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A COMMON FIXED POINT THEOREM IN SEMI-METRIC SPACE USING E.A PROPERTY

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Abstract. In this paper, we establish a common fixed point theorem for three pairs of self mappings in semi-metric spaces using implicit function and E.A property which improves and extends similar known results in the literature.

1. Introduction

Fixed point theory in semi-metric space is one of the emerging area of interdisciplinary mathematical research. Polish mathematician Stephan Banach published his contraction principle in 1922. Since then, this principle has been extended and generalized in several ways in semi-metric space. In 1928, K. Menger [10] introduced semi-metric space as generalization of metric space.In 1986, G. Jungck [7] introduced the notion of compatible mappings. Again, in 1998, G.Jungck and B.E.Rhoades [8] introduced the notion of weakly compatible mappings and showed that compatible mappings are weakly compatible but not conversely. Also in 2009, M. Imdad and J.Ali [6] introduced new class of implicit function and proved some common fixed point theorems. The significance of this type of implicit function is to relax the notion of triangle inequality. Most recently, M. Aamri and D. El. Moutawakil [1] introduced the notion of property (E.A) which is a generalization of compatible as well as non-compatible mappings. In this paper, we prove a common fixed point theorem for three pairs of self- mappings using implicit function in semi-metric space.

Definition 1 Let A and B be two self -mappings of a semi-metric space $d(X, d)$. Then A and B are said to be compatible if $lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} d(Ax_n, t) = lim_{n\to\infty} d(Ax_n, t) = 0$ for some $t \in X$

Definition 2 Let A and B be two self -mappings of a semi-metric space $d(X, d)$. Then A and B are said to satisfy the property E.A if there exists a sequence $\{x_n\}$ such that $lim_{n\to\infty} d(Ax_n, t) = lim_{n\to\infty} d(Bx_n, t) = 0$ for some $t \in X$.

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Definition 3 Let A and B be two self -mappings of a semi-metric space $d(X,d)$. A point x in X is said to be coincidence point of A and B iff $Ax = Bx$.

Definition 4 Let A and B be two self -mappings of a semi-metric space $d(X,d)$. Then A and B are said to be weakly compatible if they commute at their coincidence points.

2. Implicit Function

In this section we discuss about the notion of new class of implicit function given by M. Imdad and J. Ali [6] which is different from the one considered in V. Popa [12] with examples. Let ? be the family of lower semi-continuous functions $F : R^6 \to R$ satisfying the following conditions.

(F_1) : $F(t,0,0,t,t,0) > 0$, for all $t > 0$.

(F_2) : $F(t,0,t,0,0,t) > 0$, for all $t > 0$.

(F_3) : $F(t,t,0,0,t,t) > 0$, for all $t > 0$.

Example 1 $F(t_1,...,t_6) = t_1$-$max\ \{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$

(F_1) : $F(t,0,0,t,t,0) = \frac{t}{2} > 0$, for all $t > 0$.

(F_2) : $F(t,0,t,0,0,t) = \frac{t}{2} > 0$, for all $t > 0$.

(F_3) : $F(t,t,0,0,t,t) = 0$, for all $t > 0$.

3. Main Results.

Theorem 1 Let A,B,T,S,P and Q be self mappings of semi-metric space (X,d) which satisfy the inequality

(i)$F((d(ABx), d(Px,Qy), d(ABx,Px), d(TSy,Qy), d(Px,TSy), d(Qy,ABx)) < 0$.

Suppose that

(ii) $AB(X) \subset Q(X)$ *or* $TS(X) \subset P(X)$

(iii) $P(X)$ and $Q(X)$ are closed subset of X

(iv) the pairs (AB,P) and (TS,Q) are weakly compatible mappings

Then AB,TS,P and Q have a unique fixed point. Furthermore, if the pairs (A,B) and (T,S) are commuting pair of mappings then A,B,T,S,P and Q have a unique common fixed point.

Proof If (AB,P) satisfy the property E.A then there exists a sequence $\{x_n\}$ in X such that

$lim_{n\to\infty}\ ABx_n = lim_{n\to\infty}\ Px_n = t$ for some $t \in X$.

Since $AB(X) \subset Q(X)$, hence for each $\{x_n\}$ there exists $\{y_n\}$ in X such that

$ABx_n = Qy_n$. Letting $n \to \infty$, we have

$lim_{n\to\infty} Qy_n = lim_{n\to\infty} ABx_n = t$.

Therefore, we have $lim_{n\to\infty} ABx_n = lim_{n\to\infty} Qy_n = lim_{n\to\infty} = t$.

We assert that $lim_{n\to\infty}TSy_n = t$.
If not, we put $x = x_n$ and $y = y_n$ in inequality (i), we get

$F(d(ABx_n, TSy_n), d(Px_n, Qy_n), d(ABx_n, Px_n), d(TSy_n, Qy_n), d(Px_n, TSy_n), d(Qy_n, ABx_n)) < 0$. Letting $n \to \infty$, we have

$F(d(t, TSy_n), 0, 0, d(TSy_n, t), d(t, TSy_n), 0) < 0$.

This is a contradiction. Therefore,we get $lim_{n\to\infty}TSy_n = t$.

Hence, we have
$lim_{n\to\infty}ABx_n = lim_{n\to\infty}Qy_n = lim_{n\to\infty}Px_n = lim_{n\to\infty}TSy_n = t$.

Since $P(X)$ is closed subset of X. Hence $lim_{n\to\infty}Px_n = t \in P(X)$. Therefore, there exists $u \in X$ such that $Pu = t$.

We assume that $Pu = ABu$. If not, we put $x = u$ and $y = y_n$ in inequality (i), we get

$F(d(ABu, TSy_n), d(Pu, Qy_n), d(ABu, Pu), d(TSy_n, Qy_n), d(Pu, TSy_n), d(Qy_n, ABu)) < 0$. Letting $n \to \infty$, we have

$F(d(ABu, t), d(Pu, t), d(ABu, Pu), 0, 0, d(t, ABu)) < 0$. This is a contradiction. Therefore, we get $ABu = Pu$. Hence, u is a coincidence point of (AB, P).

Also, we have $Q(X)$ is closed subset of X.
Therefore, we get $lim_{n\to\infty}Qy_n = t \in Q(X)$.

Hence, we have $Qw = t$ for some $w \in X$. Suppose $Qw = TSw$. If not we put $x = x_n$ and $y = w$ in inequality (i) we get

$F(d(ABx_n, TSw), d(Px_n, Qw), d(ABx_n, Px_n), d(TSw, Qw), d(Px_n, TSw), d(Qw, ABx_n)) < 0$. Letting $n \to \infty$, we have

$F(d(t, TSw), d(t, Qw), d(t, t), d(TSw, Qw), d(t, TSw), d(Qw, t)) < 0$.

$F(d(Qw, TSw), 0, 0, d(TSw, Qw), d(Qw, TSw), d(Qw, t)) < 0$.
This is a contradiction.Therefore, we get $Qw = TSw$. Hence w is a coincidence point of (TS, Q).
Since, (AB, P) and (TS, Q) are weakly compatible mappings, we write.

$ABt = ABPu = PABu = Pt$ and $TSt = TSQw = QTSw = Qt$.

Since, $ABt = t$. If not we put $x = t$ and $y = y_n$ in inequality (i) we get

$F(d(ABt, TSy_n), d(Pt, Qy_n), d(ABt, Pt), d(TSy_n, Qy_n), d(Pt, TSy_n), d(Qy_n, ABt)) < 0$.

Letting $n \to \infty$, we have

$F(d(ABt, t), d(ABt, t), 0, 0, d(AB, t), d(t, ABt)) < 0$.

This is a contradiction. Therefore, we get, $AB = t$. Therefore t is a common fixed point of AB and P.

Similarly, we can show that t is a common fixed point of TS and Q.

Hence t is a common fixed point of AB, TS,P and Q.
Therefore, we get $ABt = TSt = Pt = Qt = t$.

For Uniqueness, let z be another fixed point of AB,TS,P and Q. Then by definition we get $ABz = TSz = Pz = Qz = z$.
If we put $x = t$ and $y = z$ in inequality (i), we get

$F(d(ABt, TSz), d(Pt, Qz), d(ABt, Pt), d(TSz, Qz), d(Pt, TSz), d(Qz, ABt)) < 0$.

$F(d(t, z), d(t, z), d(t, t), d(z, z), d(t, z), d(z, t) < 0$

or $F(d(t, z), d(t, z), 0, 0, d(t, z), d(z, t) < 0$.

This is a contradiction. Therefore, we get $t = z$.
Hence, we have AB,TS,P and Q have unique common fixed point.

If the pairs (A, B) and (T, S) are commuting pair of mappings then

$A(t) = A(ABt) = A(BAt) = AB(At)$. This implies $At = t$.

$B(t) = B(ABt) = BA(Bt) = AB(Bt)$. This implies $Bt = t$

Similarly $T(t) = t$ and $S(t) = t$.

This shows that A, B, T, S, P and Q have a unique common fixed point.
This completes the proof.

The proof assuming that $TS(X)$ is subset of $P(X)$ is similar to the above proof.

Example 1 Consider $X = R$ with the semi-metric space (X, d) defined by $d(x, y) = (x-y)^2$. Define a self map, A, B, T, S, P and Q as $Ax = \frac{x+2}{3}$ $Bx = -3x+4$ $S(x) = -2x + 3$, $T(x) = \frac{x+1}{2}$,$P(x) = -x + 2$ and $Q(x) = 2x - 1$.

For the sequence $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n} + 1$, the mappings satisfy all the conditions of above Theorem 1 and hence they have a unique common fixed point $x = 1$.
In the above Theorem 1, if B and S are identity mapping, then we have the following corollary

Corollarly 1 Let A,T,P and Q be self-mappings of (X, d) which satisfy the inequality
(i)$F((d(Ax, Ty), d(Px, Qy), d(Ax, Px), d(Ty, Qy), d(Px, Ty), d(Qy, Ax)) < 0$.
Suppose that

(ii) $A(X) \subset Q(X)$ *or* $T(X) \subset P(X)$

(iii) $P(X)$ and $Q(X)$ are closed subset of X

(iv) the pairs (A, P) and (T, Q) are weakly compatible mappings
Then A,T,P and Q have a unique common fixed point.
This is the result of M.Imdad and J.Ali.

Remark Our result generalizes the result of M. Imdad and J. Ali [6], extends the results of J.K. Kohli and D. Kumar [9], H.K. Pathak, R. Tiwari and M.S. Khan [11], M.Imdad, Santosh Kumar and M.S. Khan [5]and other similar results in semi-metric space.

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References

[1] M. Aamri and D. El. Moutawakil, Common fixed points under contractive conditions in symmetric space, Applied Math E-Notes, 3(2003), 156-162.

[2] S. H. Cho, G .Y .Lee andJ. S. Bae, On concidence and fixed point theorems in symmetric space,Fixed Point Theory Appl, 2008 (2008), Art. ID 562130, 9 pp.

[3] M. Imdad, J. Ali, and L.Khan, Coincidence and fixed points in semi-metric spaces under strict conditions, J. Math. Anal. Appl. 320(2006), 352-360.

[4] M. Imdad and J. Ali, Jungck's common fixed point theorem and E.A property, Acta Math. Sinica, Engish Series, 24(2008),87-94

[5] M. Imdad, S. Kumar and M. S. Khan, Remarks on fixed point theorems satisfying implicit relations, Randovi Math. J.11(2002), 135-143.

[6] M. Imdad and J. Ali, Common fixed point theorems in semi-metric space employing a new implicit function and common property (E.A.), Bull. Belgium. Soc. Simon Stevin, 16(2009),421-433.

[7] G. Jungck, Compatible mappings and common fixed point, Intern. J. Maths and Scis, 9(1986),771-779.

[8] G.Jungck, and B.E. Rhoades, Fixed points for set valued functions with out continuity, Indian J. Pure Appl. Math., 29(3) (1998), 227-238.

[9] J.K. Kohli and D. Kumar, A common fixed point theorem for six mappings via weakly compatible mappings in semi-metric space satisfying integral type implicit relations, Intern. Maths Forum, 5(2010),1-14.

[10] K. Menger, Untersuchungen uber allgemeine, Math Annalen, 100(1928),75-163.

[11] H.K.Pathak, R. Tiwari and M.S. Khan, A common fixed point theorem satisfying integral type implicit relation, Appl. Maths. E-notes 7(2007), 222-228.

[12] V.Popa, A genera fixed point theorem for weakly compatible mappings in compact metric spaces, Turkish J. Math., 25(2001),465-474.

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