Electronic Journal of Mathematical Analysis and Applications,
Vol. 2(2) July 2014, pp. 199-213.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

NONLINEAR DYNAMICS OF A MODIFIED AUTONOMOUS VAN DER POL-DUFFING CHAOTIC CIRCUIT

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Abstract. In this work, we investigate the dynamical behaviors of the modified autonomous Van der Pol-Duffing chaotic circuit (MADVP). Some stability conditions of the equilibrium points are discussed. The existence of pitchfork bifurcation is verified by using the bifurcation theory and the center manifold theorem. The occurrences of Hopf bifurcation about the equilibrium points are proved. Conditions for supercritical and subcritical Hopf bifurcations are also derived. A route to chaos in this system is shown via period-doubling bifurcations. Furthermore, the analytical conditions of the existence of homoclinic orbits and Smale horseshoe chaos in this system are obtained. Numerical simulations are used to support the theoretical predictions.

1. Introduction

Chaos is one of the most fascinating phenomena which has been extensively studied and developed by scientists since the pioneering work of Lorenz in 1963 [1]-[7]. The chaotic system has complex dynamical behaviors such as the unpredictability of the long-term future behavior and irregularity. Thus, chaos has great potential applications in many disciplines such as encryption, cryptography [8]-[9], chaos control and synchronization [10]-[17], secure communications [18], neuroscience [19], and mathematical biology [20].

As a matter of fact, nonlinear electronic circuits play an important role in studying various phenomena that undergo complex dynamical behaviors and chaos. Thus, nonlinear electronic circuits are widely used as an experimental vehicle to study nonlinear phenomena. This field of research was initiated by L.O. Chua who developed a nonlinear circuit with a piecewise nonlinear term called Chua's circuit [21], however the simplest autonomous nonlinear circuit which generates chaotic signals was presented in [22]. Recently, Chen circuit [23] and Lü circuit [24] have been implemented with quadratic nonlinear terms. Thus, our objective is to study the nonlinear dynamics of MADVP circuit. We show that the circuit's system has three equilibrium points E_0, E_+, and E_-, then we study their stability conditions. The conditions of existence of pitchfork bifurcation are derived by using center

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1991 *Mathematics Subject Classification.* 34C15, 34C28, 37M05.
Key words and phrases. Circuit implementation; Center manifold; Pitchfork bifurcation; Hopf bifurcation; Homoclinic orbits; Chaos.
Submitted April 6, 2014.

manifold theorem and the bifurcation theory [25]. The occurrences of Hopf bifurcation near the equilibrium points are also discussed. The stability conditions of the periodic solutions are obtained by using Hopf bifurcation theorems [26]-[27]. Also, we investigate the analytical conditions for the existence of the homoclinic orbits in this system by using the theorem given in [28].

The paper is organized as follows: In Section 2, a circuit realization of MADVP system is proposed. In Section 3, some stability conditions of the equilibrium points are investigated. In Section 4, pitchfork bifurcation analysis is demonstrated. In Section 5, Hopf bifurcation analysis of the equilibrium points is discussed. In Section 6, the existence of homoclinic orbits is analytically obtained. Finally, in Section 7, conclusions are drawn.

2. The circuit realization of MADVP system

The circuit implementation of MADVP circuit is shown in figure 1. The MADVP system is given as follows [13]:

$$\begin{aligned} \dot{x} &= -\upsilon(x^3 - \mu x - y), \\ \dot{y} &= x - \gamma y - z, \\ \dot{z} &= \beta y, \end{aligned} \tag{1}$$

where β, γ, υ are positive real numbers and $\mu \in \mathbb{R}$. The equilibrium points of system (1) are:

$$E_0 = (0,0,0), E_+ = (\sqrt{\mu}, 0, \sqrt{\mu}), \text{ and } E_- = (-\sqrt{\mu}, 0, -\sqrt{\mu}) \tag{2}$$

where E_+ and E_- exist if $\mu > 0$.

3. Some stability conditions of the equilibrium points

Consider the three-dimensional autonomous system

$$\frac{dX}{dt} = F(X), \quad X \in \mathbb{R}^3 \tag{3}$$

where the vector field $F(X) : \mathbb{R}^3 \to \mathbb{R}^3$ belongs to the class $C^r (r \geq 2)$ and the fixed point $X_e \in \mathbb{R}^3$ is a hyperbolic saddle focus, i.e., the eigenvalues of the Jacobian matrix J have the form:

$$\lambda_1 = \alpha, \lambda_{2,3} = \sigma \pm iw, \alpha\sigma < 0, w \neq 0, \text{ and } i = \sqrt{-1} \tag{4}$$

where α, σ, and w are real constants. The eigenvalues equation of the equilibrium point is given by the following polynomial:

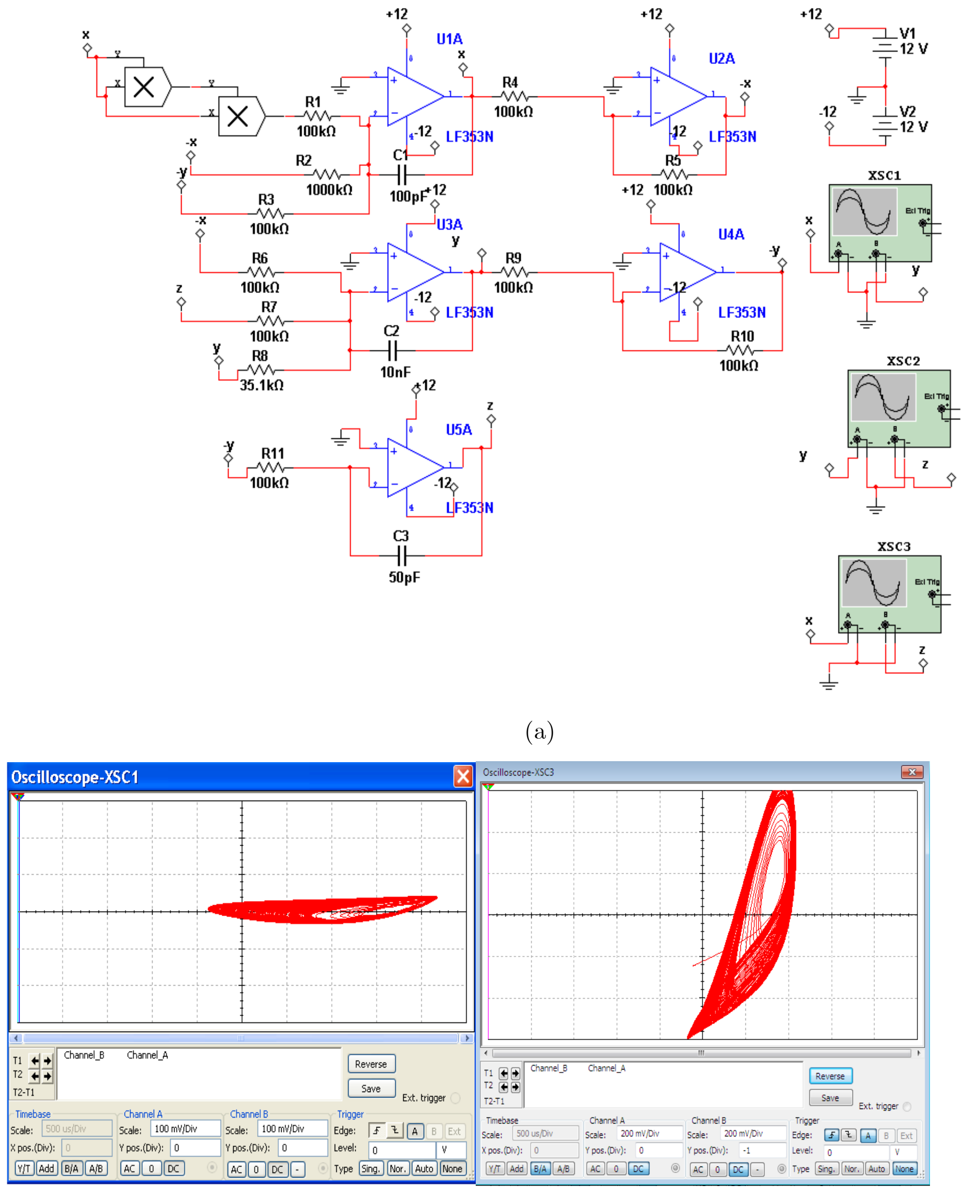
$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{5}$$

and its discriminant $D(P)$ is given by:

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4(a_1)^3a_3 - 4(a_2)^3 - 27(a_3)^2. \tag{6}$$

If $D(P) < 0$, then the characteristic equation (5) has one real root λ_0 and two complex-conjugate roots $\lambda_\pm = \theta \pm i\omega$. Hence, the characteristic equation (5) can be written as:

$$\lambda^3 - (2\theta + \lambda_0)\lambda^2 + (|\lambda_+|^2 + 2\theta\lambda_0)\lambda - |\lambda_+|^2 \lambda_0 = 0. \tag{7}$$



(a)

(b) (c)

Figure 1: (a) Circuit implementation of MADVP circuit; (b) and (c) Simulations results via oscilloscope when $\gamma = 2.85$.

The equilibrium points $E_{\pm}$ have the same characteristic equation

$$\lambda^3 + (\gamma + 2\mu\upsilon)\lambda^2 + (\beta - \upsilon + 2\mu\upsilon\gamma)\lambda + 2\mu\upsilon\beta = 0. \tag{8}$$

From the characteristic equation (8), we find that $\lambda_0 = \dfrac{-2\mu\upsilon\beta}{|\lambda_+|^2} < 0$. Thus, the conditions $D(P) < 0$ and $\gamma < \dfrac{(\upsilon-\beta)}{2\mu\upsilon}$ are sufficient conditions for the equilibrium points $E_\pm$ to be a saddle focus.

Using the parameter values $\beta = 200, \mu = 0.1, \gamma = 1.6$, and $\upsilon = 100$, we get:

$$D(P) = -0.389 \times 10^9, \lambda_0 = -23.3, \text{ and } \lambda_\pm = 0.85 \pm 13.7i, \tag{9}$$

i.e. the equilibrium points $E_\pm$ are saddle focus at this choice of the parameter values.

By applying Routh-Hurwitz criterion, the necessary and sufficient condition for the equilibrium points $E_\pm$ to be locally asymptotically stable is

$$2\mu\upsilon\gamma^2 + (\beta - \upsilon + 4\mu^2\upsilon^2)\gamma - 2\mu\nu^2 > 0. \tag{10}$$

Thus, we have the following Lemma:

Lemma 1. *The equilibrium points are asymptotically stable if and only if* $\gamma > \gamma^+$ *where*

$$\gamma^+ = \frac{-(\beta - \upsilon + 4\mu^2\upsilon^2) + \sqrt{(\beta - \upsilon + 4\mu^2\upsilon^2)^2 + 16\mu^2\nu^3}}{4\mu\upsilon}.$$

The characteristic equation of the equilibrium point E_0 is given by:

$$\lambda^3 + (\gamma - \mu\upsilon)\lambda^2 + (\beta - \upsilon - \mu\upsilon\gamma)\lambda - \mu\upsilon\beta = 0, \tag{11}$$

which implies that $\lambda_0 = \dfrac{\mu\upsilon\beta}{|\lambda_+|^2} > 0$ for $\mu > 0$. This means that E_0 is unstable when $\mu > 0$.

4. Analysis of pitchfork bifurcation

Assume that $\gamma = \dfrac{\upsilon(2-\upsilon)}{1-\upsilon}, \beta = \dfrac{\upsilon}{1-\upsilon}$, and $\mu = \mu_c = 0$ (where $\upsilon \neq 1$), then by using the characteristic equation (11), the equilibrium point E_0 has the eigenvalues $0, -\upsilon$, and $-\beta$. Therefore E_0 is not hyperbolic and consequently we can use the center manifold theorem [25] to discuss the dynamics near E_0.

By using the transformation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1-\upsilon & 1-\upsilon \\ 0 & \upsilon-1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \bar{\mu} = \mu - \mu_c,$$

system (1) is transformed into the following form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\upsilon & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} g_1 \;=\; & -x_1^3 + 3(\upsilon-1)(x_2+x_3)x_1^2 - (1-\upsilon)^2(3x_2^2 + 6x_2x_3 + 3x_3^2)x_1 \\ & +\bar{\mu}x_1 - (1-\upsilon)^3x_2^3 - 3(1-\upsilon)^3x_2^2x_3 + (\bar{\mu}(1-\upsilon) - 3(1-\upsilon)^3x_3^2)x_2 \\ & -(1-\upsilon)^3x_3^2 + \bar{\mu}(1-\upsilon)x_3, \end{aligned}$$

$$g_2 = -\frac{g_1}{\upsilon},$$
$$g_3 = (\upsilon - 1)g_2.$$

Consider the parameter $\bar{\mu}$ to be the bifurcation parameter of system (12) and also as a new independent variable of system (12). Thus, from center manifold theory, there exists a center manifold for (12) given by:

$$\begin{aligned} W^c(0) &= \{(x_1, x_2, x_3, \bar{\mu}) \in \mathbb{R}^4 \mid x_2 = h_1(x_1, \bar{\mu}), x_3 = h_2(x_1, \bar{\mu}), \\ |x_1| &< \zeta_1, |\bar{\mu}| < \zeta_2, h_i(0,0) = 0, Dh_i(0,0) = 0, i = 1, 2\}, \end{aligned} \tag{13}$$

for ζ_1 and ζ_2 sufficiently small. To compute the center manifold $W^c(0)$ we assume that

$$x_2 = h_1(x_1, \bar{\mu}) = \alpha_1 x_1^2 + \alpha_2 \bar{\mu} x_1 + \alpha_3 \bar{\mu}^2 + \alpha_4 x_1^3 + \alpha_5 \bar{\mu} x_1^2 + ..., \tag{14}$$
$$x_3 = h_2(x_1, \bar{\mu}) = \beta_1 x_1^2 + \beta_2 \bar{\mu} x_1 + \beta_3 \bar{\mu}^2 + \beta_4 x_1^3 + \beta_5 \bar{\mu} x_1^2 + ... \tag{15}$$

The center manifold must satisfy

$$\begin{aligned} \aleph(h(x_1, \bar{\mu})) \approxeq D_{x_1} h(x_1, \bar{\mu}) [Ax_1 + f(x_1, h(x_1, \bar{\mu}), \bar{\mu})] - Bh(x_1, \bar{\mu}) \\ -g(x_1, h(x_1, \bar{\mu}), \bar{\mu}) = 0, \end{aligned} \tag{16}$$

where $A = 0, f = g_1, h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, B = \begin{pmatrix} -\upsilon & 0 \\ 0 & -\beta \end{pmatrix}$, and $g = \begin{pmatrix} g_2 \\ g_3 \end{pmatrix}$.

Thus, substituting (14-15) into (16), and then equating terms of like powers to zero, we get

$$h_1(x_1, \bar{\mu}) = -\frac{1}{\upsilon^2}\bar{\mu} x_1 + \frac{1}{\upsilon^2} x_1^3 + \frac{1 - (1-\nu)^3}{\upsilon^4} \bar{\mu}^2 x_1 + ..., \tag{17}$$
$$h_2(x_1, \bar{\mu}) = \frac{1}{\beta^2}\bar{\mu} x_1 - \frac{1}{\beta^2} x_1^3 + \frac{(\upsilon - 3)(1-\nu)^3}{\upsilon^3} \bar{\mu}^2 x_1 + ... \tag{18}$$

Using equations (12) and (17-18), we obtain the vector field reduced to the center manifold as follows

$$\begin{aligned} \frac{dx_1}{dt} &= \bar{\mu} x_1 + (1 - \upsilon)(\frac{1}{\beta^2} - \frac{1}{\upsilon^2})\bar{\mu}^2 x_1 - x_1^3 + ..., \\ \frac{d\bar{\mu}}{dt} &= 0. \end{aligned} \tag{19}$$

The equilibrium point $(0, 0)$ of system (19) undergoes a pitchfork bifurcation at $\bar{\mu} = 0$, since it satisfies the conditions:

$$\begin{aligned} G(0,0) &= 0, \ \frac{\partial G}{\partial x_1}|_{(0,0)} = 0, \ \frac{\partial G}{\partial \bar{\mu}}|_{(0,0)} = 0, \ \frac{\partial^2 G}{\partial x_1^2}|_{(0,0)} = 0, \frac{\partial^2 G}{\partial x_1 \partial \bar{\mu}}|_{(0,0)} \neq 0, \\ \text{and } \frac{\partial^3 G}{\partial x_1^3} &|_{(0,0)} \neq 0, \end{aligned} \tag{20}$$

where $G(x_1, \bar{\mu}) \approx \bar{\mu} x_1 + (1 - \upsilon)(\frac{1}{\beta^2} - \frac{1}{\upsilon^2})\bar{\mu}^2 x_1 - x_1^3$.

Thus, by choosing $\gamma = \frac{\upsilon(2 - \upsilon)}{1 - \upsilon}, \beta = \frac{\upsilon}{1 - \upsilon}$ and varying the parameter μ near the critical value μ_c, system (1) has a unique locally asymptotically stable equilibrium point E_0 as $\mu < \mu_c$. When $\mu = \mu_c$, system (1) undergoes a pitchfork bifurcation at E_0. Then, by varying the parameter μ above the critical value μ_c, the equilibrium point becomes unstable and two other equilibrium points $E_\pm$ appear and they are locally asymptotically stable (near $\mu = \mu_c$).

5. Hopf bifurcation analysis of the equilibrium points

In this Section, we apply Hopf bifurcation theorems [26]–[27] to the equilibrium points of system (1). The characteristic equation of the equilibrium points E_+ and E_- is given by equation (8). So, we carry out the analysis for the equilibrium point E_+, similar analysis holds for E_-.

We shift the origin to the point E_+ by writing $x = \sqrt{\mu} + \hat{x}$, $y = \hat{y}, z = \sqrt{\mu} + \hat{z}$ into equation (1) and obtain in vector form:

$$\dot{X} = J_+ X + \hat{F}(X, \beta, \gamma, \mu, \upsilon), \tag{21}$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, J_+ = \begin{pmatrix} -2\mu\upsilon & \upsilon & 0 \\ 1 & -\gamma & -1 \\ 0 & \beta & 0 \end{pmatrix}, \hat{F} = \begin{pmatrix} -\upsilon(x^3 + 3\sqrt{\mu}x^2) \\ 0 \\ 0 \end{pmatrix},$$

and we have dropped the cap on x, y and z. We wish to determine the sufficient conditions to ensure that equation (8) will have one negative $\lambda_0(\gamma)$ and two complex-conjugate roots $\theta(\gamma) \pm i\omega(\gamma)$ whose real part $\theta(\gamma)$ vanishes at the critical point γ_c, while $\theta'(\gamma_c) \neq 0$ ($\theta'(\gamma) = \dfrac{d\theta}{d\gamma}$).

Equating the coefficients of like powers of in (7) and (8), we get:

$$\begin{aligned} 2\theta + \lambda_0 &= -(\gamma + 2\mu\upsilon), & (22)\\ |\lambda_+|^2 + 2\theta\lambda_0 &= \beta - \upsilon + 2\mu\upsilon\gamma, & (23)\\ |\lambda_+|^2 \lambda_0 &= -2\mu\upsilon\beta. & (24) \end{aligned}$$

On the other hand, equation (8) has two pure imaginary roots ($\theta(\gamma_c) = 0$) if and only if the product of the coefficients of λ^2 and λ equals the constant term, that is iff

$$2\mu\upsilon\gamma^2 + (\beta - \upsilon + 4\mu^2\upsilon^2)\gamma - 2\mu\nu^2 = 0. \tag{25}$$

Hopf bifurcation takes place when the real parts of the two complex eigenvalues vanish; therefore the roots of equation (25) are the critical values at which Hopf bifurcation takes place. Thus, the critical Hopf bifurcation point is $\gamma_c = \gamma^+$.

Multiplying (23) by λ_0, then using (22) and (24) we obtain:

$$-(\gamma + 2\mu\upsilon + 2\theta)(\beta - \upsilon + 2\mu\upsilon\gamma) = -2\mu\upsilon\beta + 2\theta(\gamma + 2\mu\upsilon + 2\theta)^2.$$

Differentiating with respect to γ and setting $\gamma = \gamma_+$, we have:

$$\frac{d\theta}{d\gamma}|_{\gamma=\gamma_+} = \frac{-(\mu\upsilon{\gamma^+}^2 + \mu\nu^2)}{2\mu\nu^2 + ({\gamma^+}^2 + 4\mu\upsilon\gamma^+)\gamma^+} < 0. \tag{26}$$

For $\gamma = \gamma^+$, λ_0 is given by $\lambda_0 = -(\gamma^+ + 2\mu\upsilon) < 0$ whereas $|\lambda_+|^2 = \dfrac{2\mu\upsilon\beta}{\gamma^+ + 2\mu\upsilon}$. Consequently, system (1) satisfies the conditions of Hopf bifurcation theorem at $E_\pm$ [26]-[27] which ensures the existence of a one-parameter family of periodic solutions in the neighborhood of the parameters $(\gamma^+, \beta, \mu, \upsilon)$.

We next study the stability of these periodic solutions. It is easy to verify that there exists a matrix:

$$T = \begin{pmatrix} \frac{1}{s} & \frac{1}{\omega(\gamma^+)} & -\frac{1}{\omega(\gamma^+)} \\ -\frac{\gamma^+}{\upsilon s} & \frac{\chi}{\upsilon s} + \frac{1}{\gamma^+\omega(\gamma^+)} & \frac{\gamma^+}{\upsilon\omega(\gamma^+)} \\ \frac{\gamma^+}{\upsilon} + \frac{1}{\gamma^+} & -\frac{\chi}{\upsilon} & \frac{\chi}{\upsilon} \end{pmatrix},$$

where $s = -\lambda_0(\gamma^+) = (\gamma^+ + 2\mu\upsilon)$ and $\chi = \omega(\gamma^+) - \dfrac{\beta}{\omega(\gamma^+)}$. It is easy to verify that the matrices T and T^{-1} (the inverse matrix of T) satisfy the following relation:

$$T^{-1}J_+|_{\gamma=\gamma^+}T = \begin{pmatrix} 0 & \omega(\gamma^+) & 0 \\ -\omega(\gamma^+) & 0 & 0 \\ 0 & 0 & -s \end{pmatrix}.$$

Let the vector $y = (y_1, y_2, y_3)$ and assume that $\bar{\omega} = \dfrac{1}{s}y_1 + \dfrac{1}{\omega(\gamma^+)}y_2 - \dfrac{1}{\omega(\gamma^+)}y_3$ which implies that

$$\hat{F}(Ty, \gamma^+) = \begin{pmatrix} -\upsilon(\bar{\omega}^3 + 3\sqrt{\mu}\bar{\omega}^2) \\ 0 \\ 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} F(y) &= T^{-1}\hat{F}(Ty, \gamma^+) \\ &= \frac{1}{d}\begin{pmatrix} -\chi(\bar{\omega}^3 + 3\sqrt{\mu}\bar{\omega}^2) \\ -\gamma^+(\bar{\omega}^3 + 3\sqrt{\mu}\bar{\omega}^2) \\ \frac{\upsilon}{\gamma^+}(\bar{\omega}^3 + 3\sqrt{\mu}\bar{\omega}^2) \end{pmatrix} = \begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix}, \end{aligned}$$

where $d = \dfrac{\chi}{\upsilon s} + \dfrac{1}{\omega(\gamma^+)}(\dfrac{\gamma^+}{\upsilon} + \dfrac{1}{\gamma^+})$ and $F(y)$ is defined by Eq. (3.6) of [27]. The quantities F^k_{ijl} and F^k_{ij} $(i, j, k = 1, 2, 3$ and $l = 1, 2)$ appearing in Eq. (3.4) of [27] can easily be obtained as follow:

$$\begin{aligned} F^1_{11} &= -\frac{6\chi\sqrt{\mu}}{ds^2}, F^1_{12} = -\frac{6\chi\sqrt{\mu}}{ds\omega(\gamma^+)}, F^1_{13} = -F^1_{12}, F^1_{22} = -\frac{6\chi\sqrt{\mu}}{d\omega^2(\gamma^+)}, F^1_{23} = -F^1_{22}, \\ F^2_{11} &= -\frac{6\gamma^+\sqrt{\mu}}{ds^2}, F^2_{12} = -\frac{6\gamma^+\sqrt{\mu}}{ds\omega(\gamma^+)}, F^2_{13} = -F^2_{12}, F^2_{22} = -\frac{6\gamma^+\sqrt{\mu}}{d\omega^2(\gamma^+)}, F^2_{23} = -F^2_{22}, \\ F^3_{11} &= \frac{6\upsilon\sqrt{\mu}}{ds^2\gamma^+}, F^3_{22} = \frac{6\upsilon\sqrt{\mu}}{d\omega^2(\gamma^+)\gamma^+}, F^3_{12} = \frac{6\upsilon\sqrt{\mu}}{ds\omega(\gamma^+)\gamma^+}, F^1_{111} = -\frac{6\chi}{ds^3}, \\ F^1_{112} &= -\frac{6\chi}{ds\omega^2(\gamma^+)}, F^2_{222} = -\frac{6\gamma^+}{d\omega^3(\gamma^+)}, \text{ and } F^2_{112} = -\frac{6\gamma^+}{ds^2\omega(\gamma^+)}. \end{aligned} \tag{27}$$

Substituting all these quantities (27) in Eq. (3.4) of [27], we obtain the following result: the bifurcating periodic solutions exist for given β, μ, and $\upsilon > 0$ in the neighborhood of $\gamma_c = \gamma^+$ is stable or unstable according to $\Gamma\theta'(\gamma^+) > 0$ or $\Gamma\theta'(\gamma^+) < 0$, respectively, and the direction of bifurcation is above or below γ^+ according to $\Gamma > 0$ or $\Gamma < 0$, respectively, where $\Gamma\theta'(\gamma^+)$ and $\theta'(\gamma^+)$ are defined in Eq. (3.4) of [27] and (26), respectively. Using these criteria, one can compute the stability of the bifurcating periodic solutions for specific numerical values of the parameters.

Remark 1. *The point E_- has the same discussion as the point E_+ because the terms F_{ij}^k of (27) equal negative F_{ij}^k of E_- and when we substitute these quantities in Eq. (3.4) of* [27]*, the negative sign will disappear. Hence, the value of $\Gamma\theta'(\gamma^+)$ in the case of E_- will be the same as in case of E_+.*

The above-mentioned analysis of Hopf bifurcation is also applied to the equilibrium point E_0 and the main results can be summarized as follows:

Lemma 2. *System (1) undergoes a Hopf bifurcation at E_0 for $\beta > 0, \mu < 0,$ and $\upsilon > 0$ and γ near to γ_0. Moreover, the bifurcating periodic solutions emanating from E_0 have two cases: (i) If $\gamma_0 > \mu\upsilon + \sqrt{\mu^2\upsilon^2 + \upsilon}$, the bifurcating periodic solutions are asymptotically orbitally stable with asymptotic phase. The direction of bifurcation is below and the bifurcation is supercritical, (ii)If $\gamma_0 < \mu\upsilon + \sqrt{\mu^2\upsilon^2 + \upsilon}$, then the bifurcating periodic solutions are unstable. The direction of bifurcation is above and the bifurcation is subcritical, where*

$$\gamma_0 = \frac{-(\beta - \upsilon + \mu^2\upsilon^2) + \sqrt{(\beta - \upsilon + \mu^2\upsilon^2)^2 + 4\mu^2\upsilon^3}}{-2\mu\upsilon}.$$

6. Numerical results

For the parameter values $\beta = 200, \mu = 0.1$, and $\upsilon = 100$, the critical Hopf bifurcation point is $\gamma^+ = 3.5078$. The equilibrium points $E_\pm$ are stable for $\gamma > \gamma^+$ and lose their stability during γ is decreased and passed γ^+. Using the criteria of the stability of periodic solution which has been discussed above, we find that $\Gamma\theta'(\gamma^+) = -1.0746 < 0$ then the periodic solutions are unstable. Since $\theta'(\gamma^+) < 0$ (see Eq. (26)), it follows that $\Gamma > 0$ and the bifurcation occurs above γ^+. In this case, the Hopf bifurcation is subcritical and as γ is decreased less than γ^+, periodic solutions with higher orders and even chaos appear. Thus, system (1) exhibits period-doubling bifurcations leading to chaos as the parameter γ is decreased (see figure 2). Furthermore, the bifurcation diagrams in figure 3 show that system (1) has rich variety of dynamical behaviors including the complete chaotic and periodic behaviors.

7. The existence of homoclinic orbits

In this Section, we are going to use an analytical approach to discuss the existence of chaotic attractor of MADVP system. The method is described as follows:

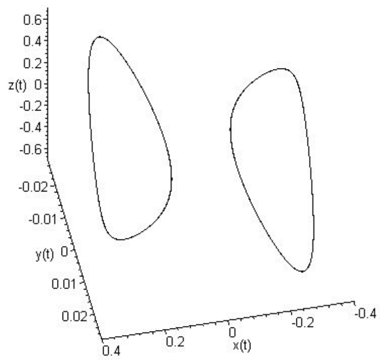
Theorem 3. [28] *Suppose that the equilibrium point X_e of system (3) is a saddle focus, whose eigenvalues satisfy $|\alpha| > |\sigma| > 0$. Suppose also that there exists a homoclinic orbit connecting X_e to itself. Hence, we have the following: (a) There are a countable number of Smale horseshoes defined in a neighborhood of the homoclinic orbit. (b) For any sufficiently small C^1-perturbation H of F the perturbed system*

$$\frac{dX}{dt} = H(x), \tag{28}$$

has at least a finite number of small horseshoes defined near the homoclinic orbit. (c) Both the original system (3) and the perturbed system (28) have horseshoe kind of chaos.

Now, we are going to apply Shil'nikov theorem to the equilibrium points $E_\pm$ of system (1). These equilibrium points have the same characteristic equation. So, it is sufficient to study the case of E_+ and similar results can be obtained for the point E_-.

For a homoclinic orbit joining E_+ to itself, the orbit is doubly asymptotic with respect to time t. Suppose that for $t > 0$



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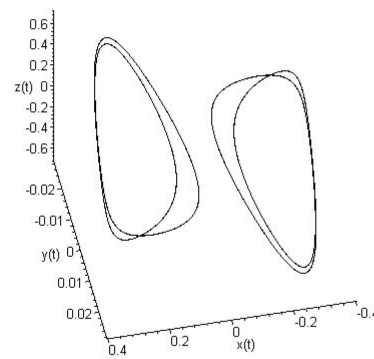

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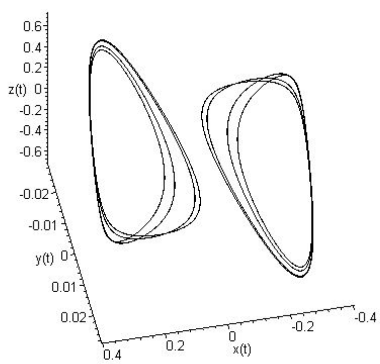

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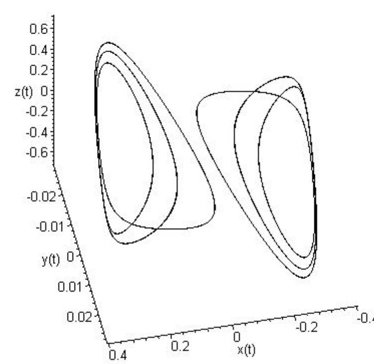

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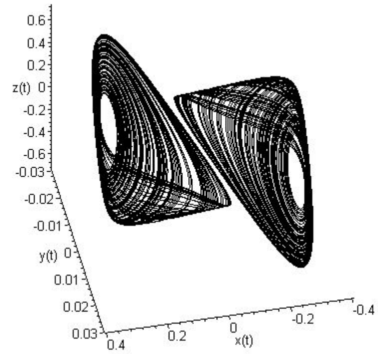

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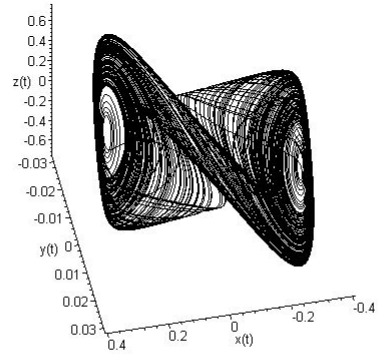

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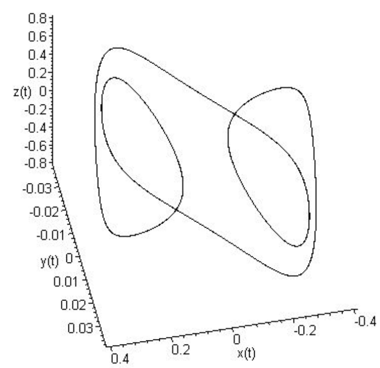

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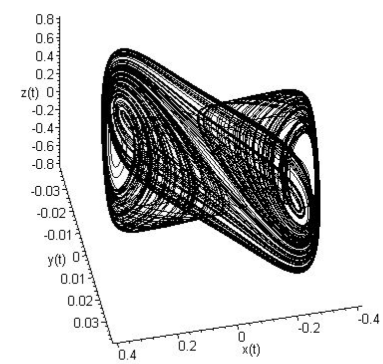
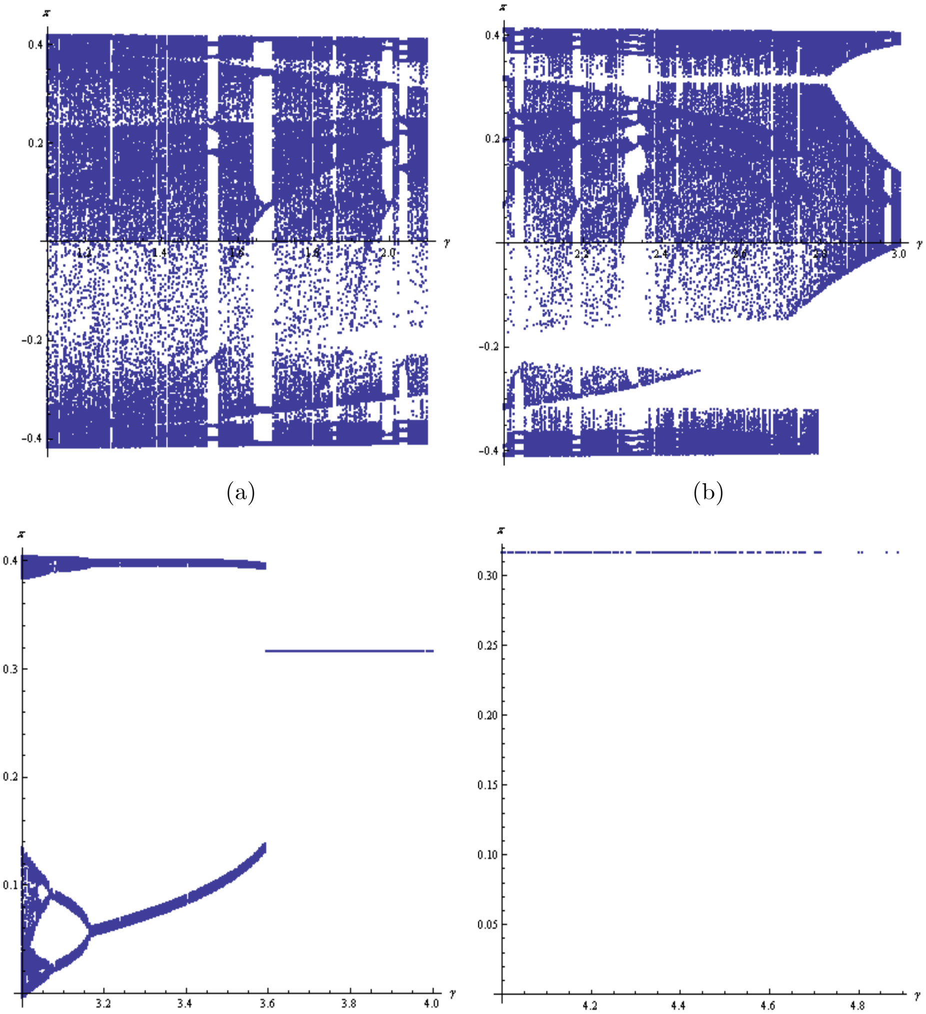


(h)

Figure 2: 3-D view of system (1) with the parameter values $\beta = 200, \mu = 0.1$, and $\upsilon = 100$, showing the appearance of; (a) Two period one limit cycles around the two equilibrium points $E_{\pm}$ for $\gamma = 3.4$. (b) Two period two limit cycles around the two equilibrium points $E_{\pm}$ for $\gamma = 3.1$. (c) Two period four limit cycles around the two equilibrium points $E_{\pm}$ for $\gamma = 3.05$.(d) Two period three limit cycles around the two equilibrium points $E_{\pm}$ for $\gamma = 2.97$. (e) One scroll chaotic attractors for $\gamma = 2.85$. (f) Double scroll chaotic attractor for $\gamma = 2.6$. (g) Homoclinic orbit for $\gamma = 1.68$. (h) Double scroll chaotic attractor for $\gamma = 1.6$.



(a) (b) (c) (d)

Figure 3(a-d): The bifurcation diagrams of system (1).

$$x(t) = \phi_x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{k\rho t}, y(t) = \phi_y(t) = b_0 + \sum_{k=1}^{\infty} b_k e^{k\rho t}, \quad (29)$$

$$z(t) = \phi_z(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{k\rho t}, \quad (30)$$

where a_k, b_k, c_k $(k \geq 0)$ are undetermined coefficients and ρ is a decaying exponent. Therefore, finding the homoclinic orbit connecting E_+ is now changed to seeking $\phi(t) = (\phi_x(t), \phi_y(t), \phi_z(t))$ such that $\phi(t) \to E_+$ as $t \to \pm\infty$.

Substituting (29)-(30) into system (1) and then comparing coefficients of $e^{k\rho t}$ $(k \geq 1)$ of like powers which yield a set of algebraic equations with the undetermined coefficients. For the constant term, we obtain:

$$-\upsilon(a_0^3-\mu a_0-b_0)=0, a_0-\gamma b_0-c_0=0, \text{ and } \beta b_0=0. \tag{31}$$

Hence $(a_0,b_0,c_0)=(\sqrt{\mu},0,\sqrt{\mu})$, By equating coefficients of $e^{\rho t}$, we get:

$$\begin{aligned} \rho a_1 &= -\upsilon(3a_0^2a_1-\mu a_1-b_1),\\ \rho b_1 &= -\upsilon(a_1-\gamma b_1-c_1),\\ \rho c_1 &= \beta b_1, \end{aligned} \tag{32}$$

which can be rewritten as:

$$(\rho I-J_+)\begin{pmatrix} a_1\\ b_1\\ c_1\end{pmatrix}=0, \tag{33}$$

where I is the identity matrix of order three.

By equating coefficients of $e^{2\rho t}$, we obtain:

$$(2\rho I-J_+)\begin{pmatrix} a_2\\ b_2\\ c_2\end{pmatrix}=\begin{pmatrix} -3\upsilon a_0a_1^2\\ 0\\ 0\end{pmatrix}. \tag{34}$$

For general $k>1$, we have:

$$(k\rho I-J_+)\begin{pmatrix} a_k\\ b_k\\ c_k\end{pmatrix}=\begin{pmatrix} \Psi_k^1(\beta,\gamma,\mu,\upsilon,\rho,\eta)\\ \Psi_k^2(\beta,\gamma,\mu,\upsilon,\rho,\eta)\\ \Psi_k^3(\beta,\gamma,\mu,\upsilon,\rho,\eta)\end{pmatrix}, \tag{35}$$

where $\Psi_k^j(\beta,\gamma,\mu,\upsilon,\rho,\eta)$, $j=1,2,3$ are some known functions depending on $\beta,\gamma,\mu,\upsilon,\rho$, and η. Using (33)-(35), we exclude the trivial solution of (a_1,b_1,c_1). Also, since J_+ has the unique negative eigenvalue $\lambda_0=\dfrac{-2\mu\upsilon\beta}{|\lambda_+|^2}$, then from (33), we deduce that there exists unique negative ρ. Hence, (a_1,b_1,c_1) can be uniquely determined in terms of the parameter η. Moreover, for $k>1$, we have $\det(k\rho I-J_+)\neq 0$ which ensures that the coefficients a_k, b_k, and c_k are uniquely determined using equation (35). Thus, $(\phi_x(t),\phi_y(t),\phi_z(t))$ have been uniquely determined for $t>0$.

For the reverse time-asymptotic trajectories, we can use the transformation $\tau=-t$ with $t>0$. Then system (1) becomes:

$$\begin{aligned} \frac{dx}{d\tau} &= \upsilon(x^3-\mu x-y),\\ \frac{dy}{d\tau} &= -x+\gamma y+z,\\ \frac{dz}{d\tau} &= -\beta y. \end{aligned} \tag{36}$$

Hence, we suppose that

$$x(\tau) = \phi'_x(\tau)=a'_0+\sum_{k=1}^{\infty}a'_ke^{-k\delta\tau}, y(\tau)=\phi'_y(\tau)=b'_0+\sum_{k=1}^{\infty}b'_ke^{-k\delta\tau}, \tag{37}$$

$$z(\tau) = \phi'_z(\tau)=c'_0+\sum_{k=1}^{\infty}c'_ke^{-k\delta\tau}. \tag{38}$$

Similar to the case of $t>0$, we compare coefficients of $e^{-k\delta\tau}$ $(k\geq 1)$ of equal powers, so we get a set of algebraic equations with the undetermined coefficients

a'_k, b'_k, c'_k and the decaying exponent δ. By comparing coefficients of the constant term, we get $(a'_0, b'_0, c'_0) = (a_0, b_0, c_0)$. However, by equating terms of equal powers at $k = 1$, we obtain:

$$(\delta I - J_+) \begin{pmatrix} a'_1 \\ b'_1 \\ c'_1 \end{pmatrix} = 0, \tag{39}$$

which implies that $\delta = \rho$. Thus, the rest of coefficients $a'_k(\beta, \gamma, \mu, \upsilon, \rho, \epsilon), b'_k(\beta, \gamma, \mu, \upsilon, \rho, \epsilon)$, and $c'_k(\beta, \gamma, \mu, \upsilon, \rho, \epsilon)$ (for $k > 1$) can also be uniquely determined. If one sets $\epsilon = \eta$, then $a'_k(\beta, \gamma, \mu, \upsilon, \rho, \eta) = a_k(\beta, \gamma, \mu, \upsilon, -\rho, \eta), b'_k(\beta, \gamma, \mu, \upsilon, \rho, \eta) = b_k(\beta, \gamma, \mu, \upsilon, -\rho, \eta)$, and $c'_k(\beta, \gamma, \mu, \upsilon, \rho, \eta) = c_k(\beta, \gamma, \mu, \upsilon, -\rho, \eta)$, $(k > 1\ )$.

The parameter η can be determined using the continuity condition of $\phi(t)$ at $t = 0$. From the previous analysis we conclude that the homoclinic orbit connecting E_+ has the following form:

$$x(t) = \begin{cases} \sqrt{\mu} + \displaystyle\sum_{k=1}^{\infty} a_k(\beta, \gamma, \mu, \upsilon, \rho, \eta) e^{k\rho t}, & t > 0 \\ \sqrt{\mu} + \displaystyle\sum_{k=1}^{\infty} a_k(\beta, \gamma, \mu, \upsilon, -\rho, \eta) e^{-k\rho t}, & t < 0 \end{cases} \tag{40}$$

$$y(t) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} b_k(\beta, \gamma, \mu, \upsilon, \rho, \eta) e^{k\rho t}, & t > 0 \\ \displaystyle\sum_{k=1}^{\infty} b_k(\beta, \gamma, \mu, \upsilon, -\rho, \eta) e^{-k\rho t}, & t < 0 \end{cases} \tag{41}$$

$$z(t) = \begin{cases} \sqrt{\mu} + \displaystyle\sum_{k=1}^{\infty} c_k(\beta, \gamma, \mu, \upsilon, \rho, \eta) e^{k\rho t}, & t > 0 \\ \sqrt{\mu} + \displaystyle\sum_{k=1}^{\infty} c_k(\beta, \gamma, \mu, \upsilon, -\rho, \eta) e^{-k\rho t}, & t > 0 \end{cases} \tag{42}$$

where a_k, b_k, c_k $(k > 1)$ are given by (35), ρ by $\det(\rho I - J_+) = 0$ and η by

$$\sum_{k=1}^{\infty} a_k(\beta, \gamma, \mu, \upsilon, \rho, \eta) = \sum_{k=1}^{\infty} b_k(\beta, \gamma, \mu, \upsilon, -\rho, \eta). \tag{43}$$

The following theorem is now proved:

Theorem 4. *If the conditions $D(P) < 0$, $\gamma < \dfrac{(\upsilon - \beta)}{2\mu\upsilon}$ are satisfied and system (1) has one homoclinic orbit whose components has the form (40)-(42), then horseshoe chaos occurs.*

Similar analysis can be obtained from the equilibrium point E_- since E_+ and E_- are images of each others under the symmetry $(x, y, z) \to (-x, -y, -z)$ and they have the same eigenvalues equation.

8. Conclusion

In this study, we have discussed the nonlinear dynamics of the chaotic MADVP circuit. Some stability conditions of the system's equilibrium points have been obtained. The existence of pitchfork bifurcation has been demonstrated by using center manifold theorem and the bifurcation theory. The conditions of Hopf bifurcation and its stability have been investigated. A route to chaos in this system has been shown via period-doubling bifurcations. The analytical conditions of existence of homoclinic orbits and occurrence of Smale horseshoe chaos in this system have been achieved. Numerical simulations show the effectiveness of the theoretical analysis.

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