Electronic Journal of Mathematical Analysis and Applications,
Vol. 2(2) July 2014, pp. 248-253.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

COMMON FIXED POINT THEOREM OF COMPATIBLE MAPPINGS OF TYPE (K) IN FUZZY METRIC SPACES

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Abstract. In this paper, we introduce the notion of compatible of type (K) in fuzzy metric space and obtain a common fixed point theorem for self mappings on complete fuzzy metric space with example. Our result generalizes and improves other similar results in literature.

1. Introduction

Fixed point theory is an important area of functional analysis. The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research. In 1965, the concept of fuzzy set was introduced by Zade[9]. Then, fuzzy metric spaces have been introduced by Kramosil and Michalek [12]. George and Veeramani [1] modified the notion of fuzzy metric spaces with the help of continuous t-norms. In 1986, G. Jungck [4] introduced notion of compatible mappings. In 1993, G. Jungck, P. P. Murthy and Y. J. Cho [5] gave a generalization of compatible mappings called compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions. In 1996, H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang [6] introduced the concept of compatible mappings of type (P) and compared with compatible mappings of type (A) and compatible mappings. In 1998, R. P. Pant [14] introduced the notion of reciprocal continuity of mappings in metric spaces. In 1998, Y.J. Cho, H.K. Pathak, S.M. Kang and J.S. Jung [17] introduced the concept of compatible mappings in fuzzy metric space. Recently, K. Jha, V. Popa and K. B. Manandhar [8] introduced the concept of compatible mappings of type (K) in metric space. Many authors have obtained fixed point theorems in fuzzy metric space using these compatible notions.
The purpose of this paper is to establish a common fixed point theorem for compatible mappings of type (E) in fuzzy metric spaces with example.

Definition 1.1 [1] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

2010 *Mathematics Subject Classification.* 47 H 10, 54 H 25.
Key words and phrases. fuzzy metric space, Compatible maps, compatible of type (K).
Submitted March 12, 2014.

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) a $*1 =$ a for all a $\in [0, 1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Definition 1.2 [1] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,
(FM1) $M(x, y, t) > 0$;
(FM 2) $M(x, y, t) = 1$ if and only if $x = y$;
(FM 3) $M(x, y, t) = M(y, x, t)$;
(FM 4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FM 5) $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous.
Then M is called a fuzzy metric on X. The function $M(x, y, t)$ denote the degree of nearness between x and y with respect to t.
Also, we consider the following condition in the fuzzy metric spaces $(X, M, *)$.
(FM6) $t \overset{lim}{\to} \infty M(x, y, t) = 1$, for all $x, y \in X$.

Example 1.1[1] Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$ and let M be fuzzy set on $X^2 \times (0, \infty)$ defined as follows:
$M(x, y, t) = \frac{t}{t+d(x,y)}$.
Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d is the standard fuzzy metric.

Definition 1.3 [1] Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be convergent to x in X if for each $\epsilon > 0$ and each $t > 0$, there exist $n_0 \in N$ such that $M(x_n, x, t) > 1 - \epsilon$ for all $n \geq n_0$.
a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.4 [17] The self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be compatible iff $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some $x\ inX$ and $t > 0$.

Definition 1.5 [5] The self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx_n, t) = 1$ and $\lim_{n\to\infty} d(SAx_n, AAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X and $t > 0$.

Definition 1.6 [6] The self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be compatible of type (P) if $\lim_{n\to\infty} d(SSx_n, AAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X and $t > 0$.

Definition 1.7 [13] The self mappings A and S of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if $\lim_{n\to\infty} ASx_n = Ax$ and $\lim_{n\to\infty} SAx_n = Sx$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$

for some x in X.
Definition 1.8 [8] The self mappings A and S of a metric space (X, d) are said to be compatible of type (K) $\lim_{n\to\infty} AAx_n = Sx$ and $\lim_{n\to\infty} SSx_n = Ax$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.

Definition 1.9 The self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be compatible of type (K) iff $\lim_{n\to\infty} M(AAx_n, Sx, t) = 1$ and $\lim_{n\to\infty} M(SSx_n, Ax, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X and $t > 0$.
The following examples show that the compatible of type (K) in fuzzy metric space is independent with compatible, compatible of type (A), compatible of type (P) and reciprocal continuous.

Example 1.2 Let $X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$, define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X, t > 0$ and $a * b = ab$ for all $a, b \in [0, 1]$ then $(X, M, *)$ is a fuzzy metric space. We define self-mappings A and S as $Ax = 2, Sx = 0$ for $x \in [0, 1] - \left\{\frac{1}{2}\right\}$, $Ax = 0, Sx = 2$ for $x = \frac{1}{2}$ and $Ax = \frac{2-x}{2}, Sx = \frac{x}{2}$ for $x \in (1, 2]$. Then, A and S are not continuous at $x = 1, \frac{1}{2}$.
Consider a sequence $\{x_n\}$ in X such that $x_n = 1 + \frac{1}{n}$ for all $n \in N$.
Then, we have $Ax_n = \frac{(2-x_n)}{2} \to \frac{1}{2} = x$ and $Sx_n = \frac{x_n}{2} \to \frac{1}{2} = x$.
Also, we have $AAx_n = A(\frac{(2-x_n)}{2}) = 2 \to 2, ASx_n = A(\frac{x_n}{2}) = 2 \to 2, S(x) = 2$ and $SSx_n = S(\frac{x_n}{2}) = 0 \to 0, SAx_n = S(\frac{2-x_n}{2}) = 0 \to 0, A(x) = 0$. Therefore, (A, S) is compatible of type (K) but the pair (A, S) is neither compatible nor compatible of type (A) (compatible of type (P), reciprocal continuous).

Example 1.3 Let $X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$, define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X, t > 0$ and $a * b = ab$ for all $a, b \in [0, 1]$ then $(X, M, *)$ is a fuzzy metric space.
Define self-mappings A and S as $Ax = x = 1$ for x $\in [0, 1), Ax = Sx = \frac{4}{3}$ for $x = 1$ and $Ax = 2 - x, Sx = x$ for x $\in (1, 2]$.
Consider a sequence $\{x_n\}$ in X such that $x_n = 1 + \frac{1}{n}$ for all $n \in N$.
Then, we have $Ax_n = (2 - x_n) \to 1 = x$, and $Sx_n = x_n \to 1 = x$. Since, $2 - x_n < 1$ for all $n \in N$, we have $AAx_n = A(2 - x_n) = 1 \to 1$, $ASx_n = A(x_n) = 2 - x_n \to 1$ and $SSx_n = S(x_n) = x_n \to 1, SAx_n = S(2 - x_n) = 1 \to 1$.
Also, we have $A(x) = \frac{4}{3} = S(t)$ but $AS(x) = A(1) = A(\frac{4}{3}) = \frac{2}{3}, SA(x) = A(1) = \frac{4}{3} = \frac{4}{3}$.
However, we have $\frac{2}{3} = AS(x) \neq SA(x) = \frac{4}{3}$, at $x = 1$. Therefore, (A, S) is not compatible of type (K) but it is compatible, compatible of type (A) and compatible of type (P).

Lemma 1.1 [10] Let $(X, M, *)$ be a fuzzy metric space. Then for all x, y in X, $M(x, y, .)$ is non-decreasing.

Lemma 1.2 [15] Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t)$ for all x, y and $t > 0$ then $x = y$.

Lemma 1.3 [3] The only t-norm $*$ satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is, $a * b = \min \{a, b\}$ for all $a, b \in [0, 1]$.

2. Main Result

Theorem 2.1 Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be a self mappings of X satisfying the following conditions:
(i)$A(X) \subset T(X), B(X) \subset S(X)$,
(ii) $M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Bx, Ty, t) * M(Ax, Ty, t)$, for all $x, y \in X$, $k \in (0, 1)$ and $t > 0$, and
(iii) S and T are continuous.
If (A, S) and (B, T) compatible of type of (K), then A, B, S and T have a unique common fixed point.
Proof: Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, so for any $x_0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this x_1, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we define a sequence $\{y_n\}$ in X such that
$y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}$ and $y_{2n} = Bx_{2n-1} = Sx_{2n}$, for all $n = 1, 2, ...$
From (ii), we get
$M(y_{2n+1}, y_{2n+2}, kt) = M(Ax_{2n}, Bx_{2n+1}, kt)$
$\geq M(Sx_{2n}, Tx_{2n+1}, t)*M(Ax_{2n}, Sx_{2n}, t)*M(Bx_{2n}, Tx_{2n+1}, t)*M(Ax_{2n}, Tx_{2n+1}, t)$
$= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t)$
$\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$
From lemma 1.1 and 1.3 we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \tag{1}$$

Similarly, we have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) \tag{2}$$

From (1) and (2), we have

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t) \tag{3}$$

From (3), we have $M(y_{n+1}, y_{n+2}, t) \geq M(y_n, y_{n+1}, \frac{t}{k}) \geq M(y_{n-1}, y_n, \frac{t}{k^2}) \geq ... \geq M(y_1, y_2, \frac{t}{k^n}) \to 1 as n \to \infty$.
So, $M(y_n, y_{n+1}, t) \to 1 as n \to \infty$ for any $t > 0$. For each $\epsilon > 0$ and each $t > 0$, we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1 - \epsilon for all n > n_0$. For $m, n \in N$, we suppose $m \geq n$. Then, we have that
$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, \frac{t}{m-n}) * M(yn + 1, y_{n+2}, \frac{t}{m-n}) * ... * M(y_{m-1}, y_m, \frac{t}{m-n})$
$\geq (1 - \epsilon) * (1 - \epsilon) * ...(m - n)$ times. This implies
$M(y_n, y_m, t) \geq (1 - \epsilon)$ and hence $\{y_n\}$ is a Cauchy sequence in X.
Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $z \in X$, and so that $\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z. Since (A, S) and (B, T) are compatible of type (K), we have

$$AAx_{2n-2} \to Sz, SSx_{2n} \to Az, BBx_{2n-1} \to Tz, TTx_{2n-1} \to Bz \tag{4}$$

From (ii), we get
$M(AAx_{2n-2}, BBx_{2n-1}, kt) \geq M(SAx_{2n-2}, TBx_{2n-1}, t)*M(AAx_{2n-2}, SAx_{2n-2}, t)* M(BBx2n-1, TBx_{2n-1}, t) * M(AAx_{2n-2}, TBx_{2n-1,t})$ Taking limit as $n \to \infty$ and using (4), we have
$M(Sz, Tz, kt) \geq M(Sz, Tz, t) * M(Sz, Sz, t) * M(Tz, Tz, t) * M(Sz, Tz, t)$

$(Sz, Tz, t) * 1 * 1 * M(Sz, Tz, t) \geq M(Sz, Tz, t)$.
It follows that

$$Sz = Tz \tag{5}$$

Now, from (ii), we get $M(Az, BBx_{2n-1}, kt) \geq M(Sz, TBx_{2n-1}, t) * M(Az, Sz, t) * M(Bz, TBx_{2n-1}, t) * M(Az, TBx_{2n-1}, t)$
Again, taking limit as $n \to \infty$ and using (4) and (5), we have
$M(Az, Tz, kt) \geq M(Sz, Sz, t) * M(Az, Tz, t) * M(Tz, Tz, t) * M(Az, Tz, t)(Az, Tz, t)$.
and hence

$$Az = Tz \tag{6}$$

From (ii), (5) and (6), we get $M(Az, Bz, kt) \geq M(Sz, Tz, t) * M(Az, Sz, t) * M(Bz, Tz, t) * M(Az, Tz, t) = M(Az, Az, t) * M(Az, Az, t) * M(Bz, Az, t) * M(Az, Az, t) \geq M(Az, Bz, t)$.
and hence

$$Az = Bz \tag{7}$$

From (5), (6) and (7), we have

$$Az = Bz = Tz = Sz \tag{8}$$

Now, we show that $Bz = z$. From (ii), we get
$M(Ax_{2n}, Bz, kt) \geq M(Sx_{2n}, Tz, t) * M(Ax_{2n}, Sx_{2n}, t) * M(Bz, Tz, t) * M(Ax_{2n}, Tz, t)$.
And, taking limit as $n \to \infty$ and using (5) and (6), we have
$M(z, Bz, kt) \geq M(z, Tz, t) * M(z, z, t) * M(Bz, Tz, t) * M(z, Tz, t)$
$= M(z, Bz, t) * 1 * M(Az, Az, t) * M(z, Bz, t) \geq M(z, Bz, t)$.
And hence $Bz = z$. Thus from (8), we get $z = Az = Bz = Tz = Sz$ and so z is a common fixed point of A, B, S and T.
In order to prove the uniqueness of fixed point, let w be another common fixed point of A, B, S and T. Then, $Aw = Bw = Sw = Tw$, therefore, using (ii), we get $M(z, w, kt) = M(Az, Bw, kt) \geq M(Sz, Tw, t) * M(Az, Sz, t) * M(Bw, Tw, t) * M(Az, Tw, t) \geq M(z, w, t)$. From Lemma 2, we get $z = w$. This completes the proof of theorem.
We have the following example.

Example 2.1 Let $X = [2, 10]$ with the metric d defined by $d(x, y) = |x - y|$ and define $M(x, y, t) = \frac{t}{d(x,y)}$ for all $x, y \in X, t > 0$. Clearly $(X, M, *)$ is a complete fuzzy metric space. Define A, B, S and $T : X \to X$ as follows: $Ax = 2$ if $x \leq 3, Ax = 3$ if $x > 3$; $Bx = 2$ if $x \leq 5, Bx = 3$ if $x > 5$ and $Sx, Tx = x$ for all $x \in X$,
Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$.
If $A = B$ and $T = S$, then we get following result.

Corollary 2.2. Let $(X, M, *)$ be a complete fuzzy metric space and A and S be a self mappings of X satisfying the following conditions:
(i) $A(X) \subset S(X)$,
(ii) $M(Ax, Ay, kt) \geq M(Sx, Sy, t) * M(Ax, Sx, t) * M(Ax, Sy, t) * M(Ax, Sy, t)$,
for all $x, y \in X, k \in (0, 1)$ and $t > 0$, and
(iii) S is continuous. If (A, S) compatible of type of (K), then A and S have a unique common fixed point.

Remarks. Our result extends and generalizes the results of Cho[15], Koireng and Rohon[11] and Jha et al. [8]. Also, our result improves other similar results in literature.

Acknowledgement: The first author is thankful to the University Grant Commission (UGC), Nepal for providing financial support as a Ph.D. scholar at Kathmandu University, Nepal.

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