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NEW SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. A certain subclass $TQ^n_{r,s,\lambda}(\eta,\beta)$ consisting of analytic functions with negative coefficients in the open unit disk $\mathbb{U}$ is introduced. In this paper we obtain coefficient inequalities, extreme points, integral inequalities and the $(n,\delta)-$neighborhood.

1. Introduction

Euler, Gauss and Riemann were among the earliest mathematicians who studied the hypergeometric functions. It starts of with the real functions and eventually it becomes more effective in the complex domain. Because of its variety of applications, the theory of hypergeometric becomes the favourite topics to discuss by many mathematicians. For instance, we can find this applications in a wide range of subjects such as combinatorics, numerical analysis, dynamical systems and mathematical physics. We can generalize numerous results of the classical hypergeometric functions to the $q-$hypergeometric level. A generalized $q-$Taylor's formula in fractional $q-$calculus has recently been introduced by Purohit and Raina [13]. They also derived $q-$generating functions for $q-$hypergeometric functions. In this work some of the properties of the generalized differential operator are discussed.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and normalized in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.
Further, let T denote the subclass of $\mathcal{A}$ consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an analytic and univalent function f is in T if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} a_k z^k. \tag{2}$$

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A $q-$hypergeometric function is a power series in one complex variable z with power series coefficients which depend, apart from q on r complex upper parameters $a_i, b_j, (i = 1, ..., r, j = 1, ..., s, b_j \in \mathbb{C}\backslash\{0, -1, -2, ...\})$ as follows

$$ {}_r\Omega_s\left(a_1, ...a_r; b_1, ...b_s, q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, q)_k ... (a_r, q)_k}{(q, q)_k (b_1, q)_k ... (b_s, q)_k} \left[(-1)^k q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)} \right]^{1+s-r} z^k, \tag{3} $$

with $\left(\begin{array}{c} k \\ 2 \end{array}\right) = \frac{k(k-1)}{2}$, where $q \neq 0$ when $r > s+1, (r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U})$, $\mathbb{N}$ denote the set of positive integers and $(a, q)_q$ is the $q-$shifted factorial defined by

$$ (a, q)_k = \begin{cases} 1, & k = 0; \\ (1-a)(1-aq)(1-aq^2)...(1-aq^{k-1}), & k \in \mathbb{N}. \end{cases} $$

By using the ratio test, one recognize that, if $0 < |q| < 1$, the series (3) converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s + 1$. For brief survey on $q-$hypergeometric functions, one may refer to [1, 2, 3], see also [18, 19].

Now for $z \in \mathbb{U}, 0 < |q| < 1$, and $r = s + 1$, the $q-$ hypergeometric function defined in (3) takes the form

$$ {}_r\upsilon_s\left(a_1, ..., a_r; b_1, ..., b_s, q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, q)_k ... (a_r, q)_k}{(q, q)_k (b_1, q)_k ... (b_s, q)_k} z^k $$

which converges absolutely in the open unit disk $\mathbb{U}$.

Corresponding to a function ${}_r\Lambda_s(a_i; b_j; q, z)$ defined by

$$ {}_r\Lambda_s(a_i; b_j; q, z) = z\, {}_r\upsilon_s(a_i; b_j; q, z) = z + \sum_{k=2}^{\infty} \frac{(a_1, q)_{k-1} ... (a_r, q)_{k-1}}{(q, q)_{k-1} (b_1, q)_{k-1} ... (b_s, q)_{k-1}} z^k. $$

We will use the following operator which defined and studied by the authors (see [7]).

$$ \begin{aligned} \mathcal{M}^0_{r,s,\lambda}(a_i, b_j; q) f(z) =& f(z) * {}_r\Lambda_s(a_i, b_j; q; z) \\ \mathcal{M}^1_{r,s,\lambda}(a_i, b_j; q) f(z) =& (1-\lambda) f(z) * {}_r\Lambda_s(a_i, b_j; q; z) + \lambda z D_q \left(f(z) * {}_r\Lambda_s(a_i, b_j; q; z)\right) \\ &\vdots \\ \mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q) f(z) =& \mathcal{M}^1_{r,s,\lambda}\left(\mathcal{M}^{n-1}_{r,s,\lambda}(f(z))\right) \\ =& z + \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda\right]^n \Upsilon_k a_k z^k \end{aligned} \tag{4} $$

where $*$ denotes the usual Hadamard product of analytic functions and

$$ \Upsilon_k = \frac{(a_1, q)_{k-1} ... (a_r, q)_{k-1}}{(q, q)_{k-1} (b_1, q)_{k-1} ... (b_s, q)_{k-1}}. \tag{5} $$

Remark

- When $n = 0$ we get the linear operator introduced and studied recently by Mohammed and Darus [4].

- When $n = 0, a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, ..., (i = 1, ..., r, j = 1, ..., s)$ and $q \to 1$, we receive the well-known Daziok-Srivastava linear operator [8] $(for\ r = s + 1)$.

- And when $r = 1, s = 0, a_1 = q$ and $\lambda = 1$, we obtain *Sălăgean* differential operator (see [5]).

Many other differential operators studied by various authors can be seen in the literature (see for examples [9],[10]).

In the following definitions, we introduce new subclass of analytic functions containing $q-$hypergeometric functions $\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z)$.

Definition 1.1 Let $f \in \mathcal{A}$. Then $f \in Q^n_{r,s,\lambda}(\eta, \beta)$ if and only if

$$Re\left\{(1-\eta)\frac{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z)}{z} + \eta\left(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z)\right)'\right\} > \beta, \tag{6}$$

where $0 \leq \beta < 1, \eta \geq 0, z \in \mathbb{U}$ and $\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z)$ is given by (4).

We further let $TQ^n_{r,s,\lambda}(\eta, \beta) = Q^n_{r,s,\lambda}(\eta, \beta) \cap T$.

We present some examples by using specializing the values of $r, s, a_1, a_2...a_r, b_1, b_2, ...b_s, q$ and λ.

Example 1 For $n = 0, r = s+1, a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, ...(i = 1, 2, ...r, j = 1, ..., s)$ and $q \to 1$, then

$$Q^0_{s+1,s}(\eta, \beta) = H(a_i, b_j, \eta, \beta)$$

$$= Re\left\{(1-\eta)\frac{H(\alpha_1, ...\alpha_r, \beta_1, ...\beta_s)f(z)}{z} + \eta\left(H(\alpha_1, ...\alpha_r, \beta_1, ...\beta_s)f(z)\right)'\right\} > \beta.$$

Example 2 For $r = 1, s = 0, a_1 = q$, and $\lambda = 1$, then

$$Q_{1,0,1}(\eta, \beta) = S^n(\eta, \beta) = Re\left\{(1-\eta)\frac{S^n f(z)}{z} + \eta(S^n f(z))'\right\} > \beta.$$

Example 3 For $n = 0$, then

$$Q^0_{r,s}(\eta, \beta) = \mathcal{M}^s_r(\eta, \beta) = Re\left\{(1-\eta)\frac{\mathcal{M}^s_r(a_i, b_j; q)f(z)}{z} + \eta(\mathcal{M}^s_r(a_i, b_j; q)f(z))'\right\} > \beta.$$

Example 4 For $\lambda = 0, r = 1, s = 0, a_1 = q$ and $q \to 1$, then

$$Q_{1,0,0}(\eta, \beta) = Q(\eta, \beta) = Re\left\{(1-\eta)\frac{f(z)}{z} + \eta(f(z))'\right\} > \beta$$

where $Q(\eta, \beta)$ denote the class of analytic functions which was studied by Ding et al. [11].

2. Coefficient inequalities

First, we prove a sufficient coefficient bound.

Theorem 2.1. If $f(z) \in \mathcal{A}$ be given by (1) satisfies

$$\sum_{k=1}^{\infty} \Phi(\eta, \lambda, n, k)|a_k| \leq 1 - \beta \tag{7}$$

for some $\beta(0 \leq \beta < 1), \eta \geq 0$ and $\lambda \geq 0$, where

$$\Phi(\eta, \lambda, n, k) = (|1 - \eta| + \eta k) (1 + (k-1)\lambda)^n |\Upsilon_k|, \tag{8}$$

and Υ_k given by (5), then $f \in Q^n_{r,s,\lambda}(\eta, \beta)$.

Proof. Let, the expression (7) be true for $f \in \mathcal{A}$. By using the fact $Rew > \beta \leftrightarrow |1 - \beta + w| > |1 + \beta - w|$. It suffices to show that,

$$\begin{aligned} &\left|(1-\beta)z + (1-\eta)\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z) + \eta z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z))'\right| - \\ &\left|(1+\beta)z - (1-\eta)\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z) - \eta z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; qf(z)))'\right| > 0. \end{aligned} \tag{9}$$

So, we have

$$\left|(1-\beta)z + (1-\eta)\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z) + \eta z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z))'\right| -$$
$$\left|(1+\beta)z - (1-\eta)\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(z) - \eta z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; qf(z)))'\right|$$

$$= \left|(1-\beta)z + (1-\eta)z\left(1 + \sum_{k=2}^{\infty}(1+(k-1)\lambda)^n \Upsilon_k a_k z^{k-1}\right) + \eta z\left(1 + \sum_{k=2}^{\infty} k(1+(k-1)\lambda)^n \Upsilon_k a_k z^{k-1}\right)\right|$$
$$- \left|(1+\beta)z - (1-\eta)z\left(1 + \sum_{k=2}^{\infty}(1+(k-1)\lambda)^n \Upsilon_k a_k z^{k-1}\right) - \eta z\left(1 + \sum_{k=2}^{\infty} k(1+(k-1)\lambda)^n \Upsilon_k a_k z^{k-1}\right)\right|$$

We impose

$$\geq (2-\beta)|z| - |1-\eta||z|\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n|\Upsilon_k||a_k||z|^{k-1} - \eta|z|\sum_{k=2}^{\infty}k(1+(k-1)\lambda)^n|\Upsilon_k||a_k||z|^{k-1}$$
$$-\beta|z| - |1-\eta||z|\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n|\Upsilon_k||a_k||z|^{k-1} - \eta|z|\sum_{k=2}^{\infty}k(1+(k-1)\lambda)^n|\Upsilon_k||a_k||z|^{k-1}$$

After simplification and by using (7), we get

$$2(1-\beta) - 2\left(|1-\eta|\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n|\Upsilon_k||a_k| + \eta\sum_{k=2}^{\infty}k(1+(k-1)\lambda)^n||\Upsilon_k||a_k|\right) \geq 0.$$

This completes the proof of Theorem 2.1.

Corollary 2.2 A function $f \in A$, is in $S^n(\eta, \beta)$ if

$$\sum_{k=2}^{\infty}\left(|1-\eta|k^n + \eta k^{n+1}\right)|a_k| < 1 - \beta$$

where $n \geq 0, \eta \geq 0$ and $0 \leq \beta < 1$.

Corollary 2.3 A function $f \in A$, is in $\mathcal{M}_r^s(\eta,\beta)$ if

$$\sum_{k=2}^{\infty} (|1-\eta|+\eta k)\,|\Upsilon_k||a_k| < 1-\beta$$

where Υ_k is given by (5), $\eta \geq 0$ and $0 \leq \beta < 1$.

We next show that condition (7) is also necessary for functions in $TQ^n_{r,s,\lambda}(\eta,\beta)$.

Theorem 2.4 Let the function $f(z) \in T$ be given by (2). Then $f(z) \in TQ^n_{r,s,\lambda}(\eta,\beta)$ if and only if (7) is satisfied.

Proof. In view of Theorem 2.1, it is sufficient to prove the "only if " part. Let us assume that $f(z)$ defined by (2) is in $TQ^n_{r,s,\lambda}(\eta,\beta)$. We have

$$Re\left\{(1-\eta)\frac{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)}{z} + \eta\left(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)\right)'\right\} > \beta.$$

Since $Rez \leq |z|$, we have

$$\left|(1-\eta)\frac{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)}{z} + \eta\left(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)\right)'\right| > \beta.$$

By a computation, we obtain

$$\left|1-(1-\eta)\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n\,\Upsilon_k a_k z^{k-1} - \eta\sum_{k=2}^{\infty}k\,(1+(k-1)\lambda)^n\,\Upsilon_k a_k z^{k-1}\right| > \beta. \tag{10}$$

Set $z = re^{i\theta}(\theta \in R)$ in (10). Hence

$$\sum_{k=2}^{\infty}(|1-\eta|+\eta k)(1+(k-1)\lambda)^n|\Upsilon_k||a_k|r^{k-1} \leq 1-\beta. \tag{11}$$

Letting $r \to 1^-$ in (11), we get (7). Thus, this completes the proof of the theorem.

Corollary 4.5 If $f \in TQ^n_{r,s,\lambda}(\eta,\beta)$, then

$$|a_k| \leq \frac{1-\beta}{\Phi(\eta,\lambda,n,k)}, \quad 0 \leq \beta < 1, \eta \geq 0, \lambda \geq 0, \, and \, n \geq 0.$$

Equality holds for the function

$$f(z) = z - \frac{1-\beta}{\Phi(\eta,\lambda,n,k)}z^n.$$

3. Extreme points

The determination of the extreme points of a family $f(z)$ of univalent functions enables us to solve many external problems for $f(z)$ (see[6]).

Theorem 3.1 Let

$$f_1(z) = z \quad and \quad f_k(z) = z - \frac{1-\beta}{\Phi(\eta,\lambda,n,k)}z^k, \quad (k \geq 2).$$

Then $f \in TQ^n_{r,s,\lambda}(\eta,\beta)$, if and only if, it can be represented in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \quad (\mu_k \geq 0, \sum_{k=1}^{\infty} \mu_k = 1). \tag{12}$$

Proof. Suppose $f(z)$ can be expressed as in (12). Then

$$\begin{aligned} f(z) =& \sum_{k=1}^{\infty} \mu_k f_k(z) \\ =& \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) \\ =& \mu_1 z + \sum_{k=2}^{\infty} \mu_k (z - \frac{1-\beta}{\Phi(\eta,\lambda,n,k)} z^k) \\ =& z - \sum_{k=2}^{\infty} \mu_k \frac{1-\beta}{\Phi(\eta,\lambda,n,k)} z^k. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\Phi(\eta,\lambda,n,k)}{(1-\beta)} \quad \frac{(1-\beta)}{\Phi(\eta,\lambda,n,k)} \mu_k \\ &= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \end{aligned}$$

So by Theorem 2.1, $f \in TQ^n_{r,s,\lambda}(\eta,\beta)$.

Conversely, we suppose $f \in TQ^n_{r,s,\lambda}(\eta,\beta)$. Since

$$|a_k| \leq \frac{1-\beta}{\Phi(\eta,\lambda,n,k)}, \quad k \geq 2.$$

We set

$$\mu_k = \frac{\Phi(\eta,\lambda,n,k)}{1-\beta} |a_k|, \quad k \geq 2$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

Then we have

$$\begin{aligned} f(z) =& \sum_{k=1}^{\infty} \mu_k f_k(z) \\ =& \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) \end{aligned}$$

and the proof is complete.

Corollary 3.2 The extreme points of $TQ^n_{r,s,\lambda}(\eta,\beta)$ are the functions $f_1(z) = z$ and

$$f_k(z) = z - \frac{1-\beta}{\Phi(\eta,\lambda,n,k)} z^k, \quad (k \geq 2),$$

for $0 \leq \beta < 1$ and $n \geq 0$.

4. Integral means inequalities

For any two functions f and g analytic in $\mathbb{U}$, f is said to be subordinate to g in $\mathbb{U}$, denote by $f \prec g$ if there exists an analytic function ω defined $\mathbb{U}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z)), z \in \mathbb{U}$.

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. In 1925, Littlewood [12] proved the following subordination theorem.

Theorem 4.1[12]. If f and g are any two functions, analytic in $\mathbb{U}$, with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}, (0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\mu d\theta \le \int_0^{2\pi} |g(z)|^\mu d\theta.$$

Now

Theorem 4.2 Let $f \in TQ^n_{r,s,\lambda}(\eta, \beta)$ and f_k be defined by

$$f_k(z) = z - \frac{(1-\beta)}{\Phi(\eta, \lambda, n, k)} z^k \quad (k = 2, 3, ...,).$$

If there exists an analytic function $\omega(z)$ given by

$$[\omega(z)]^{k-1} = \sum_{k=2}^{\infty} \frac{\Phi(\eta, \lambda, n, k)}{(1-\beta)} a_k z^{k-1}$$

then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \le \int_0^{2\pi} |f_k(re^{i\theta})|^\mu d\theta.$$

Proof. We want to prove that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{\infty} a_k z^{k-1}\right|^\mu d\theta \le \int_0^{2\pi} \left|1 - \frac{1-\beta}{\Phi(\eta, \lambda, n, k)} z^{k-1}\right|^\mu d\theta.$$

By Theorem 4.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 - \frac{1-\beta}{\Phi(\eta, \lambda, n, k)} z^{k-1}.$$

We may write

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} = 1 - \frac{1-\beta}{\Phi(\eta, \lambda, n, k)} [\omega(z)]^{k-1}$$

which implies

$$[\omega(z)]^{k-1} = \sum_{k=2}^{\infty} \frac{\Phi(\eta, \lambda, n, k)}{1-\beta} a_k z^{k-1}.$$

Clearly, $\omega(0) = 0$. By (7), we have

$$\begin{aligned} |\omega(z)|^{k-1} &= \left|\sum_{k=2}^{\infty} \frac{\Phi(\eta, \lambda, n, k)}{1-\beta} a_k z^{k-1}\right| \\ &\le \sum_{k=2}^{\infty} \frac{\Phi(\eta, \lambda, n, k)}{(1-\beta)} |a_k||z|^{k-1} \le |z| < 1. \end{aligned}$$

5. Neighborhoods of the class $TQ^n_{r,s,\lambda}(\eta,\beta)$

The concept of neighborhoods was first introduced by Goodman in [14] and then generalized by Ruscheweyh in [9]. Also refer to Silverman [15], Ahuja and Nunokawa [16] and Frasin and Darus [17].

We would like to investigate the $(n,\delta)-$neighborhoods of the subclass $TQ^n_{r,s,\lambda}(\eta,\beta)$.

First, we define $(n,\delta)-$neighborhoods of the function $f \in T$ as the following:

Definition 5.1 For any $f(z) \in T$ and $\delta \geq 0$, we define

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad \sum_{k=2}^{\infty} k \cdot |a_k - b_k| \leq \delta \right\}. \tag{13}$$

So, for $e(z) = z$, we observe that

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad \sum_{k=2}^{\infty} k \cdot |b_k| \leq \delta \right\}. \tag{14}$$

Next we give the following:

Theorem 5.2 Let

$$\delta = \frac{2(\beta-1)}{(1+\lambda)^n \left(|1-\eta| + 2\eta\right) ||\Upsilon_2|}$$

where $\Upsilon_2 = \frac{(1-a_1)\ldots(1-a_r)}{(1-q)(1-b_1)\ldots(1-b_s)}$, then $TQ^n_{r,s,\lambda}(\eta,\beta) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. For $f(z) \in TQ^n_{r,s,\lambda}(\eta,\beta)$ and making use of the condition (7), we obtain

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{1-\beta}{(1+\lambda)^n \left(|1-\eta| + 2\eta\right) |\Upsilon_2|}. \tag{15}$$

On the other hand, we also find from (7) and (15) that

$$(1+\lambda)^n |\Upsilon_2| \sum_{k=2}^{\infty} \left(|1-\eta| + \eta k\right) |a_k| \leq 1-\beta$$

$$\eta(1+\lambda)^n |\Upsilon_2| \sum_{k=2}^{\infty} k|a_k| \leq (1-\beta) - (1+\lambda)^n |\Upsilon_2||1-\eta| \sum_{k=2}^{\infty} |a_k|.$$

Thus,

$$\sum_{k=2}^{\infty} k \cdot a_k \leq \frac{2(1-\beta)}{(1+\lambda)^n \left(|1-\eta| + 2\eta\right) |\Upsilon_2|} = \delta$$

which in view of (14), proves Theorem 5.2.

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