

COMPOSITE FINITE DIFFERENCE SCHEME APPLIED TO A COUPLE OF NONLINEAR EVOLUTION EQUATIONS

M. S. EL-AZAB, I. L. EL-KALLA AND S. A. EL MORSY

ABSTRACT. In this paper, a new finite difference scheme called Composite Finite Difference Scheme (CFDS) is improved to solve linear and nonlinear high order partial differential equations. The CFDS is applied to a class of nonlinear evolution equations especially Korteweg de Vries Burger equation (KdVB). Comparisons between the classical explicit finite difference method (FDM) and the new suggested scheme are presented. Using Von Neumann stability analysis, we study the stability of each method.

1. INTRODUCTION

The search for a better and easy tool for the solution of nonlinear evolution equations illuminating the nonlinear phenomena of our life keeps continuing. A variety of methods therefore were proposed to find solutions of these kinds of equations [1]-[13]. Several numerical methods have been intensively investigated for the numerical solution of partial differential equations. Spectral methods are often an efficient and highly accurate schemes when compared with local methods. There are three primary types of spectral methods based on the choice of test functions, namely, the Galerkin, tau and collocation methods. A. Bhrawy use the Jacobi-Gauss-Lobatto collocation method for solving generalized Fitzhugh-Nagumo equation [13]. KdVB, have especial importance as it describe various physical phenomenas. The KdVB equation represent a couple of two important equations the Kortewegde Vries equation (KdV) and Burger equation. The KdV describe the behavior of long waves in shallow water waves and waves of the plasma. It was discovered by Kortewegde Vries in 1895. As it is an important equation many papers try to present its analytic or numerical solutions. Adomian decomposition method (ADM) used in solving it in [14], Variational iteration method (VIM) [15], Homotopy perturbation method (HPM) [16] and many other analytical solution methods such as inverse scattering transform (IST) [17] and traveling wave solution [17]-[19]. In [20] A. Biswas solve the generalized KdV equation with time-dependent damping and dispersion. The Burgers equations have been found to describe various kind of phenomena such as a mathematical model of turbulence [21] and the approximate theory of flow

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through a shock wave traveling in viscous fluid [22, 23]. Fletcher using the Hopf-Cole transformation [24] gave an analytic solution of the system of two dimensional Burger's equations, several numerical methods of this equation system have been given such as algorithms based on cubic spline function technique [25], Wubs apply an explicitimplicit method [26], implicit finite-difference scheme [27]. As far as we know that little numerical works has been done to solve the KdVB equation. Recently a numerical method proposed for solving the KdVB equation by Zaki [28], he uses the collocation method with quintic B-spline finite element. Soliman [29] use the collocation solution of the KdV equation using septic splines as element shape function.

In this paper, our aim is to solve a class of nonlinear evolution equations using CFDS. In section 2, we use the CFDS and the explicit finite-difference technique to solve KdVB equation. A network of grid points is first established throughout the region occupied by the independent variables. In section 3, the stability of the CFDS was discussed using Von Neumann approach and we found that the CFDS is unconditionally stable for solving KdVB equation. In the last section, some illustrative numerical examples are given and numerical comparisons between both methods are introduced.

2. COMPOSITE FINITE DIFFERENCE SCHEME (CFDS)

Consider the KdVB equation has the form,

$$u_t + \epsilon uu_x + \nu u_{xx} + \mu u_{xxx} = 0, \quad (x, t) \in Q_T \quad (1)$$

Here $Q_T = \Omega \times I$, $\Omega \equiv (a, b)$, $I \equiv (0, T)$, a and b are real positive constants, ϵ , μ and ν are parameters. We consider equation (1) associated with initial condition $u(x, 0) = u_0(x)$. In Finite difference method (FDM) the domain is discretized to a finite number of points forming a mesh with horizontal step size $h = \frac{b-a}{N}$, N is the number of intervals, $0 \leq i \leq N$ and k is the time step such that $T = kj$, $0 \leq j \leq M$. The derivatives are replaced by difference formulas [30, 31] as follows, for $i = 1, 2$ we use the forward formulas

$$\begin{aligned} (u_x)_i^j &= \frac{-3u_i^j + 4u_{i+1}^j - u_{i+2}^j}{2h}, \\ (u_{xx})_i^j &= \frac{2u_i^j - 5u_{i+1}^j + 4u_{i+2}^j - u_{i+3}^j}{h^2}, \\ (u_{xxx})_i^j &= \frac{-5u_i^j + 18u_{i+1}^j - 24u_{i+2}^j + 14u_{i+3}^j - 3u_{i+4}^j}{2h^3}. \end{aligned} \quad (2)$$

while for $i = 3 : N - 2$ we use central formulas

$$\begin{aligned} (u_x)_i^j &= \frac{u_{i+1}^j - u_{i-1}^j}{2h}, \\ (u_{xx})_i^j &= \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{2h^2}, \\ (u_{xxx})_i^j &= \frac{u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j}{2h^3} \end{aligned} \quad (3)$$

and for $i = N - 1, N$ we use the backward formulas

$$\begin{aligned}(u_x)_i^j &= \frac{3u_i^j - 4u_{i-1}^j + u_{i-2}^j}{2h}, \\ (u_{xx})_i^j &= \frac{2u_i^j - 5u_{i-1}^j + 4u_{i-2}^j - u_{i-3}^j}{h^2}, \\ (u_{xxx})_i^j &= \frac{5u_i^j - 18u_{i-1}^j + 24u_{i-2}^j - 14u_{i-3}^j + 3u_{i-4}^j}{2h^3}.\end{aligned}\quad (4)$$

We assume that $F(u)$ denote any continuous and differentiable function, multiply equation (1) by the derivative of F , we have

$$\frac{\partial F}{\partial u} \frac{\partial u}{\partial t} = -F'(u) (\epsilon u u_x + v u_{xx} + \mu u_{xxx}) \quad (5)$$

and

$$\frac{\partial F}{\partial u} = -F'(u) (\epsilon u u_x + v u_{xx} + \mu u_{xxx}) \quad (6)$$

The usual forward difference formula leads to, $\frac{\partial F}{\partial u} = \left(\frac{F(u_i^{(j+1)}) - F(u_i^j)}{k} \right)$, k is the time step. Substitute in (6), we have

$$F(u_i^{j+1}) = F(u_i^j) - k F'(u_i^j) (\epsilon u_i^j (u_x)_i^j + v (u_{xx})_i^j + \mu (u_{xxx})_i^j) \quad (7)$$

To obtain the logarithmic finite difference method (Log FDM) assume $F(u) = e^u$, and substitute in (7) we have,

$$F(u_i^{j+1}) = F(u_i^j) \left(1 - k (\epsilon (u_i^j) (u_x)_i^j + v (u_{xx})_i^j + \mu (u_{xxx})_i^j) \right) \quad (8)$$

taking the inverse function (ln) to both sides of (8), we obtain

$$u_i^{j+1} = u_i^j + \ln \left(1 - k (\epsilon (u_i^j) (u_x)_i^j + v (u_{xx})_i^j + \mu (u_{xxx})_i^j) \right) \quad (9)$$

In (7) if we choose $F(u) = \ln u$, we obtain the Exponential finite difference method (Exp FDM) that is developed by Bhattachary [32, 33]. He used Exp FDM to solve the one dimensional heat conduction in a solid slab. In [34] R. F. Handschuh and T.G. Keith, apply the Exp FDM technique to some classes of partial differential equations. A. R. Bahadir [35] apply Exp FDM to KdV equation.

3. Von Neumann stability

In this section, we will study the stability of explicit finite difference method and Composite finite difference methods using Von Neumann stability analysis.

3.1. Stability of explicit finite difference method. To study the stability of the KdVB equation by Von Neumann analysis [36]. Define,

$$Z_{i,j} = e^{\alpha t} e^{I\beta x} = e^{\alpha j k} e^{I\beta i h} = \xi^j e^{I\beta i h} \quad (10)$$

where $\xi = e^{\alpha k}$. Rewrite the nonlinear term in (1) $\epsilon u u_x$ in the form $\frac{\epsilon}{2} u_x^2$, we obtain

$$u_t + \frac{\epsilon}{2} u_x^2 + v u_{xx} + \mu u_{xxx} = 0 \quad (11)$$

From (10) substitute in (11), we get

$$Z_i^j = Z_i^{j-1} - \frac{\epsilon k}{4h} \left((Z_{i+1}^j)^2 - (Z_{i-1}^j)^2 \right) - \frac{vk}{h^2} (Z_{i+1}^j - 2Z_i^j + Z_{i-1}^j) - \frac{\mu k}{2h^3} (Z_{i+2}^j - 2Z_{i+1}^j + 2Z_{i-1}^j - Z_{i-2}^j) \quad (12)$$

For simplicity we will use the linearized form of KdVB, thus (12) transformed to

$$Z_i^j = Z_i^{j-1} - \frac{\epsilon k}{4h} (Z_{i+1}^j - Z_{i-1}^j) - \frac{vk}{h^2} (Z_{i+1}^j - 2Z_i^j + Z_{i-1}^j) - \frac{\mu k}{2h^3} (Z_{i+2}^j - 2Z_{i+1}^j + 2Z_{i-1}^j - Z_{i-2}^j) \quad (13)$$

Substitute (10) in (13), we obtain

$$\begin{aligned} & \xi^j e^{I\beta ih} - \xi^{j-1} e^{I\beta ih} + r \left(\xi^j e^{I\beta(i+1)h} - \xi^j e^{I\beta(i-1)h} \right) \\ & + s \left(\xi^j e^{I\beta(i+1)h} - \xi^j e^{I\beta(i-1)h} - 2\xi^j e^{I\beta ih} \right) \\ & + p \left(\xi^j e^{I\beta(i+2)h} - 2\xi^j e^{I\beta(i+1)h} + 2\xi^j e^{I\beta(i-1)h} - \xi^j e^{I\beta(i-2)h} \right) = 0, \end{aligned} \quad (14)$$

where, $r = \frac{\epsilon k}{4h}$, $s = \frac{vk}{h^2}$ and $p = \frac{\mu k}{2h^3}$. Cancellation of $\xi^j e^{I\beta ih}$ in (14) leads to

$$1 - \frac{1}{\xi} + 2Ir \sin \beta h - 2s(1 - \cos \beta h) + 2Ip(\sin 2\beta h - 2 \sin \beta h) = 0, \quad (15)$$

$$1 - \frac{1}{\xi} + 2Ir \sin \beta h - 4s \sin^2 \frac{\beta h}{2} + 2Ip(2 \sin \beta h \cos \beta h - 2 \sin \beta h) = 0, \quad (16)$$

$$1 - \frac{1}{\xi} + 2Ir \sin \beta h - 4s \sin^2 \frac{\beta h}{2} - 4Ip \sin \beta h (1 - \cos \beta h) = 0, \quad (17)$$

$$1 - \frac{1}{\xi} + 2Ir \sin \beta h - 4s \sin^2 \frac{\beta h}{2} - 8Ip \sin \beta h \sin^2 \frac{\beta h}{2} = 0, \quad (18)$$

$$1 - 4s \sin^2 \frac{\beta h}{2} + 2I \sin \beta h \left(r - 4p \sin^2 \frac{\beta h}{2} \right) = \frac{1}{\xi}. \quad (19)$$

Take $v \leq 0$, then $s \leq 0$ and,

$$\xi = \frac{1}{1 - 4s \sin^2 \frac{\beta h}{2} + 2I \sin \beta h \left(r - 4p \sin^2 \frac{\beta h}{2} \right)} \quad (20)$$

It is clear that $|\xi| \leq 1 \forall s, r$ and p .

3.2. Stability of Composite finite difference scheme. In this subsection we investigate the stability of the CFDS for the KdVB equation in its linearized form.

3.2.1. Stability of Exponential finite difference method. Consider the linearized version of KdVB equation that takes the form,

$$u_t + \frac{\epsilon}{2} u_x + v u_{xx} + \mu u_{xxx} = 0. \quad (21)$$

Multiply (21) by $\frac{1}{u}$, we get

$$\frac{1}{u} \frac{\partial u}{\partial t} = -\frac{1}{u} \left(\frac{\epsilon}{2} u_x + v u_{xx} + \mu u_{xxx} \right) \quad (22)$$

or,

$$\frac{\partial \ln u}{\partial t} = -\frac{1}{u} \left(\frac{\epsilon}{2} u_x + v u_{xx} + \mu u_{xxx} \right), \quad (23)$$

$$\frac{\partial \ln u}{\partial t} = \frac{\partial}{\partial t} \ln (e^{\alpha t} e^{I\beta x}) = \frac{\partial}{\partial t} (\alpha t + I\beta x) = \alpha. \quad (24)$$

Substitute from (24) into (22), we obtain

$$\alpha = -\frac{1}{u} \left(\frac{\epsilon}{2} u_x + v u_{xx} + \mu u_{xxx} \right). \quad (25)$$

Using (10) and substituting in (25) we have,

$$\alpha = \frac{-1}{k \xi^j e^{I\beta i h}} \left[r \left(\xi^j e^{I\beta(i+1)h} - \xi^j e^{I\beta(i-1)h} \right) + s \left(\xi^j e^{I\beta(i+1)h} + \xi^j e^{I\beta(i-1)h} - 2\xi^j e^{I\beta i h} \right) \right. \\ \left. + p \left(\xi^j e^{I\beta(i+2)h} - 2\xi^j e^{I\beta(i+1)h} - 2\xi^j e^{I\beta(i-1)h} - \xi^j e^{I\beta(i-2)h} \right) \right] \quad (26)$$

$$\alpha k = -(2rI \sin \beta h + s(2 \cos \beta h - 2) + 2pI(\sin 2\beta h - 2 \sin \beta h)) \quad (27)$$

$$\text{Real}(\alpha k) = -s(2 \cos \beta h - 2) = \frac{4vk}{h^2} \left(\sin^2 \frac{\beta h}{2} \right) \quad (28)$$

as $v \leq 0$, this leads to $\text{Real}(\alpha k) \leq 0$ and from the relation, $\xi = e^{\alpha k}$ this implies $|\xi| \leq 1$ and the method is unconditionally stable.

3.2.2. Stability of Logarithmic finite difference method. For logarithmic finite difference method multiply (21) by e^u ,

$$e^u \frac{\partial u}{\partial t} = -e^u \left(\frac{\epsilon}{2} u_x + v u_{xx} + \mu u_{xxx} \right). \quad (29)$$

$$e^u \frac{\partial u}{\partial t} = \frac{\partial e^u}{\partial t} = \frac{\partial}{\partial t} e^{\alpha t} e^{I\beta x} = \alpha (e^{\alpha t} e^{I\beta x}) e^{\alpha t} e^{I\beta x} = \alpha u e^u. \quad (30)$$

Substitute (30) in (29), we get

$$\alpha u e^u = -e^u \left(\frac{\epsilon}{2} u_x + v u_{xx} + \mu u_{xxx} \right) \quad (31)$$

$$\alpha = -\frac{1}{u} \left(\frac{\epsilon}{2} u_x + v u_{xx} + \mu u_{xxx} \right) \quad (32)$$

Equation (32) is exactly the same as equation (25), similarly; we will have the same stability condition (28).

4. NUMERICAL EXPERIMENTS

Case 1:

For purpose of illustration of the CFDS for solving the KdVB equation (1), $-10 \leq x \leq 10$ in case of $\epsilon = 1, v = -2, \mu = 1$ and $k = 0.0001$, start with an initial approximation,

$$u(x, 0) = \frac{6v^2}{25\epsilon\mu} \left[1 - \tanh \left(\frac{v}{10\mu} x \right) + \frac{1}{2} \text{sech}^2 \left(\frac{v}{10\mu} x \right) \right]. \quad (33)$$

Case 2:

Consider the Burger equation (1) with $\epsilon = 1, v = -2, \mu = 0, h = 1$ and $k = 0.0001$, and have initial approximation $u(x, 0) = 2x$. The exact solution, $u(x, t) = \frac{2x}{1+2t}$.

Tables (1-4) illustrate the numerical results of solving KdVB and Burger equations using FDM and CFDS at $t = 0.01, t = 0.1, t = 1.0$ and $t = 5.0$ in comparison with the analytical solution.

Figures (1- 4) illustrate the numerical results of solving KdVB and Burger equations using FDM and CFDS at $t = 0.01, t = 0.1, t = 1.0$ and $t = 5.0$ in comparison with the analytical solution.

TABLE 1. Solution of KdVB equation at $t = 0.01$

$\epsilon = 1, v = -2, \mu = 1$ with $h = 1$ and $k = 0.0001$ at $t = 0.01$			
x	FDM	CFDS	Exact
-10	1.9193	1.9193	1.9193
-8	1.9170	1.9170	1.9170
-6	1.9068	1.9068	1.9062
-4	1.8661	1.8661	1.8640
-2	1.7364	1.7364	1.7305
0	1.4418	1.4418	1.4307
2	1.0081	1.0081	0.9951
4	0.5925	0.5925	0.5823
6	0.3071	0.3071	0.3010
8	0.1479	0.1480	0.1448
10	0.0686	0.0686	0.0672

TABLE 2. Solution of KdVB equation at $t = 0.1$

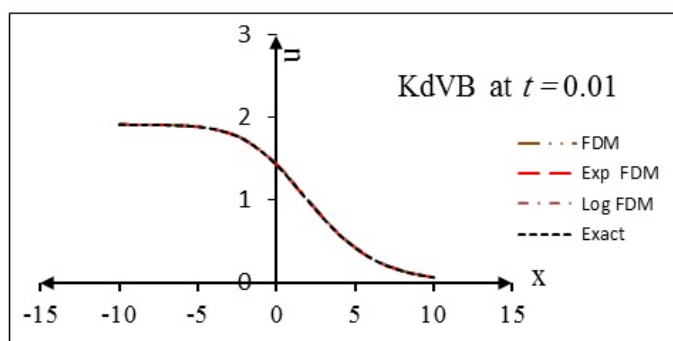
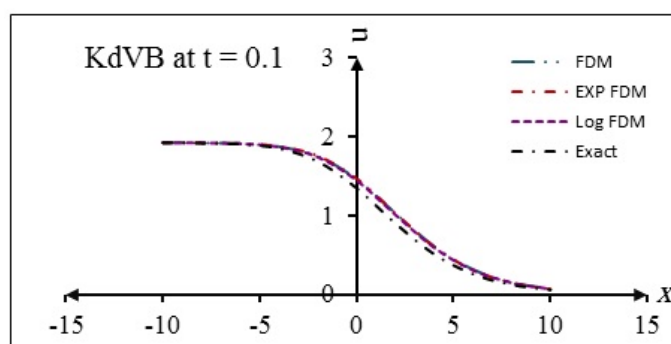
$\epsilon = 1, v = -2, \mu = 1$ with $h = 1$ and $k = 0.0001$ at $t = 0.1$			
x	FDM	CFDS	Exact
-10	1.91946	1.91937	1.91909
-8	1.91731	1.9171	1.91575
-6	1.90774	1.90692	1.90117
-4	1.86941	1.86651	1.8458
-2	1.74522	1.73718	1.68141
0	1.45822	1.4426	1.34373
2	1.02747	1.00812	0.9951
4	0.60792	0.59187	0.50929
6	0.31648	0.306415	0.25816
8	0.15289	0.14753	0.12294
10	0.07086	0.06846	0.05676

TABLE 3. Solution of Burger equation at $t = 1.0$

$\epsilon = 1, v = -2, \mu = 0$ with $h = 1$ and $k = 0.0001$ at $t = 1$			
x	FDM	CFDS	Exact
-10	-16.6662	-16.6664	-16.6667
-8	-13.3329	-13.3331	-13.3333
-6	-9.9997	-9.99985	-10.000
-4	-6.66646	-6.66657	-6.66667
-2	-3.33323	-3.33328	-3.33333
0	0	0	0
2	3.33323	3.33328	3.33333
4	6.6664	6.66657	6.66667
6	9.9997	9.99985	10.000
8	13.332	13.3331	13.3333
10	16.6662	16.6664	16.6667

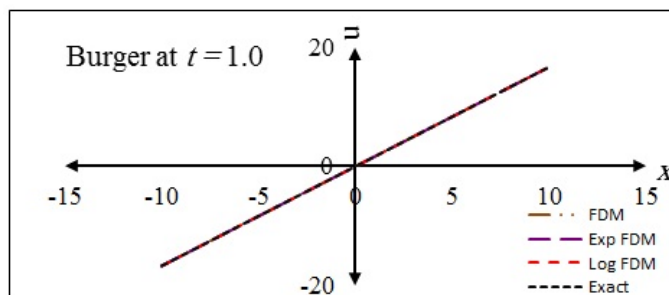
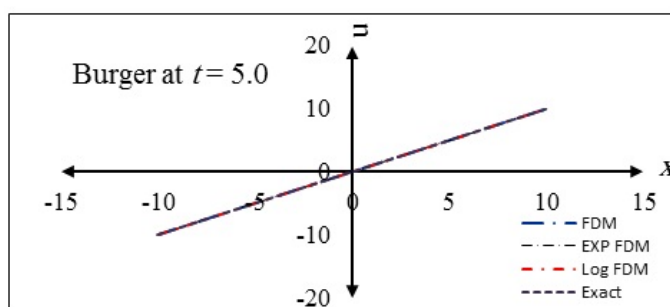
TABLE 4. Solution of Burger equation at $t = 5.0$

$\epsilon = 1, v = -2, \mu = 1$ with $h = 1$ and $k = 0.0001$ at $t = 5$			
x	FDM	CFDS	Exact
-10	-9.99931	-9.99965	-10
-8	-7.99945	-7.99972	-8
-6	-5.99958	-5.99979	-6
-4	-3.99972	-3.99986	-4
-2	-1.99986	-1.99993	-2
0	0	0	0
2	1.99986	1.99993	2
4	3.99972	3.99986	4
6	5.99958	5.99979	6
8	7.99945	7.99972	8
10	9.99931	9.99965	10

FIGURE 1. KdVB at $t = 0.01$ FIGURE 2. KdVB at $t = 0.1$

5. CONCLUSION

The CFDS is effective for solving linear and nonlinear partial differential equations especially for small time intervals. A comparison between CFDS, the classical

FIGURE 3. Burger at $t = 1$ FIGURE 4. Burger at $t = 5$

FDM and the exact solution is performed. The numerical results show that the solution using CFDS give high accuracy and no more conditions or restrictions are needed. Von Neumann stability showed that CFDS is unconditionally stable method.

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M. S. EL-AZAB

MATHEMATICAL AND PHYSICAL SCIENCES DEPARTMENT, FACULTY OF ENGINEERING, MANSOURA UNIVERSITY, MANSOURA , EGYPT

E-mail address: ms_elazab@hotmail.com

I. L. EL-KALLA

MATHEMATICAL AND PHYSICAL SCIENCES DEPARTMENT, FACULTY OF ENGINEERING, MANSOURA UNIVERSITY, MANSOURA , EGYPT

E-mail address: al_kalla@mans.edu.eg

S. A. EL MORSY

MATHEMATICAL AND PHYSICAL SCIENCES DEPARTMENT, FACULTY OF ENGINEERING, MANSOURA UNIVERSITY, MANSOURA , EGYPT

E-mail address: salwazaghrot@yahoo.com