Electronic Journal of Mathematical Analysis and Applications,
Vol. 3(1) Jan. 2015, pp. 52-65.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

EFFECTIVENESS OF HADAMARD PRODUCT OF BASIC SETS OF POLYNOMIALS OF SEVERAL COMPLEX VARIABLES IN HYPERELLIPTICAL REGIONS

M. A. ABUL-DAHAB, A. M. SALEEM, Z. M. KISHKA

Abstract. In this paper, we study some extended results for Hadamard product set for many simple monic sets of polynomials of several complex variables in hyperelliptical regions, then we obtain the effectiveness conditions for Hadamard product in hyperelliptical regions. Moreover an upper bound for the order of Hadamard product set is given. Our new results extend and improve a lot of known results (see [2, 11]).

1. Introduction

The development of Hadamard product bases arises in a wide variety of ways, namely in convolution of periodic functions, characteristic functions in probability theory (e.g. Bochners theorem) but also in combinatorics in the study of association schemes. It should be also stressed the recent usage of Hadamard product base technique in the study of automorphic L-functions arising in number theory (see e.g. Lagarias, Suzuki/Journal of Number Theory 118 (2006), 98-116 and the references given there). The idea of studying effectiveness properties of the Hadamard product set of sets of polynomials in a single complex variable was introduced in [9, 10]. In [11] Nassif and Rizk introduced an extension of this product in the case of two complex variables using spherical regions. It should be mentioned here the recent study of Hadamard product set in Clifford Analysis (see [1]).

In the present paper, we aim to investigate the extent of a generalization of Hadamard product set in $\mathbb{C}^n$ using hyperelliptical regions.

2010 *Mathematics Subject Classification.* 30G35, 32A17, 32A99.
Key words and phrases. Bases of polynomial, Hadamard product, Hyperelliptical regions.
Submitted Jun 1, 2014.

To avoid lengthy scripts, the following notations are adopted throughout this work (see [3, 4, 5, 6, 7, 8, 12]).

$$\begin{aligned}
&\mathbf{m} = (m_1, m_2, ..., m_n); \quad < \mathbf{m} >= m_1 + m_2 + ... + m_n;\\
&\mathbf{h} = (h_1, h_2, ..., h_n); \qquad < \mathbf{h} >= h_1 + h_2 + ... + h_n;\\
&\mathbf{z} = (z_1, z_2, ..., z_k); \quad \mathbf{z^m} = z_1^{m_1}.z_2^{m_2}.....z_n^{m_n}; \quad \mathbf{0} = (0, 0, ..., 0);\\
&|\mathbf{z}|^2 = |z_1|^2 + |z_2|^2 + ... + |z_n|^2; \quad \mathbf{t^m} = t_1^{m_1}.t_2^{m_2}.....t_n^{m_n};\\
&\mathbf{r} = (r_1, r_2, ..., r_n); \quad \mathbf{r}^* = \mathbf{r} \text{ if } r_s = r \ \ \forall \ \ s \in I; \quad I = \{1, 2, 3, ..., n\}.
\end{aligned}$$

In these notations $m_1, m_2, ..., m_k$ and $h_1, h_2, ..., h_k$ are non-negative integers while $t_1, t_2, ..., t_n$ are non-negative numbers, $0 < t_s < 1$, $|\mathbf{t}| = (\sum_{s=1}^n \ t_s^2)^{(\frac{1}{2})} = 1$. Also, square brackets are used here in functional notation to express the fact that the function is either a function of several complex variables or one related to such function. In the space of several complex variables $\mathbb{C}^n$; an open hyperelliptical region of radii r_s, is here denoted by $\mathbf{E}_{[\mathbf{r}]}$ and its closure by $\overline{\mathbf{E}}_{[\mathbf{r}]}$, where r_s; $s \in I$ are positive numbers. In terms of the introduced notations, these regions satisfy the following inequalities:

$$\begin{aligned}
\mathbf{E}_{[\mathbf{r}]} &= \{\mathbf{w} : |\mathbf{w}| < 1\}\\
\overline{\mathbf{E}}_{[\mathbf{r}]} &= \{\mathbf{w} : |\mathbf{w}| \leq 1\},
\end{aligned} \tag{1.1}$$

where $\mathbf{w} = (w_1, w_2, ..., w_k),\ w_s = \frac{z_s}{r_s};\ \ s \in I$.

Suppose now that the function $f(\mathbf{z})$, given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}}\ \mathbf{z^m}, \tag{1.2}$$

is regular in $\overline{\mathbf{E}}_{[\mathbf{r}]}$ and

$$M[f;\ [\mathbf{r}]] = \sup_{\overline{\mathbf{E}}_{[\mathbf{r}]}} |f(\mathbf{z})|.$$

From (1.1) we easily see that $\{|z_s| \leq r_s\ t_s :\ |\mathbf{t}| = 1\} \subset \overline{\mathbf{E}}_{[\mathbf{r}]}$, where $\mathbf{t}$ is the vector $(t_1, t_2, ..., t_n)$. Hence it follows that

$$|a_{\mathbf{m}}| \leq \ \sigma_{\mathbf{m}}\ \frac{M[f; [\rho]]}{\Pi_{s=1}^k (\rho_s)^{m_s}}, \tag{1.3}$$

for all $0 < \rho_s < r_s;\ \ s \in I$, where

$$\sigma_{\mathbf{m}} = \inf_{|\mathbf{t}|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\{< \mathbf{m} >\}^{\frac{<\mathbf{m}>}{2}}}{\Pi_{s=1}^n\ m_s^{\frac{m_s}{2}}}, \tag{1.4}$$

and $1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{n})^{<\mathbf{m}>}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0;\ s \in I$.

Thus, it follows that

$$\limsup_{<\mathbf{m}>\to\infty} \left\{\frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}\ \Pi_{s=1}^k\ (\rho_s)^{<\mathbf{m}>-m_s}}\right\}^{\frac{1}{<\mathbf{m}>}} \leq \ \frac{1}{\Pi_{s=1}^k\ \rho_s}. \tag{1.5}$$

Since ρ_s can be chosen arbitrary near to $r_s; s \in I$, we conclude that

$$\limsup_{<\mathbf{m}>\to\infty} \left\{\frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}\ \Pi_{s=1}^n\ (r_s)^{<\mathbf{m}>-m_s}}\right\}^{\frac{1}{<\mathbf{m}>}} \leq \ \frac{1}{\Pi_{s=1}^n\ r_s}. \tag{1.6}$$

Then, it can be easily proved that the function $f(\mathbf{z})$ is regular in the open hyperelliptical $\mathbf{E}_{[\mathbf{r}]}$. The numbers r_s, given by (1.6), is thus conveniently called the radii of regularity of the function $f(\mathbf{z})$.

Definition 1.1. [3, 4, 5, 6, 7, 8, 12] A set of polynomials

$$\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0[\mathbf{z}], P_1[\mathbf{z}], P_2[\mathbf{z}], ..., P_n[\mathbf{z}], ...\},$$

is said to be basic when every polynomial in the complex variables z_s, $s \in I$, can be uniquely expressed as a finite linear combination of the elements of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to [12] the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and and only if there exists a unique row-finite matrix $\overline{P}$ such that

$$\overline{P}P = P\overline{P} = \mathbf{I}, \tag{1.7}$$

where $P = [P_{\mathbf{m};\mathbf{h}}]$ is the matrix of coefficients, $\overline{P}$ is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and $\mathbf{I}$ is the unit matrix.

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\{\overline{P}_{\mathbf{m}}[\mathbf{z}]\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m};\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}, \tag{1.8}$$

$$\overline{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m};\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}, \tag{1.9}$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m};\mathbf{h}}\ P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m};\mathbf{h}}\ \overline{P}_{\mathbf{h}}[\mathbf{z}]. \tag{1.10}$$

Thus, for the function $f(\mathbf{z})$ given in (1.2) we get

$$f(\mathbf{z}) = \sum_{\mathbf{m}} \Pi_{\mathbf{m}}\ P_{\mathbf{m}}[\mathbf{z}], \tag{1.11}$$

where

$$\Pi_{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h};\mathbf{m}}\ a_{\mathbf{h}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h};\mathbf{m}} \frac{f^{\mathbf{h}}(\mathbf{0})}{\mathbf{h}!}, \tag{1.12}$$

and $h! = h(h-1)(h-2)...3.2.1$. The series $\sum_{\mathbf{m}}^{\infty} \Pi_{\mathbf{m}}\ P_{\mathbf{m}}[\mathbf{z}]$ is the associated basic series of $f(\mathbf{z})$.

Definition 1.2. [3, 4, 5, 6, 7, 8, 12]. The associated basic series $\sum_{\mathbf{m}}^{\infty} \Pi_{\mathbf{m}}\ P_{\mathbf{m}}[\mathbf{z}]$ is said to represent $f(\mathbf{z})$ in

(i) $\overline{\mathbf{E}}_{[\mathbf{r}]}$ when it converges uniformly to $f(\mathbf{z})$ in $\overline{\mathbf{E}}_{[\mathbf{r}]}$,
(ii) $\mathbf{E}_{[\mathbf{r}]}$ when it converges uniformly to $f(\mathbf{z})$ in $\mathbf{E}_{[\mathbf{r}]}$,
(iii) $\mathbf{D}(\overline{\mathbf{E}}_{[\mathbf{r}]})$ when it converges uniformly to $f(\mathbf{z})$ in some hyperelliptical surrounding the hyperelliptical $\overline{\mathbf{E}}_{[\mathbf{r}]}$, not necessarily the former hyperelliptical.

Definition 1.3. [3, 4, 5, 6, 7, 8, 12] The set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be simple set, when the polynomial $P_{\mathbf{m}}[\mathbf{z}]$ is of degree $< \mathbf{m} >$, that is to say

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})\ =\mathbf{0}}^{(\mathbf{m})} P_{\mathbf{m};\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}. \tag{1.13}$$

A simple set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be absolutely monic if the coefficient $\mathbf{P}_{\mathbf{m},\mathbf{m}}$ of $z_1^{m_1} z_2^{m_2} ... z_s^{m_s}$ in (1.13) is unity.

Definition 1.4. [2, 7, 8] The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be algebraic of degree j when its matrix of coefficients P satisfies the usual identity

$$\alpha_0\ P^j + \alpha_1\ P^{j-1} + ... + \alpha_j I = \mathbf{0}.$$

Hance, we have a relation of the form

$$\overline{P}_{\mathbf{m},\mathbf{h}} = \delta_{\mathbf{m},\mathbf{h}}\gamma_0 + \sum_{r=1}^{j-1} \gamma_r\ P^{(r)}_{\mathbf{m},\mathbf{h}}, \tag{1.14}$$

where $P^{(r)}_{\mathbf{m},\mathbf{h}}$ are the elements of the power matrix P^r and $\gamma_r,\ r = 1, 2, ..., j-1$ are constant numbers.

Definition 1.5. [3, 4, 5, 6, 7, 8, 12] Let $N_{\mathbf{m}} = N_{m_1,m_2,...,m_n}$ be the number of non-zero coefficients $\overline{P}_{\mathbf{m};\mathbf{h}}$ in the representation (1.9). A basic set satisfying the condition

$$\lim_{<\mathbf{m}>\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = 1, \tag{1.15}$$

is called a Cannon set and if

$$\lim_{<\mathbf{m}>\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = a > 1,$$

then the set is called a general basic set.

Now, let $D_{\mathbf{m}} = D_{m_1,m_2,...,m_n}$ be the degree of the polynomial of the highest degree in the representation (1.9), that is to say, if $D_{\mathbf{h}} = D_{h_1,h_2,...,h_n}$ is the degree of the polynomial $P_{\mathbf{m}}$, then $D_{\mathbf{h}} < D_{\mathbf{m}}\ \forall\ h_s < m_s$. Since the elements of the basic set are linearly independent, then $N_{\mathbf{m}} \leq 1 + 2 + 3 + ... + (D_{\mathbf{m}+1}) \leq \lambda_1\ D^2_{\mathbf{m}}$, where λ_1 is a constant.Therefore, the conditions (1.15) for a basic set to be a Cannon set implies the following condition (see [2]):

$$\lim_{<\mathbf{m}>\to\infty} \{D_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = 1. \tag{1.16}$$

For any function $f(\mathbf{z})$ of several complex variables, there is formally an associated basic series $\sum_{\mathbf{h}=\mathbf{0}}^{\infty} \Pi_{\mathbf{h}}\ P_{\mathbf{h}}[\mathbf{z}]$. When this associated series converges uniformity to $f(\mathbf{z})$ in some domain it is said to represent $f(\mathbf{z})$ in that domain. In other words, as in the classical terminology of Whittaker (see [13]), the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in that domain. The convergence properties of basic sets of polynomials are classified according to the classes of functions represented by their associated basic series and also according to the domain in which they are represented.
To study the convergence properties of such basic sets of polynomials in hyperelliptical regions (c.f.[2, 4]), we consider the following notations for Cannon sums:

$$\Omega[P_{\mathbf{m}}, \overline{\mathbf{E}}_{[\mathbf{r}]}] = \ \sigma_{\mathbf{m}}\ \Pi_{s=1}^{n}\{r_s\}^{<\mathbf{m}>-m_s}\ \sum_{\mathbf{h}} |\overline{P}_{\mathbf{m},\mathbf{h}}|\ M(P_{\mathbf{m}}, \overline{\mathbf{E}}_{[\mathbf{r}]}). \tag{1.17}$$

Also, the Cannon function for the basic sets of polynomials in hyperelliptical regions was defined as follows:

$$\Omega[P, \overline{\mathbf{E}}_{[\mathbf{r}]}] = \limsup_{<\mathbf{m}>\to\infty} \{\Omega[P_{\mathbf{m}}, \overline{\mathbf{E}}_{[\mathbf{r}]}]\}^{\frac{1}{<\mathbf{m}>}}. \tag{1.18}$$

Concerning the effectiveness of the basic set of polynomials of several complex variables in hyperelliptical regions, we have from [3], the following results.

Theorem 1.1. *The necessary and sufficient condition for the Cannon basic set $\{P_{\boldsymbol{m}}[\boldsymbol{z}]\}$ of polynomials of several complex variables to be effective in the closed hyperellipse $\overline{\boldsymbol{E}}_{[\boldsymbol{r}]}$ is that*

$$\Omega[P, \overline{\boldsymbol{E}}_{[\boldsymbol{r}]}] = \prod_{s=1}^{n} r_s.$$

If $\mathbf{r} \to \mathbf{0}^+$, then we will obtain the effectiveness at the origin as in the following corollary:

Corollary 1.1. *The necessary and sufficient condition for the basic set $\{P_{\boldsymbol{m}}[\boldsymbol{z}]\}$ of polynomials of several complex variables to be effective at the origin is that*

$$\Omega[P, \overline{\boldsymbol{E}}_{[\boldsymbol{0}^+]}] = \boldsymbol{0}.$$

For more information about the study of basic sets of polynomials, we refer to [3, 4, 5, 6, 7, 8, 12].

2. Hadamard product set

In [9, 10], Hadamard product of simple monic sets of polynomials of a single complex variable was introduced and its effectiveness properties were studied. Also, in [11] Nassif and Rizk extended this study in the case of two complex variables and they introduced the following definition:

Definition 2.1. Let $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ be two simple monic sets of polynomials, where

$$P_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} P_{i,j}^{m,n} z^i w^j,$$

$$q_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} q_{i,j}^{m,n} z^i w^j.$$

Then Hadamard product of the sets $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ is the simple monic set $\{U_{m,n}(z,w)\}$ given by

$$U_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} U_{i,j}^{m,n} z^i w^j,$$

where

$$U_{i,j}^{m,n} = \frac{\sigma_{m,n}}{\sigma_{i,j}}\, q_{i,j}^{m,n}\, q_{i,j}^{m,n}; \quad ((i,j) \leq (m,n)).$$

In this work we will give an inevitable extension in the definition of Hadamard product of basic sets of polynomials of two complex variables as to yield favorable results in the case of several complex variables in hyperelliptical regions in $\mathbb{C}^n$, by using k basic sets of polynomials instead of two sets. Now, we will extend the above product by using k basic sets of polynomials of several complex variables, so we will denote these polynomials by $\{P_{1,\mathbf{m}}(\mathbf{z})\}$, $\{P_{2,\mathbf{m}}(\mathbf{z})\}$,...,$\{P_{k,\mathbf{m}}(\mathbf{z})\}$ and in general write $\{P_{s,\mathbf{m}}(\mathbf{z})\};\ s \in I_1,\ I_1 = \{1, 2, 3, ..., k\}$.

Definition 2.2. Let $\{P_{s,\mathbf{m}}(\mathbf{z})\}$; $s \in I_1$ be simple monic sets of polynomials of several complex variables, where

$$P_{s,\mathbf{m}}(\mathbf{z}) = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s,\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}. \tag{2.1}$$

Then Hadamard product of the sets $\{P_{s,\mathbf{m}}(\mathbf{z})\}$ is the simple monic set $\{H_{\mathbf{m}}(\mathbf{z})\}$ given by

$$H_{\mathbf{m}}(\mathbf{z}) = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}, \tag{2.2}$$

where

$$H_{\mathbf{m},\mathbf{h}} = \Big(\frac{\sigma_{\mathbf{m}}}{\sigma_{\mathbf{h}}}\Big)^{k-1} \Bigg(\Pi_{s=1}^{k}\ P_{s,\mathbf{m},\mathbf{h}}\Bigg). \tag{2.3}$$

If, we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 2.1.

3. Effectiveness of Hadamard product set

In this section, we will study the effectiveness of the extended Hadamard product of simple monic sets of polynomials of several complex variables defined by (2.2) and (2.3) in closed hyperelliptical regions and at the origin.

Let $\{P_{s,\mathbf{m}}(\mathbf{z})\}$ be simple monic sets of polynomials of the several complex variables z_s; $s \in I_1$, so that we can write

$$P_{s,\mathbf{m}}(\mathbf{z}) = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s,\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}, \tag{3.1}$$

where

$$P_{s,\mathbf{m},\mathbf{m}} = 1; \quad s \in I_1.$$

The normalizing functions of the sets $\{P_{s,\mathbf{m}}(\mathbf{z})\}$ are defined by

$$\mu[P_s; [\mathbf{r}]] = \limsup_{<\mathbf{m}>\to\infty} \Big\{\sigma_{\mathbf{m}}\ \Pi_{s=1}^{k} r_s^{<\mathbf{m}>-m_s}\ \mathbf{M}(P_{s,\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]})\Big\}^{\frac{1}{<\mathbf{m}>}}, \tag{3.2}$$

where $\mathbf{M}(P_{s,\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]})$ is defined as follows:

$$\mathbf{M}(P_{s,\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]}) = \sup_{\overline{\mathbf{E}}_{[\mathbf{r}]}} |P_{s,\mathbf{m}}(\mathbf{z})|.$$

We observe that, since the sets $\{P_{s,\mathbf{m}}(\mathbf{z})\}$ are monic, then by applying Cauchy's inequality in (2.1), we have

$$|P_{s,\mathbf{m},\mathbf{h}}| \leq \frac{\sigma_{\mathbf{m}}}{\Pi_{s=1}^{k} r_s^{h_s}} \sup_{\overline{\mathbf{E}}_{[\mathbf{r}]}} |P_{s,\mathbf{m}}(\mathbf{z})|,$$

which implies that,

$$\mathbf{M}(P_{s,\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]}) \geq \frac{\Pi_{s=1}^{k} r_s^{m_s}}{\sigma_{\mathbf{m}}}.$$

So that (2.2) gives

$$\mu[P_s; [\mathbf{r}]] \geq \Pi_{s=1}^{k} r_s. \tag{3.3}$$

Next, we show if R_s are positive numbers greater than r_s; $s \in I_1$, then

$$\mu[P_s;[\mathbf{R}]] \leq \frac{\Pi_{s=1}^k R_s}{\Pi_{s=1}^k r_s}\ \mu[P_s;[\mathbf{r}]], \quad R_s > r_s. \tag{3.4}$$

In fact, this relation follows by applying (2.2) to the inequality

$$\mathbf{M}(P_{s,\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{R}]}) \leq\ K(\frac{\Pi_{s=1}^k R_s}{\Pi_{s=1}^k r_s})^{<\mathbf{m}>}\ \mathbf{M}(P_{s,\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]}),$$

which in its turn, is derivable from (3.2), Cauchys inequality and the supremum of $|z^{\mathbf{m}}|$, where $K = O(<\mathbf{m}> +1)$.

Now, let $\{P_{s,\mathbf{m}}(\mathbf{z})\}$ be simple monic sets of polynomials of several complex variables and that $\{H^{\star}_{\mathbf{m}}(\mathbf{z})\}$ is the set defined as follows

$$H^{\star}_{\mathbf{m}}(\mathbf{z}) = \Pi_{s=1}^k P_{s,\mathbf{m}}(\mathbf{z}). \tag{3.5}$$

The following fundamental lemma is proved

Lemma 3.1. *If, for any* $r_s > 0;\ \ s \in I_1$

$$\mu[P_s;[\boldsymbol{r}]] = \Pi_{s=1}^k r_s. \tag{3.6}$$

Then

$$\mu[H^{\star};[\boldsymbol{r}]] = \Pi_{s=1}^k r_s. \tag{3.7}$$

Proof. We first observe that, if R_s be any finite numbers greater than r_s; $\ s \in I_1$, then by (3.2), (3.3) and (3.4), we obtain that

$$\mu[P_s;[\mathbf{R}]] = \Pi_{s=1}^k R_s. \tag{3.8}$$

Now, given $r^{\star}_s > r_s$, we choose finite numbers $r^{'}_s$ such that

$$r_s < r^{'}_s < r^{*}_s. \tag{3.9}$$

Then by (3.2) and (3.6), we obtain that

$$\mathbf{M}(P_{s,\mathbf{h}};\overline{\mathbf{E}}_{[\mathbf{r}]}) <\ \frac{\zeta}{\sigma_{\mathbf{h}}}(\Pi_{s=1}^k r^{'}_s)^{h_s}; \quad \zeta > 1. \tag{3.10}$$

Also from (3.5), we can writ

$$H^{\star}_{\mathbf{m}}(\mathbf{z}) = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})}\ \Pi_{s=1}^k\ P_{s,\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}} \tag{3.11}$$

Hence (3.9) and (3.10) lead to

$$\mathbf{M}(H^{\star}_{\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]}) \leq\ \zeta K\Big(1-(\frac{r^{'}_s}{r^{\star}_s})^k\Big)^{-k}\ \mathbf{M}(P_{s,\mathbf{h}};\overline{\mathbf{E}}_{[\mathbf{r}^{\star}]}); \quad s \in I_1. \tag{3.12}$$

Making $<\mathbf{m}>\to\infty$ and applying (3.6), we get

$$\begin{aligned}\mu[H^{\star};[\mathbf{r}]] &= \limsup_{<\mathbf{m}>\to\infty}\Big\{\sigma_{\mathbf{m}}\ \Pi_{s=1}^k r_s^{<\mathbf{m}>-m_s}\ \mathbf{M}(H^{\star}_{\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]})\Big\}^{\frac{1}{<\mathbf{m}>}} \\ &\leq \mu[P_s;[\mathbf{r}^{\star}] = \Pi_{s=1}^k r^{\star}_s,\end{aligned} \tag{3.13}$$

which leads to the equality (3.7), by the choice of $r^{\star}_s$ near to r_s, $\ s \in I_1$, and our lemma is therefore proved ☐

From Lemma 3.1 if we consider the simple monic sets $\{P_{s,\mathbf{m}}(\mathbf{z})\}$ accord to condition (2.6), then it is not hard to prove by induction for the j-power sets $\{P^{(j)}_{s,\mathbf{m}}(\mathbf{z})\}$ that

$$\mu[P^{(j)}_s;[\mathbf{r}]=\Pi^k_{s=1}r_s. \tag{3.14}$$

Before getting the results for the effectiveness in the closed hyperelliptical $\overline{\mathbf{E}}_{[\mathbf{r}]}$ and at the origin, we need the following technical lemmas.

Lemma 3.2. *Let $\{P_{s,\boldsymbol{m}}(\boldsymbol{z})\}$; $s\in I_1$ be simple monic algebraic sets of polynomials of several complex variables, which accord to condition (3.6). Then the set will be effective in the closed hyperelliptical $\overline{\mathbf{E}}_{[\boldsymbol{r}]}$.*

Proof. Suppose that the monomial $\mathbf{z}^{\mathbf{m}}$ admit the representation

$$\mathbf{z}^{\mathbf{m}}=\sum_{\mathbf{h}}\overline{P}_{\mathbf{m};\mathbf{h}}\ P_{\mathbf{h}}(\mathbf{z}).$$

Since, the set $\{P_{1,\mathbf{m}}(\mathbf{z})\}$ is algebraic, then there exists a relation of the form

$$\overline{P}_{1,\mathbf{m};\mathbf{h}}=\sum_{j=1}^{N}\gamma_j\ P^{(j)}_{1,\mathbf{m};\mathbf{h}};\quad ((\mathbf{h})\leq(\mathbf{m})), \tag{3.15}$$

where N is a finite positive integer which together with the coefficients $(\gamma_j)^N_{j=1}$, is independent of the indices $(\mathbf{h})$ and $(\mathbf{m})$. The coefficients $P^{(j)}_{1,\mathbf{m};\mathbf{h}}$ are defined by

$$P^{(j)}_{1,\mathbf{m}}(\mathbf{z})=\sum_{(\mathbf{h})=1}^{(\mathbf{m})}P_{s,\mathbf{m},\mathbf{h}}\ \mathbf{z}^{\mathbf{h}};\ \ 1\leq j\leq N. \tag{3.16}$$

So that,

$$|P^{(j)}_{1,\mathbf{m},\mathbf{h}}|\leq\frac{\sigma_{\mathbf{m}}}{\Pi^k_{s=1}r_s^{<\mathbf{m}>-m_s}}\ \sup_{\overline{\mathbf{E}}_{[\mathbf{r}]}}|P^{(j)}_{1,\mathbf{m}}(\mathbf{z})|, \tag{3.17}$$

or

$$|P^{(j)}_{1,\mathbf{m},\mathbf{h}}|\ \Pi^k_{s=1}r_s^{<\mathbf{h}>-h_s}\leq\ \sigma_{\mathbf{h}}\ \mathbf{M}(P^{(j)}_{1,\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]}), \tag{3.18}$$

According to (3.14) for given $r^\star_s>r_s,\ s\in I_1$ and from the definition corresponding to $\mu[P^{(j)}_1;[\mathbf{r}]]$ we may deduce that

$$\mathbf{M}(P^{(j)}_{1,\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]})<\ \frac{K}{\sigma_{\mathbf{m}}}\ (\Pi^k_{s=1}r^\star_s)^{m_s}, \tag{3.19}$$

Applying (3.18) and (3.19) in (3.16), it follows that

$$|\overline{P}^{(j)}_{1,\mathbf{m},\mathbf{h}}|<\ \eta\ K\ \alpha\ \frac{\sigma_{\mathbf{h}}}{\sigma_{\mathbf{m}}}\ \frac{(\Pi^k_{s=1}r^\star_s)^{m_s}}{(\Pi^k_{s=1}r_s)^{h_s}}, \tag{3.20}$$

where $\alpha=\max\{|\gamma_j|;\ 1\leq j\leq N\}$ and η be a constant.

In view of the representation

$$\mathbf{z}^{\mathbf{m}}=\sum_{\mathbf{h}}\overline{P}_{\mathbf{m};\mathbf{h}}\ P_{\mathbf{h}}(\mathbf{z}).$$

The Cannon sum of the set $\{P^{(j)}_{1,\mathbf{m}}(\mathbf{z})\}$ in the hyperelliptical regions $\overline{\mathbf{E}}_{[\mathbf{r}]}$ will be

$$\Omega[P^{(j)}_{1,\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]}] = \sigma_{\mathbf{m}}\, \Pi^{k}_{s=1} r_s^{<\mathbf{m}>-m_s} \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} |P^{(j)}_{1,\mathbf{m},\mathbf{h}}|\ \mathbf{M}(P^{(j)}_{1,\mathbf{h}};\overline{\mathbf{E}}_{[\mathbf{r}]}). \tag{3.21}$$

From (3.19), (3.20) and (2.21) (for $r^\star_s > r_s$), we find that

$$\Omega[P^{(j)}_{1,\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]}] < \eta\, K\, \alpha\, (\Pi^{k}_{s=1} r^\star_s)^{<\mathbf{m}>}. \tag{3.22}$$

Hence the Cannon function of the set $\{P^{(j)}_{1,\mathbf{m}}(\mathbf{z})\}$ in the hyperelliptical regions $\overline{\mathbf{E}}_{[\mathbf{r}]}$ turns out to be

$$\Omega[P^{(j)}_{1};\overline{\mathbf{E}}_{[\mathbf{r}]}] = \limsup_{<\mathbf{m}>\to\infty} \left\{\Omega[P^{(j)}_{1,\mathbf{m}};\overline{\mathbf{E}}_{[\mathbf{r}]}]\right\}^{\frac{1}{<\mathbf{m}>}} = \Pi^{k}_{s=1} r^\star_s, \tag{3.23}$$

which, by the choice of $r^\star_s,\ s \in I_1$, implies that

$$\Omega[P^{(j)}_{1};\overline{\mathbf{E}}_{[\mathbf{r}]}] = \Pi^{k}_{s=1} r_s.$$

As very similar, we can obtain that the sets $\{P^{(j)}_{\nu,\mathbf{m}}(\mathbf{z})\};\ \nu = 2,3,4,...,k$ will be effective in the hyperelliptical regions $\overline{\mathbf{E}}_{[\mathbf{r}]}$. Our lemma is therefore proved. □

The following theorem will give us the effectiveness of the Hadamard product set in the closed hyperelliptical $\overline{\mathbf{E}}_{[\mathbf{r}]}$.

Theorem 3.1. *Let $\{P_{s,\mathbf{m}}(\mathbf{z})\}$; $s \in I_1$ be simple monic sets of polynomials of several complex variables, which are effective in the hyperelliptical $\overline{\mathbf{E}}_{[\mathbf{r}]}$ for all $r_s \geq \rho^{(s_1)}_s > 0$; $s, s_1 \in I_1$. Then the extended Hadamard product set $\{H_{\mathbf{m}}(\mathbf{z})\}$ defined as in (2.3), (2.3) and is algebraic, will be effective in the hyperelliptical $\overline{\mathbf{E}}_{[\mathbf{r}]}$ for all*

$$\Pi^{k}_{s=1} r_s \geq \Pi^{k}_{s=1} \rho^{(s_1)}_s;\quad s_1 \in I_1$$

and this result is best possible.

Proof. Since the set $\{P_{1,\mathbf{m}}(\mathbf{z})\}$ is effective in the hyperelliptical $\overline{\mathbf{E}}_{[\mathbf{r}]}$ for $\Pi^{k}_{s=1} r_s \geq \Pi^{k}_{s=1} \rho^{(1)}_s$, then for the Cannon function of this set, we have

$$\Omega[P_1;\overline{\mathbf{E}}_{[\mathbf{r}]}] = \Pi^{k}_{s=1} r_s.$$

Hence, if $\Pi^{k}_{s=1}\rho^{(1,*)}_s$ be any positive numbers greater then $\Pi^{k}_{s=1}\rho^{(1)}_s$, it follows that

$$\Omega[P_1;\overline{\mathbf{E}}_{[\rho]}] = \Pi^{k}_{s=1}\rho^{(1)}_s < \Pi^{k}_{s=1}\rho^{(1,*)}_s. \tag{3.24}$$

Therefore, for the Cannon sum of the set $\{P_{1,\mathbf{m}}(\mathbf{z})\}$, we get

$$\Omega[P^{(j)}_{1,\mathbf{m}};\overline{\mathbf{E}}_{[\rho]}] < \eta\, K\, \alpha\, (\Pi^{k}_{s=1}\rho^{(1,*)}_s)^{<\mathbf{m}>}. \tag{3.25}$$

Since the simple set $\{P_{1,\mathbf{m}}(\mathbf{z})\}$ is monic, then in view of (3.18), we infer that

$$|P_{1,\mathbf{m},\mathbf{h}}| < K \frac{\sigma_{\mathbf{h}}}{\sigma_{\mathbf{m}}}\, \frac{(\Pi^{k}_{s=1}\rho^{(1,*)}_s)^{m_s}}{(\Pi^{k}_{s=1}\rho^{(1)}_s)^{h_s}}. \tag{3.26}$$

Similarly, we can obtain that

$$|P_{\nu,\mathbf{m},\mathbf{h}}| < K \frac{\sigma_{\mathbf{h}}}{\sigma_{\mathbf{m}}}\, \frac{(\Pi^{k}_{s=1}\rho^{(\nu,*)}_s)^{m_s}}{(\Pi^{k}_{s=1}\rho^{(\nu)}_s)^{h_s}};\quad \nu = 2,3,4,...,k. \tag{3.27}$$

Introducing (3.27) in (3.3). we can obtain

$$|H_{\mathbf{m},\mathbf{h}}| < K^{\kappa}\frac{\sigma_{\mathbf{h}}}{\sigma_{\mathbf{m}}}\,\Pi_{s=1}^{k}\,\frac{(\rho_s^{(s_1,*)})^{m_s}}{(\rho_s^{(s_1)})^{h_s}};\quad s_1\in I_1. \tag{3.28}$$

It follows therefore, that

$$\begin{aligned}\mathbf{M}[H_{\mathbf{m}});\overline{\mathbf{E}}_{[\rho]}] = \sup_{\overline{\mathbf{E}}_{[\rho]}}|H_{\mathbf{m}}(\mathbf{z})| &\le \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})}|H_{\mathbf{m},\mathbf{h}}|\,\mathbf{M}[P_{\mathbf{h}};\overline{\mathbf{E}}_{[\rho]}]\\ &< \lambda\,K^{\kappa}\frac{\sigma_{\mathbf{h}}}{\sigma_{\mathbf{m}}}\left(\Pi_{s=1}^{k}\,\frac{(\rho_s^{(s_1,*)})^{m_s}}{(\rho_s^{(s_1)})^{h_s}}\right)\left(\frac{1}{\sigma_{\mathbf{h}}}\,\Pi_{s=1}^{k}\,\rho_s^{(s_1)})^{h_s}\right);\quad s_1\in I_1,\end{aligned} \tag{3.29}$$

where λ is a constant. Taking the limit as $<\mathbf{m}>\to\infty$ we obtain the normalizing function of the set $\{H_{\mathbf{m}}(\mathbf{z})\}$, as follows

$$\mu[H;[\rho]] \le \Pi_{s=1}^{k}\,\rho_s^{(s_1,*)};\quad s_1\in I_1.$$

Since $\rho_s^{(s_1,*)}$ can be arbitrary chosen near to $\rho_s^{(s_1)}$, we conclude that

$$\mu[H;[\rho]] \le \Pi_{s=1}^{k}\,\rho_s^{(s_1)};\quad s_1\in I_1.$$

Since the set $\{H_{\mathbf{m}}(\mathbf{z})\}$ is algebraic, then according to Lemma 3.2 above, we conclude that the simple set $\{H_{\mathbf{m}}(\mathbf{z})\}$ is effective in the hyperelliptical region $\overline{\mathbf{E}}_{[\rho]}$ and consequently, in the hyperelliptical region $\overline{\mathbf{E}}_{[\mathbf{r}]}$ for all

$$\Pi_{s=1}^{k}r_s \ge \Pi_{s=1}^{k}\rho_s^{(s_1)};\quad s_1\in I_1,$$

and the first assertion of Theorem 3.1 is proved. To complete the proof of Theorem 3.1, we give the following example

Example 3.1. *Let the set*

$$P_{\boldsymbol{0}}(\boldsymbol{z}) = 1;\quad P_{\boldsymbol{m}}(\boldsymbol{z}) = \boldsymbol{z}^{m_s} + \frac{1}{\sigma_{\boldsymbol{m}}}(\Pi_{s=1}^{k}\rho_s^{*})^{m_s},$$

$$q_{\boldsymbol{0}}(\boldsymbol{z}) = 1;\quad q_{\boldsymbol{m}}(\boldsymbol{z}) = \boldsymbol{z}^{m_s} + \frac{1}{\sigma_{\boldsymbol{m}}}(\Pi_{s=1}^{k}r_s)^{m_s},$$

where m_s are not zeros, be effective in $\overline{\boldsymbol{E}}_{[\boldsymbol{r}]}$ for all $\Pi_{s=1}^{k}r_s \ge \Pi_{s=1}^{k}\rho_s^{*}$. Now, according to Definition 2.2 of Hadamard product, we have

$$H_{\boldsymbol{0}}(\boldsymbol{z}) = 1;\quad P_{\boldsymbol{m}}(\boldsymbol{z}) = \boldsymbol{z}^{m_s} + \frac{1}{\sigma_{\boldsymbol{m}}}(\Pi_{s=1}^{k}\rho_s^{*}\ r_s)^{m_s},$$

It is easy to see that this set is not effective in $\overline{\boldsymbol{E}}_{[\rho]}$, where $0 < \Pi_{s=1}^{k}\rho_s < \Pi_{s=1}^{k}\rho_s^{*}\ r_s$ as required. Theorem 3.1, is therefore established.

□

From Theorem 3.1, we deduce the following result

Corollary 3.1. *Let $\{P_{s,\boldsymbol{m}}(\boldsymbol{z})\}$; $s\in I_1$ be simple monic sets of polynomials of several complex variables, which are effective at the origin. Then, the Hadamard product set $\{H_{\boldsymbol{m}}(\boldsymbol{z})\}$ defined as in (2.2), (2.3) and is algebraic will be effective at the origin.*

To get the results concerning the effectiveness in the hyperspherical regions $\overline{\mathbf{S}}_r$ (cf. [2, 11])as special cases from the results concerning effectiveness in the hyperelliptical regions $\overline{\mathbf{E}}_{[\mathbf{r}]}$, put $r = r_s;\ s \in I_1$ in Theorem 3.1, we can arrive to the following result

Corollary 3.2. *The effectiveness of the sets* $\{P_{s,\boldsymbol{m}}(\boldsymbol{z})\};\ s \in I_1$ *in the equiellipse* $\overline{\boldsymbol{E}}_{[\boldsymbol{r}]^*}$ *yields the effectiveness of the set* $\{H_{\boldsymbol{m}}(\boldsymbol{z})\}$ *in the hyperspherical* $\overline{\boldsymbol{S}}_r$.

Remark 3.1. Similar results for Hadamard product of algebraic general basic set of polynomials of several complex variables in hyperelliptical regions and hyperspherical regions can be obtained.

4. Growth of Hadamard product set

The mode of increase of a basic set $P_n(z)$ is determined by the order and type of the basic set. For a simple basic set $\{P_n(z)\}$, the order ρ is defined by [13]

$$\rho = \lim_{r\to\infty}\ \limsup_{n\to\infty} \frac{\log \omega_n(r)}{n \log n},$$

where $\omega_n(r)$ stands as usual for the Cannon sum, given above. If $0 < \rho < 1$, the type τ is given by

$$\tau = \limsup_{r\to\infty} \frac{e}{\rho}\ \limsup_{n\to\infty} \frac{[\omega_n(r)]^{\frac{1}{n\rho}}}{n}.$$

It has been shown in [12] that the upper bound of the class of entire functions of several complex variables represented by a given basic set is determined by the mode of increase of the basic set. The significance of the order and type of a basic set lies in the fact that they define the class of entire functions represented by the basic set.

To complete our study, we investigate the order of the Hadamard product set $\{H_{\mathbf{m}}(\mathbf{z})\}$ in the equi-hyperellipse $\overline{\mathbf{E}}_{[\mathbf{r}]^*}$.

Suppose that $\{P_{s,\mathbf{m}}(\mathbf{z})\};\ s \in I_1$, is a simple monic set for which

$$P_{s,\mathbf{m};\mathbf{h}} = M_s\ \frac{<\mathbf{m}>^{\varrho_s(<\mathbf{m}>-<\mathbf{h}>)}}{[(\mathbf{m})+1]}\sigma_{\mathbf{h}}; \quad \mathbf{0} \le (\mathbf{h}) < (\mathbf{m}), \tag{4.1}$$

where ϱ_s are positive constants and $M_s \ge 1$, are finite numbers, $s \in I_1$.

Then for the maximum modulus $\mathbf{M}(H_{\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]^*})$ of Hadamard product set $\{H_{\mathbf{m}}(\mathbf{z})\}$, it follows that

$$\begin{aligned}
\mathbf{M}(H_{\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]^*}) = \sup_{\overline{\mathbf{E}}_{[\mathbf{r}]^*}} |H_{\mathbf{m}}(\mathbf{z})| &\le \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} |\Big(\frac{\sigma_{\mathbf{m}}}{\sigma_{\mathbf{h}}}\Big)^{k-1} \left(\Pi_{s=1}^k\ P_{s,\mathbf{m},\mathbf{h}}\right)|\ \frac{r^{<\mathbf{h}>}}{\sigma_{\mathbf{h}}} \\
&\le\ M \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} (\frac{r^{\frac{1}{\varrho}}}{<\mathbf{m}>})^{\varrho<\mathbf{h}>}\ <\mathbf{m}>^{\varrho<\mathbf{m}>} \\
&<\ M\ <\mathbf{m}>^{\varrho<\mathbf{m}>}, \quad for\ <\mathbf{m}>^{\varrho} > r,
\end{aligned} \tag{4.2}$$

where $M = \Pi_{s=1}^k\ M_s$ and $\varrho = \Pi_{s=1}^k\ \varrho_s$.

Now, If the set $\{H_{\mathbf{m}}(\mathbf{z})\}$ is algebraic, then we have a relation of the form

$$\overline{H}_{\mathbf{m};\mathbf{h}} = \sum_{j=1}^{N} \gamma_j\ H^{(j)}_{\mathbf{m};\mathbf{h}}; \quad ((\mathbf{h}) \le (\mathbf{m})). \tag{4.3}$$

Introducing (4.1), (4.2) and (4.3) in the Cannon sum $\Omega[H_{\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]^*}]$ of Hadamard product set $\{H_{\mathbf{m}}(\mathbf{z})\}$, we find that

$$\begin{aligned}
\Omega[H_{\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]^*}] &= \sigma_{\mathbf{m}} \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} |\overline{H}_{\mathbf{m};\mathbf{h}}| \, \mathbf{M}(H_{\mathbf{h}}; \overline{\mathbf{E}}_{[\mathbf{r}]^*}) \\
&= \sigma_{\mathbf{m}} \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \sum_{j=0}^{N} |\gamma_j \, H^{(j)}_{\mathbf{m};\mathbf{h}}| \, \mathbf{M}(\mathbb{P}\mathbb{E}_{\mathbf{h}}; \overline{\mathbf{E}}_{[\mathbf{r}]^*}) \\
&< K \, \beta \, \sigma_{\mathbf{m}} \, <\mathbf{m}>^{\varrho <\mathbf{m}>} < K \, (\sqrt{2})^{(} <\mathbf{m}>) \, <\mathbf{m}>^{\lambda <\mathbf{m}>} .
\end{aligned}$$

Thus

$$\Gamma = \lim_{r\to\infty} \limsup_{<\mathbf{m}>\to\infty} \frac{\log \, \Omega[H_{\mathbf{m}}; \overline{\mathbf{E}}_{[\mathbf{r}]^*}]}{<\mathbf{m}> \, \log \, <\mathbf{m}>} \leq \varrho. \tag{4.4}$$

It is easy to see that $\Gamma \geq \varrho$, therefore the set $\{H_{\mathbf{m}}(\mathbf{z})\}$ is of order $\Gamma = \varrho = \Pi_{s=1}^{k} \varrho_s$ at most. This completes the proof of the following theorem:

Theorem 4.1. *The Hadamard product algebraic set $\{H_{\boldsymbol{m}}(\boldsymbol{z})\}$ will be of order $\Gamma = \varrho = \Pi_{s=1}^{k} \varrho_s$ at most when the simple monic sets $\{P_{s,\boldsymbol{m}}(\boldsymbol{z})\}$; $s \in I_1$ accord to the restriction (4.1).*

Remark 4.1. It is worthy ensure that all results obtained in this work are also true for the inverse sets, power sets and inverse power sets of the concerned sets.

References

[1] M. A. Abul-Ez, Hadamard product of bases of polynomials in Clifford analysis, *Complex variables theory Appl.*, **43**, (2000), 109-128.

[2] A. El-Sayed, Extended results of the Hadamard product of simple sets of polynomials in hypersphere, *Annal. Soci. Math. Polon.*, **2**, (2006), 201-213.

[3] A. El-Sayed and Z. M. G. Kishka, On the effectiveness of basic sets of polynomials of several complex variables in elliptical regions., In Proceedings of the 3rd International ISAAC Congress, pages 265-278, Freie Universitaet Berlin, Germany. Kluwer. Acad. Publ (2003).

[4] G. F. Hassan, Ruscheweyh differential operator sets of basic sets of polynomials of several complex variables in hyperelliptical regions, *Acta Math. Acad. Paed. Nyiregyhazi.*, **22**, (2006), 247-264.

[5] Z. M. Kishka, M. A. Saleem and M. Abul-Dahab, On simple exponential sets of polynomials. Published online: 014 June 2013 in *Medite. J. Math.*

[6] Z. M. Kishka and A. El-sayed, On the order and type of basic and composite sets of polynomial in complete Reinhardt domains, *Perio. Math. Hung.*, **46**, (2003), 67-79.

[7] Z. M. Kishka, Exponential and Algebraic General Basic Sets of Polynomials in Several Complex Variables. *Sohag Pure and App. Sci. Bull.*, **8**, (1992), 85-105.

[8] Z. M. Kishka, Composite sets of polynomials of several complex variables. *Proc. Conf. Oper. Res and Math. Methods. Bull. Fac. Sc. Alex.*, **26**, (1986), 92-103.

[9] S. Z. Melek and A. E. El-Said, On Hadamard product of basic sets of polynomials, *Bull. Fac. of Engineering, Ain Shams Univ.*, **16**, (1985), 1-14.

[10] M. N. Mikhail, Basic sets of polynomials and their reciprocal, product and quontient sets, *Duke Math.J.*, **20**, (1953),459-480.

[11] M. Nassif and S. W. Rizk, Hadamard product of simple sets of polynomials of two complex variables, *Bull. Fac. of Engineering, Ain Shams Univ.*, **18**, (1988), 97-116.

[12] M. Nassif, Composite sets of polynomials of several complex variables, *Publ. Math. Debrecen.*, **18**, (1971), 43-52.

[13] J. M. Whittaker, Sur Les Series De Base De Polynomes Quelconques, *Gauthier-Villars, Paris.*, (1949).

M. A. Abul-Dahab
Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt
E-mail address: mamabuldahab@yahoo.com

A. M. Saleem
Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
E-mail address: abuelhassan@yahoo.com

Z. M. Kishka
Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
E-mail address: zanhomkishka@yahoo.com