON CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

YURDAL SEVER AND ERDİNÇ DÜNDAR

Abstract. In this work, we deal with various kinds of convergence for double sequences of functions with values in $\mathbb{R}$. We introduce the concepts of uniformly convergent and uniformly Cauchy sequences for double sequences of functions and show the relation between them.

1. Introduction and Definitions

Balcerzak et al. [2] discussed various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in $\mathbb{R}$ or in a metric space. Gezer and Karakuş [8] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and then they examined the relation between them. Gökhan et al. [9] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. Also, some useful results on double sequences and double sequences of functions may be found in [3, 4, 5, 6, 7, 10, 12, 13, 15].

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers.

Now, we recall the concept of convergence of the double sequences, the double sequences of functions and basic definitions and concepts. (See [1, 7, 9, 11, 13, 14]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in the Pringsheim's sense (P-convergent) if for any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|x_{mn} - L| < \varepsilon,$$

whenever $m, n \geq N$. In this case we write

$$P - \lim_{m,n\to\infty} x_{mn} = L \text{ or } \lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|x_{mn} - x_{jk}| < \varepsilon$$

for all $m, n, j, k \geq N$.

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It is known that a double sequence (x_{mn}) of real numbers is a Cauchy sequence if and only if it is convergent.

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$ for all $m, n \in \mathbb{N}$. That is,

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Now, we give the pointwise convergent and uniformly convergent for double sequences of functions.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to f on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N = N(x, \varepsilon)$ such that

$$|f_{mn}(x) - f(x)| < \varepsilon$$

for all $m, n \geq N$. In this case we write

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ } or \text{ } f_{mn} \to f, \text{ } on \text{ } S.$$

Throughout the paper we take convergent instead of pointwise convergent.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly convergent to f on a set $S \subset \mathbb{R}$, if for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $m, n \geq N$ implies

$$|f_{mn}(x) - f(x)| < \varepsilon, \text{ for all } x \in S.$$

In this case we write

$$f_{mn} \rightrightarrows_S f.$$

2. Main Results

Theorem 2.1. *Let $\{f_{mn}\}$ be a double sequence of functions and f be a function on $S \subset \mathbb{R}$. Then*

$$f_{mn} \rightrightarrows_S f$$

if and only if

$$\lim_{m,n\to\infty} p_{mn} = 0,$$

where

$$p_{mn} = \mathop{sup}_{x\in S} |f_{mn}(x) - f(x)|.$$

Proof. The proof is straightforward and so is omitted. □

Definition 2.2. *A double sequence of functions $\{f_{mn}\}$ on $S \subset \mathbb{R}$ is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that*

$$|f_{mn}(x) - f_{jk}(x)| < \varepsilon, \text{ } for \text{ } all \text{ } x \in S$$

for all $m, n, j, k \geq N$.

Now, we give Cauchy criteria for uniform convergence.

Theorem 2.3. *Let $\{f_{mn}\}$ be a sequence of functions on $S \subset \mathbb{R}$. $\{f_{mn}\}$ is uniformly convergent if and only if it is uniformly Cauchy on S.*

Proof. Assume that $f_{mn} \rightrightarrows_S f$. Then, for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $m, n \geq N$ implies

$$|f_{mn}(x) - f(x)| < \frac{\varepsilon}{2}, \text{ for all } x \in S.$$

Therefore, we have

$$\begin{aligned} |f_{mn}(x) - f_{jk}(x)| &\leq |f_{mn}(x) - f(x)| + |f_{jk}(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $x \in S$ and for all $m, n, j, k \geq N$. This provides that $\{f_{mn}\}$ is uniformly Cauchy on S.

Conversely assume that $\{f_{mn}\}$ is uniformly Cauchy on S. Then, for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_{mn}(x) - f_{jk}(x)| < \frac{\varepsilon}{2}, \text{ for all } x \in S \tag{2.1}$$

for all $m, n, j, k \geq N$. Since double sequence of numbers $\{f_{mn}(x)\}$ is Cauchy sequence for every $x \in S$, then

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x).$$

Now, we show that $f_{mn} \rightrightarrows_S f$. By (2.1) fixing $m, n \geq N$ and applying limit operator for $j, k \to \infty$ ($\lim_{j,k\to\infty} f_{jk}(x) = f(x)$), we have

$$|f_{mn}(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon, \text{ for all } x \in S$$

for all $m, n \geq N$. This provides that $f_{mn} \rightrightarrows_S f$. □

Theorem 2.4. *Let $\{f_{mn}\}$ be a double sequence of functions and f be a function on $S \subset \mathbb{R}$ and $f_{mn} \rightrightarrows_S f$. Assume that $x \in \overline{S}$ and*

$$\lim_{t\to x} f_{mn}(t) = a_{mn}, \quad m, n \in \mathbb{N} \tag{2.2}$$

for all $t \in S$. Then, double sequence (a_{mn}) is convergent and

$$\lim_{t\to x} f(t) = \lim_{m,n\to\infty} a_{mn} \tag{2.3}$$

for all $t \in S$. That is,

$$\lim_{t\to x} \lim_{m,n\to\infty} f_{mn}(t) = \lim_{m,n\to\infty} \lim_{t\to x} f_{mn}(t)$$

for all $t \in S$.

Proof. Let $f_{mn} \rightrightarrows_S f$. Then, for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that for all $m, n, j, k \geq N$, we have

$$|f_{mn}(t) - f_{jk}(t)| < \varepsilon, \text{ for all } t \in S. \tag{2.4}$$

By (2.4) fixing $m, n, j, k \geq N$ and applying the limit operator for $t \to x \in \overline{S}$, by (2.2) we have

$$|a_{mn} - a_{jk}| \leq \varepsilon.$$

Therefore, double sequence (a_{mn}) is Cauchy sequence and so (a_{mn}) is convergent, say $\lim_{m,n\to\infty} a_{mn} = a$. Since

$$f_{mn} \rightrightarrows_S f \text{ and } \lim_{m,n\to\infty} a_{mn} = a,$$

then there exists $N^{'}=N^{'}(\varepsilon)$ such that for all $m,n\geq N^{'}$

$$|f_{mn}(t)-f_{jk}(t)|<\frac{\varepsilon}{3},\ \text{for all } t\in S$$

and we have

$$|a_{mn}-a|<\frac{\varepsilon}{3}.$$

For fixed $m,n\geq N^{'}$, since

$$\lim_{t\to x}f_{mn}(t)=a_{mn},$$

then, there exists a punctured neighborhood $\tilde{U}(x)$ of x such that

$$|f_{mn}(t)-a_{mn}|<\frac{\varepsilon}{3},$$

for all $t\in\tilde{U}(x)\cap S$. Therefore, for every $m,n\geq N^{'}$ and for every $t\in\tilde{U}(x)\cap S$, we have

$$\begin{aligned} |f_{mn}(t)-a| &\leq |f(t)-f_{mn}(t)|+|f_{mn}(t)-a_{mn}|+|a_{mn}-a| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \end{aligned}$$

and so (2.3) is obtained. □

Theorem 2.5. *Let $\{f_{mn}\}$ be double sequence of continuous functions on $S\subset\mathbb{R}$. If $\{f_{mn}\}$ is uniformly convergent to f on $S\subset\mathbb{R}$, then f is continuous on $S\subset\mathbb{R}$. That is,*

$$f_{mn}\in C(S),m,n\in\mathbb{N}\ \text{and}\ f_{mn}\rightrightarrows_S f\Rightarrow f\in C(S),\quad m,n\in\mathbb{N},$$

where $C(S)$ denote the set of continuous functions on $S\subset\mathbb{R}$.

Proof. Let $x\in S$ be a arbitrary limit point of S. Since $f_{mn}\in C(S)$, then for all $m,n\in\mathbb{N}$

$$\lim_{t\to x}f_{mn}(t)=f_{mn}(x)$$

for all $t\in S$. By Theorem 2.4, $\{f_{mn}\}$ is convergent and for all $t\in S$,

$$\lim_{t\to x}f(t)=\lim_{m,n\to\infty}f_{mn}(x)=f(x),\ \ m,n\in\mathbb{N}.$$

Therefore, we have f is continuous at x. Since x is arbitrary point of S, so f is continuous on $S\subset\mathbb{R}$. □

Theorem 2.6. *Let S be a compact subset of $\mathbb{R}$, $\{f_{mn}\}$ be double sequence of continuous functions on S. Assume that $\{f_{mn}\}$ be monotonic decreasing on S, i.e.,*

$$f_{(m+1),(n+1)}(x)\leq f_{mn}(x),\ (m,n=1,2,...),\ \text{for every } x\in S,$$

f is continuous and

$$\lim_{m,n\to\infty}f_{mn}(x)=f(x)$$

on S. Then

$$f_{mn}\rightrightarrows_S f.$$

Proof. Let $x \in S$, $\{f_{mn}\}$ be monotonic decreasing on S and

$$g_{mn}(x) = f_{mn}(x) - f(x).$$

Then, $g_{mn}(x)$ is continuous on S, $g_{mn}(x) \to 0$ $(x \in S)$, and $g_{mn}(x)$ is monotonic decreasing, for every $x \in S$. We show that $\{g_{mn}\}$ is uniformly convergent to 0 on S. Let $\varepsilon > 0$. Since for $x \in S$, $g_{mn}(x) \to 0$ there exist $m_x, n_x \in \mathbb{N}$ such that for every $x \in S$

$$0 \le g_{m_x n_x}(x) < \frac{\varepsilon}{2}. \tag{2.5}$$

Since $g_{m_x n_x}(t)$ is continuous at $x \in S$ and by 2.5 there exists an open neighborhood $U(x)$ of x such that

$$0 \le g_{m_x n_x}(t) < \varepsilon$$

for every $t \in U(x) \cap S \equiv K(x)$. Since $\{g_{mn}(t)\}$ is monotonic decreasing for every $t \in K(x)$, we have

$$0 \le g_{mn}(t) \le \varepsilon$$

for all $m \ge m_x, n \ge n_x$. Since S is a compact subset of $\mathbb{R}$, there exists finite set $\{x_1, x_2, ..., x_i\}$ such that

$$S \subset K(x_1) \cup K(x_2) \cup ... \cup K(x_i).$$

We define

$$M = \text{maks}\left\{m_{x_1}, m_{x_2}, ..., m_{x_i}\right\},$$
$$N = \text{maks}\left\{n_{x_1}, n_{x_2}, ..., n_{x_i}\right\}.$$

Then, for every $m \ge M$, $n \ge N$ we have

$$0 \le g_{mn}(t) < \varepsilon$$

for every $t \in S$ and so

$$g_{mn} \rightrightarrows_S 0.$$

□

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Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar\Turkey
E-mail address: yurdalsever@hotmail.com, ysever@aku.edu.tr
E-mail address: erdincdundar79@gmail.com, edundar@aku.edu.tr