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ENTROPY FORMULAS AND THEIR APPLICATIONS ON TIME DEPENDENT RIEMANNIAN METRICS

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Abstract. In this paper we discuss entropy formulas under the abstract geometric flow. We adopt Perelman's approach for the Ricci flow to obtain unified energy and entropy functionals, which are monotonically nondecreasing along the flow. We demonstrate their applications to rule out existence of geometric breathers (steady, shrinking and expanding) other than gradient solitons.

1. Introduction

A classical problem in differential geometry is to find canonical metrics of Riemannian manifolds. By a canonical metric we mean a metric of constant curvature whose existence often yields useful geometric and topological information. A well known example is the classification of Gauss curvature metrics of simply connected Riemannian surfaces, i.e., the uniformization theorem. By now it is well known that the geometric flows play fundamental roles in achieving this objective. In this paper we study the generalized abstract geometric flow via entropy formulas. Our major aim is to prove a unified approach to the treatment of numerous geometric flows that have been developed by several authors. We say that a one parameter family of time-dependent Riemannian metrics $g(t)$, $t \in [0,T]$, is a generalized geometric flow if it is evolving by the following system of initial value problem

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij}(x,t) &= -2h_{ij}(x,t) \\ g_{ij}(x,0) &= g_0(x), \end{aligned} \tag{1}$$

where $x \in M$ and h_{ij} is a general time-dependent symmetric $(0,2)$-tensor. The scaling factor 2 in (1) is insignificant while the negative sign may be important in some specific applications for the purpose of keeping the flow either forward or backward in time. In this paper, we define unified energy functionals and entropy formulas which turn out to be monotone under certain conditions as long as the geometric flow exists. The energy and entropy functionals are generalization of classical entropy of Shannon arising from Thermodynamics and Fisher information entropy from information theory for Ricci flat manifold. Perelman's energy and

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functional for the Ricci flow [13] provide us with a great deal of motivations. The geometric implication of the monotonicity formulas derived here is that there are no compact geometric breathers excepts those that are gradient solitons. See [4] and [9] for other useful applications, for instance, the monotonicity of compact eigenvalue and monotone volume. For further application of entropies see [11] on static metrics and [3, 12] on evolving manifolds.

Most of our calculations are done in local coordinates, where $\{x^i\}$ is fixed in a neighbourhood of every point $x \in M$. The Riemannian metric $g(x)$ at any point $x \in M$ is a bilinear symmetric positive definite matrix denoted in local coordinates by

$$g_{ij} = ds^2 = g_{ij}dx^i dx^j.$$

The Laplace-Beltrami operator acting on a smooth function f on M is defined as a dot product of divergence and gradient of f, where

$$(grad\ f)^i = (\nabla f)^i = g^{ij}\frac{\partial}{\partial x^j} f \quad and \quad divF = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}F^i).$$

F being a smooth vector field. Also, we have the metric norm $|\nabla f|_g^2 = g^{ij}\nabla_i f \nabla_j f$. The Riemannian structure allows us to define Riemannian volume measure $d\mu$ on M, $d\mu = \sqrt{|g_{ij}(x)|}dx^i$. By the divergence theorem we have the following integration by parts formulas for functions $f, h \in C^2(M)$

$$\int_M f\Delta_g h\ dV = -\int_M \langle \nabla f, \nabla h\rangle_g dV = \int_M \Delta_g f\ hdV.$$

For any smooth function f on M, we have the Bochner's formula defined as

$$\Delta(|\nabla f|^2) = 2|\nabla\nabla f|^2 + 2\langle \nabla f, \nabla \Delta f\rangle + 2Rc(\nabla f, \nabla f), \tag{2}$$

where Rc is the Ricci curvature of M whose tensor components will be written in local coordinates as R_{ij}. Interestingly, when a manifold is being evolved under a geometric flow all the associated quantities also evolve along the flow. For examples, the Riemannian volume measure $d\mu$ of (M, g) evolves by

$$\partial_t d\mu = -\mathcal{H} d\mu$$

and $\mathcal{H}$ by

$$\partial_t \mathcal{H} = g^{ij}\partial_t h_{ij} + 2|h_{ij}|^2$$

where g^{ij} is the inverse of the metric g_{ij}, $\mathcal{H} = g^{ij}h_{ij}$, i.e., the metric trace of $(0, 2)$-tensor h_{ij} and $|h_{ij}|^2 = g^{ik}g^{jl}h_{ij}h_{kl}$. Denote $\beta := g^{ij}\partial_t h_{ij}$, in particular, under the Ricci flow, where $h_{ij} = R_{ij}$ and $\mathcal{H} = R$, we have $\beta = \Delta R$. Here in this paper we will assume that

$$\beta - \Delta\mathcal{H} \geq 0. \tag{3}$$

This is motivated by an error term appearing in a result of Müller [9, Lemma 1.6]. For our case the error term reads; for any time-dependent vector field X

$$\mathcal{D}(X) := 2(R_{ij} - h_{ij})(X^i, X^j) + 2\langle \nabla\mathcal{H} - 2div\ h, X\rangle + \partial_t\mathcal{H} - \Delta\mathcal{H} - 2|h_{ij}|^2, \tag{4}$$

where div is the divergence operator, i.e., $(div\ h)_k = g^{ij}\nabla_i h_{jk}$. Clearly the last three terms in (4) above is the same as the quantity $\beta - \Delta\mathcal{H}$. It does make sense to assume (3) whenever $\mathcal{D}(X)$ is nonnegative. The application of this is that we are on a steady or shrinking soliton (self-similar solution to the geometric flow) if the equality in (3) holds. Note that we can also express $|h_{ij}|^2 \geq \frac{1}{n}\mathcal{H}^2$ since $|g^{ij}h_{ij}|^2 = \mathcal{H}^2$. Using the condition that $\beta - \Delta\mathcal{H} \geq 0$, we have a governing differential inequality for the evolution of $\mathcal{H}$ as follows

$$\frac{\partial}{\partial t}\mathcal{H} \geq \Delta\mathcal{H} + \frac{2}{n}\mathcal{H}^2. \tag{5}$$

Suppose $\mathcal{H} \geq \mathcal{H}_{min}$, we can apply the maximum principle to show that

$$\mathcal{H}_{g(t)} \geq \frac{\mathcal{H}_{min}(0)}{1 - \frac{2}{n}\mathcal{H}_{min}(0)t} \tag{6}$$

for all $t \geq 0$ as long as the flow exists.

The next section gives a brief review on the concepts of geometric breathers and solitons. We devote Sections 3 - 5 to the treatment of energy and entropy monotonicity formulas and their geometric applications. The last section presents some geometric flows available in literature where our approach is applicable.

2. Solitons and Breathers for the geometric flow

Generally speaking, a soliton is a self-similar solution to an evolution equation which evolves along the symmetry group of the flow. In the case of the geometric flow, the symmetries are scalings and diffeomorphisms. Soliton solutions are very crucial to the study of behaviour of solutions near singularities in special applications of geometric flows where singularity models are more obvious. The periodic solutions modulo scaling and diffeomorphisms are called breathers. A priori, we do not usually expect to have periodic solutions since the geometric flow is a heat-type equation. These special solutions (solitons and breathers) motivate the analysis of geometric flow through entropies and monotonicity formulas. We show as geometric applications of entropy formulas that there are no compact breathers except the trivial ones which are essentially gradient solitons. Details about Ricci solitons and Ricci breathers are found in [1] and [2] for instance.

2.1. **Geometric Solitons.** In this case we modify the flow by a one-parameter group of diffeomorphisms ϕ_t and define a *time*−dependent vector field X from it.

Definition 2.1. *Let $\{\phi_t\}, t \in I$ be a one-parameter family of diffeomorphisms, $\varphi_t : M \to M$, and $\{g(t)\}_{t\in I}$ be a one-parameter family of Riemannian metrics defined on M. Given a smooth scalar function $\beta(t) > 0$, such that*

$$g(t) = \beta(t)\phi_t^* g_0 \ \ \text{and} \ \ h_{ij}(g) = h_{ij}(\phi_t^* g_0). \tag{7}$$

Any geometrici flow (i.e., solution $g(t)$ of (1)) with this property is called a **geometric soliton**.

This simply means that on a geometric soliton all the Riemannian manifolds (M^n, g) are isometric up to a scale factor that is allowed to vary with *time*. Therefore, the geometric flow equation is equivalent to

$$h_{ij}(g_0) + \frac{1}{2}\mathcal{L}_X g_0 = \sigma g_0 \tag{8}$$

for any $\sigma(t) = -\frac{1}{2}\beta'(t)$, where X is a vector field on M and $\mathcal{L}_X g_0$ is the Lie derivative of the evolving metric. If the vector field X is the gradient of a function, say f, then the solution is called a gradient soliton and (8) becomes

$$h_{ij} + \nabla_i\nabla_j f = \sigma g_{ij}, \tag{9}$$

where σ is the homothety constant. The case $\beta'(t) < 0$, $\beta'(t) = 0$ or $\beta'(t) > 0$ corresponds to shrinking, steady or expanding gradient soliton. Clearly, a 2-tensor h_{ij} is a multiple of the metric $g_{ij}(t)$ if X vanishes identically.

2.2. **Geometric Breathers.**

Definition 2.2. *(**Breathers**): A metric $g_{ij}(t)$ which evolves by the generalized geometrici flow is called a breather if for some t_1, t_2, such that $t_1 < t_2$, the metric*

$$g_{ij}(t_2) = \alpha\phi_t^* g_{ij}(t_1) \quad \text{and} \quad h_{ij}(g_{ij}(t_2)) = \phi_t^* h_{ij}(g_{ij}(t_2) \tag{10}$$

for some constant $\alpha > 0$ and diffeomorphism $\phi_t : M \to M$. The cases $\alpha < 1, \alpha = 1$ and $\alpha > 1$ correspond to shrinking, steady and expanding breathers. Clearly, steady, shrinking or expanding gradient solitons are trivial breathers for which metric $g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by diffeomorphism and scaling for t_1 and t_2.

Remark 2.3. *If we consider the generalized geometric flow as a dynamic system on the space of Riemannnian metrics modulo diffeomorphism and scaling, the breathers correspond to the periodic orbits while the solitons are fixed points.*

3. Energy and entropy functionals for geometric flow

Let $g = g(t) \in \Gamma(S_+^2(T^*M))$ be a Riemannian metric solving the geometric flow (1) on a closed manifold M and $f : M \to \mathbb{R}$ be a gradient function. Let $u = u(t,x)$ be a positive solution to the geometric heat-type equation

$$\square^* u(t,x) = \Big(-\frac{\partial}{\partial t} - \Delta + \mathcal{H}\Big) u(t,x) = 0, \quad (t,x) \in [0,T) \times M \tag{11}$$

with the normalization condition $\int_M u(t,x)\ d\mu_{g(t)} = 1$. Define the classical Boltzmann entropy as

$$\mathcal{E}(u(t)) = \int_M u(t,x) \log u(t,x)\ d\mu_{g(t)}. \tag{12}$$

We have the following proposition;

Proposition 3.1. *With the notation above*

$$\square^*(u \log u) = \frac{|\nabla u|_g^2}{u} + \mathcal{H}u \tag{13}$$

and

$$\begin{aligned} \square^*\Big(\frac{|\nabla u|_g^2}{u} + \mathcal{H}u\Big) &= \frac{2}{u}(h + Rc)(\nabla u, \nabla u) + \frac{2}{u}\Big(\nabla\nabla u - \frac{\nabla u \otimes \nabla u}{u}\Big)^2 \\ &+ 4\langle \nabla\mathcal{H}, \nabla u\rangle_g + 2\Delta\mathcal{H}u + 2\Big(|h_{ij}|_g^2 + \frac{1}{2}(B - \Delta\mathcal{H})\Big)u \end{aligned} \tag{14}$$

Proof.

$$\begin{aligned} \square^*(u \log u) &= (-\partial_t - \Delta_g + \mathcal{H})(u \log u) \\ &= (-\partial_t - \Delta_g + \mathcal{H})u \log u - \partial_t u - \Delta u - \frac{|\nabla u|}{u} \\ &= -\Big(\frac{|\nabla u|^2}{u} + \mathcal{H}u\Big). \end{aligned}$$

The first result follows immediately by making use of the fact that

$$(-\partial_t - \Delta_g + \mathcal{H})u = 0 = -(\partial_t + \Delta_g - \mathcal{H})u.$$

Similarly, by direct calculations (using Bochner's formula (2)), we have

$$\Box^*(\mathcal{H}u) = -2(|h|^2 + \frac{1}{2}(B - \Delta\mathcal{H}) + \Delta\mathcal{H}\Big)u \tag{15}$$

and

$$\Box^*\Big(\frac{|\nabla u|^2}{u}\Big) = -\frac{2}{u}(Rc + h)(\nabla u, \nabla u) - \frac{2}{u}\Big(\nabla\nabla u - \frac{\nabla u \otimes \nabla u}{u}\Big)^2 \\ -2\langle \nabla u, \nabla\mathcal{H}\rangle. \tag{16}$$

Hence, we arrive at the second result by putting together (15) and (16). □

By the proposition above we have proved the first and second derivative of the Boltzmann entropy along the geometric flow.

Theorem 3.2. *(Evolution of Boltzmann entropy) With the above notations we have*

$$\frac{d}{dt}\mathcal{E}(u(t)) = \int_M \Big(\frac{|\nabla u|^2}{u} + \mathcal{H}\Big)ud\mu \tag{17}$$

$$\frac{d^2}{dt^2}\mathcal{E}(u(t)) = \int_M \Big(\Big|h - \nabla(\frac{\nabla u}{u})\Big|^2 + \Theta(\frac{\nabla u}{u})\Big)u\ d\mu. \tag{18}$$

The proof of the first part is a consequence of (13) while that of second part follows from (14) and will be completed under Theorem 3.4. □

3.1. $\mathcal{F}$-energy functional and its monotonicity.

Definition 3.3. *We define Perelman-type $\mathcal{F}$-energy functional as the integral of* $\Box^*(u \log u)$ *i.e., the first derivative of Boltzmann entropy*

$$\mathcal{F} = \int_M \Big(\frac{|\nabla u|^2}{u} + \mathcal{H}\Big)ud\mu \tag{19}$$

and Perelman-type $\mathcal{W}$-entropy as the combination of $\mathcal{F}$-energy, Boltzmann entropy and certain positive scaling factor τ as

$$\mathcal{W} = \tau\mathcal{F} - \mathcal{E} - \frac{n}{2}\log(4\pi\tau) - n. \tag{20}$$

Details on $\mathcal{W}$-entropy is delayed till Section 4. Let $f = -\log u$, $f \in C^\infty(M)$ (f is called a potential function), then f satisfies the backward heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - \mathcal{H}$$

with the constraints $\int_M e^{-f}d\mu = 1$. Then we have

Theorem 3.4. *Let $(g_{ij}(t), f(t))$ solves the following coupled system*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2h_{ij} \\ \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - \mathcal{H}. \end{cases} \tag{21}$$

Then, the energy functional $\mathcal{F}$ defined by

$$\mathcal{F} = \int_M (|\nabla f|^2 + \mathcal{H})e^{-f}d\mu \tag{22}$$

evolves by

$$\frac{d}{dt}\mathcal{F} = \int_M 2\Big(|h_{ij} + \nabla_i\nabla_j f|^2 + \Theta(-\nabla f)\Big)e^{-f}d\mu. \tag{23}$$

Moreover $\mathcal{F}$ is monotonically nondecreasing provided $\Theta(-\nabla f) \geq 0$. The monotonicity is strict unless $h_{ij} + \nabla_i\nabla_j f = 0$ and $\Theta(-\nabla f) = 0$.

Proof. By direct computation

$$\frac{d}{dt}\mathcal{F} = \int_M \Big(\frac{\partial}{\partial t} - \mathcal{H}\Big)\Big(\frac{|\nabla u|^2}{u} + \mathcal{H}\Big)d\mu = \int_M \Box^*\Big(\frac{|\nabla u|^2}{u} + \mathcal{H}\Big)d\mu.$$

Then by Proposition 3.1

$$\frac{d}{dt}\mathcal{F} = \int_M \Big[\frac{2}{u}\Big(\nabla\nabla u - \frac{\nabla u \otimes \nabla u}{u}\Big)^2 + \frac{2}{u}(h + Rc)(\nabla u, \nabla u)$$
$$+ 2\langle\nabla\mathcal{H}, \nabla u\rangle + 2\Big(|h_{ij}|_g^2 + \frac{1}{2}(B - \Delta\mathcal{H})\Big)u\Big]d\mu,$$

where we have used

$$\int_M (2\langle\langle\nabla\mathcal{H}, \nabla u\rangle + 2u\Delta\mathcal{H})d\mu = 0$$

due to integration by parts in (14). Notice also that

$$\frac{2}{u}\Big(\nabla\nabla u - \frac{\nabla u \otimes \nabla u}{u}\Big)^2 = 2u\Big|\nabla\Big(\frac{\nabla u}{u}\Big)\Big|^2 = 2u|\nabla\nabla \log u|^2.$$

Then

$$\frac{d}{dt}\mathcal{F} = \int_M \Big[2u|\nabla\nabla \log u|^2 + 2u(h + Rc)(\nabla \log u, \nabla \log u)$$
$$+ 2\langle\nabla\mathcal{H}, \nabla u\rangle + 2\Big(|h_{ij}|_g^2 + \frac{1}{2}(B - \Delta\mathcal{H})\Big)u\Big]d\mu$$
$$= \int_M \Big[2u(|h|^2 + |\nabla\nabla \log u|^2) + 2u(h + Rc)(\nabla \log u, \nabla \log u)$$
$$+ 2\langle\nabla\mathcal{H}, \nabla u\rangle + (B - \Delta\mathcal{H})u\Big]d\mu$$
$$= \int_M \Big[2u(|h - \nabla\nabla \log u|^2 + 4u\langle h, \nabla\nabla \log u\rangle + 2\langle\nabla\mathcal{H}, \nabla u\rangle$$
$$2u(h + Rc)(\nabla \log u, \nabla \log u) + (B - \Delta\mathcal{H})u\Big]d\mu.$$

Since we are on a closed manifold we have by the divergence theorem that

$$\int_M div\ (uh\nabla \log u)d\mu = 0,$$

where the integrand is evaluated as

$$div\ (uh\nabla \log u) = h(\nabla \log u, \nabla u) + u\langle divh, \nabla \log u\rangle + u\langle h, \nabla\nabla \log u\rangle. \tag{24}$$

Using the expression (24) in the above we have

$$\frac{d}{dt}\mathcal{F} = \int_M \Big[2u(|h - \nabla\nabla \log u|^2 + 2u(Rc - h)(\nabla \log u, \nabla \log u)$$
$$+ (B - \Delta\mathcal{H})u + 2\langle\nabla\mathcal{H}, \nabla u\rangle - 4\langle divh, \nabla u\rangle\Big]d\mu$$
$$= 2\int_M \Big(|h - \nabla\nabla \log u|^2 u d\mu + 2\int_M \Big[(Rc - h)(\nabla \log u, \nabla \log u)$$

$$+ \langle \nabla\mathcal{H} - 2\ divh, \nabla \log u \rangle + \frac{1}{2}(B - \Delta\mathcal{H})\Big] u d\mu.$$

Putting in $f = -\log u$, we obtain the desired result with

$$\Theta(-\nabla f) = (Rc - h)(\nabla f, \nabla f) + \langle \nabla\mathcal{H} - 2\ divh, \nabla f \rangle + \frac{1}{2}(B - \Delta\mathcal{H}).$$

□

Remark 3.5. *This also completes the proof of Theorem 3.2 (the second part). This can be seen by putting $f = -\log u$ or $u = e^{-f}$ for a potential function $f \in C^\infty(M)$ and $\nabla f = -|\nabla u|/u$ in the second equation in the theorem. Thus*

$$\frac{d}{dt}\mathcal{F}(g(t), f) = \frac{d^2}{dt^2}\mathcal{E}(f(t)).$$

Notice also that when $\frac{d}{dt}\mathcal{F}(g(t), f) = 0$ each term on the right hand side (23) will be identically zero which implies $h_{ij} + \nabla_i\nabla_j f = 0$, $h_{ij} = 0$ and $f \equiv const$. Then $\Theta(-\nabla f) = \Theta(0) = 0$ which also implies $(B - \Delta\mathcal{H}) = 0$.

3.2. **The family of $\mathcal{F}_k$-energy functional.**

Definition 3.6. *Let (M^n, g) be a closed n-dimensional Riemannian Manifold, define for any $f \in C^\infty(M)$ with $\int_M e^{-f} d\mu = 1$, a family of energy functional $\mathcal{F}_k$ as*

$$\mathcal{F}_k = \int_M \Big(|\nabla f|^2 + k\mathcal{H}\Big) e^{-f} d\mu, \tag{25}$$

where $k \geq 1$. When $k = 1$, we simply get Perelman's $\mathcal{F}$ energy.

In the next, we obtain the monotonicity formula for this family of functional $\mathcal{F}_k(g, f)$.

Theorem 3.7. *Let $(M, g_(t))$ evolve by the generalized geometric flow (1) such that (g, f) solves the coupled system (26).*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2h_{ij} \\[2ex] \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - \mathcal{H}. \end{cases} \tag{26}$$

Then,

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_k(g_{ij}, f) = & \int_M \Big(|h_{ij} + \nabla_i\nabla_j f|^2 + (k-1)|h_{ij}|^2\Big) e^{-f} d\mu \\ & + 2\int_M \Big(\Theta(-\nabla f) + \frac{(k-1)}{2}(B - \Delta\mathcal{H})\Big) e^{-f} d\mu. \end{aligned} \tag{27}$$

Furthermore $\mathcal{F}_k$ is monotonically nondecreasing in time for $k \geq 1$, $\Theta(-\nabla f) \geq 0$ and $(B - \Delta\mathcal{H}) \geq 0$. The monotonicity is strict unless $h_{ij} = 0$, f is constant and $(B - \Delta\mathcal{H}) = 0$

Proof. Rewrite $\mathcal{F}_k$ as

$$\mathcal{F}_k(g, f) = \int_M (|\nabla f|^2 + \mathcal{H}) e^{-f} d\mu + (k-1)\int_M \mathcal{H} e^{-f} d\mu.$$

A straightforward differentiation yields

$$\frac{d}{dt}\mathcal{F}_k(g,f) = \frac{\partial}{\partial t}\Big(\int_M (|\nabla f|^2+\mathcal{H})e^{-f}d\mu\Big) + (k-1)\frac{\partial}{\partial t}\Big(\int_M \mathcal{H}e^{-f}d\mu\Big) \tag{28}$$

By Theorem 3.4 we have

$$\frac{\partial}{\partial t}\Big(\int_M (|\nabla f|^2+\mathcal{H})e^{-f}d\mu\Big) = \int_M 2\Big(|h_{ij}+\nabla_i\nabla_j f|^2 + \Theta(-\nabla f)\Big)e^{-f}d\mu. \tag{29}$$

Then we are left to evaluating

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_M \mathcal{H}e^{-f}d\mu\Big) &= \int_M \Big(\frac{\partial \mathcal{H}}{\partial t}e^{-f} - \mathcal{H}\frac{\partial f}{\partial t}e^{-f} - \mathcal{H}^2 e^{-f}\Big)d\mu \\ &= \int_M \Big((2|h_{ij}|^2+B) + \mathcal{H}(\Delta f - |\nabla f|^2 + \mathcal{H}) - \mathcal{H}^2\Big)e^{-f}d\mu \\ &= \int_M \Big((2|h_{ij}|^2+B)e^{-f} - \mathcal{H}\Delta(e^{-f})\Big)d\mu \\ &= \int_M \Big((2|h_{ij}|^2 + (B-\Delta\mathcal{H})\Big)e^{-f}d\mu.\end{aligned}$$

The result then follows by using the last equality and (29) in (28) . □

The energy functional $\mathcal{F}_k$ has been studied by Li [7] under the Ricci flow, where it was used to rule out existence of nontrivial Ricci breathers other than steady gradient soliton. We also remark that the extra terms obtained in the monotonicity formula for $\mathcal{F}_k$ can be used to obtain further useful geometric information about the underlying manifold.

4. $\mathcal{W}$-entropy functional and its monotonicity

Recall that Perelman's $\mathcal{F}$-energy functional is the derivative of Boltzmann-Shannon entropy $\mathcal{E}$. We now present $\mathcal{W}$ -entropy which is a modification of $\mathcal{F}$-energy as discussed in the previous section with inclusion of a positive scaling parameter τ and combination of entropy $\mathcal{E}$. These combine nicely and the resulting entropy yields useful applications. We discuss variation of this entropy and its monotonicity, which will be used to prove nonexistence of nontrivial shrinking breathers on compact manifold.

Definition 4.1. *Let $(M, g(t))$ be a closed manifold evolving by the generalized geometric flow (1). For any function $f \in C^\infty$, we define Perelman's $\mathcal{W}$-entropy as*

$$\mathcal{W}(g,f,\tau) := \int_M \Big[\tau(|\nabla f|^2+\mathcal{H}) + f - n\Big](4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu, \tag{30}$$

where $g(t)$ is a Riemannian metric on n-compact manifold M, f is a smooth function on M and $\tau = T - t$ is a positive scale parameter.

Here we denote $u := (4\pi\tau)^{-\frac{n}{2}}e^{-f}$, the solution of the heat-type equation (11) with the condition $\int_M (4\pi\tau)^{-\frac{n}{2}}e^{-f} = 1$. Then Boltzmann-Shannon entropy $\mathcal{E}$ becomes

$$\mathcal{E}(f(t)) = -\int_M fe^{-f}(4\pi\tau)^{-\frac{n}{2}}d\mu - \frac{n}{2}\log(4\pi\tau).$$

Hence, we write

$$\mathcal{W}(g,f,\tau) = \tau\mathcal{F} - \mathcal{E}(f(t)) - \frac{n}{2}\log(4\pi\tau) - n. \tag{31}$$

We note that the $\mathcal{W}$-entropy is invariant under simultaneous scaling of τ and g.

Lemma 4.2. *Let $\eta > 0$ be any constant and $\phi : M \to M$ be any diffeomorphism. Then*

$$\mathcal{W}(\eta \cdot g, f, \eta \cdot \tau) = \mathcal{W}(g, f, \tau) \quad \text{and} \quad \mathcal{W}(\phi_t^* g, \phi_t^* f, \tau) = \mathcal{W}(g, f, \tau).$$

We now find the monotonicity of $\mathcal{W}$-entropy under the coupled system

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2h_{ij} \\ \frac{\partial f}{\partial t} &= -\Delta f + |\nabla f|^2 - \mathcal{H} + \frac{n}{2\tau} \\ \frac{d\tau}{dt} &= -1. \end{aligned} \tag{32}$$

By a straightforward computation using (31) we have

$$\begin{aligned} \frac{d}{dt}\mathcal{W}(g, f, \tau) &= \tau \frac{d}{dt}\mathcal{F}(g, f) - \mathcal{F}(g, f) - \frac{d}{dt}\mathcal{E}(f) + \frac{n}{2} \\ &= 2\tau \int_M \Big(|h_{ij} + \nabla_i\nabla_j f|^2 + \theta(-\nabla f)\Big)(4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu \\ &\qquad\qquad - 2\int_M (\mathcal{H} + |\nabla f|^2)(4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu + \frac{n}{2\tau} \\ &= 2\tau \int_M \Big(|h_{ij} + \nabla_i\nabla_j f|^2 + \theta(-\nabla f) - \frac{1}{\tau}(\Delta f + \mathcal{H}) + \frac{n}{4\tau^2}\Big) \\ &\qquad\qquad \times (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu \\ &= 2\tau \int_M \Big(|h_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}|^2 + \theta(-\nabla f)\Big)(4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu, \end{aligned}$$

where we have used the identity $\int_M (\Delta f - |\nabla f|^2)e^{-f} = 0$ on a closed manifold. By this we have proved the following

Theorem 4.3. *Let $(g(t), f(t), \tau(t))$ be a solution to the coupled system (32). Then, we have the monotonicity formula*

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int_M \Big(|h_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}|^2 + \theta(-\nabla f)\Big) \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu. \tag{33}$$

If $\Theta(-\nabla f) \geq 0$, $\mathcal{W}(g, f, \tau)$ is monotonically nondecreasing and there is equality if and only if

$$h_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij} = 0 \quad \text{and} \quad \theta(-\nabla f) = 0$$

which implies $\mathcal{H} + \Delta f - \frac{n}{2\tau} = 0$ and $\mathcal{F} - \frac{n}{2} = 0$.

The monotonicity of $\mathcal{W}$ can be applied to rule out existence of nontrivial shrinking breathers in the general geometric flow on the condition that $\theta(-\nabla f) \geq 0$.

Proposition 4.4. *$\mu(g_{ij}(t), \tau - t)$ is nondecreasing along the geometric flow and the monotonicity is strict unless we are on a shrinking gradient soliton. A shrinking breather is necessarily a shrinking gradient soliton.*

5. $\mathcal{W}_+$-entropy over expander

The $\mathcal{W}_+$-entropy is dual to Perelman $\mathcal{W}$-entropy for the shrinkers. The duality is due to the difference in sign which is caused by the antiderivative of $1/\tau$ depending on the circumstance either $t > T$ or $t < T$.

Definition 5.1. *Let $(M, g(t))$ be a closed manifold evolving by the generalized geometric flow (1). For any function $f_+ \in C^\infty$, we define $\mathcal{W}_+$-entropy by*

$$\mathcal{W}_+(g, f_+, \sigma) := \int_M \Big[\sigma(|\nabla f_+|^2 + \mathcal{H}) - f_+ + n\Big](4\pi\sigma)^{-\frac{n}{2}} e^{-f_+} d\mu, \tag{34}$$

with the constraint $\int_M (4\pi\sigma)^{-\frac{n}{2}} e^{-f_+} d\mu = 1$ and $\sigma = t - T$.

An entropy of this form was introduced in [3] by Feldman, Ilmanen and Ni. Let $(4\pi\sigma)^{-\frac{n}{2}} e^{-f_+}$ solves the heat-type equation (11), then f_+ satisfies the backward heat equation

$$\frac{\partial f_+}{\partial t} = -\Delta f_+ + |\nabla f_+| - \mathcal{H} - \frac{n}{2\sigma}. \tag{35}$$

Theorem 5.2. *Under the coupled system*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2h_{ij} \\[2mm] \dfrac{\partial f_+}{\partial t} = -\Delta f_+ + |\nabla f_+|^2 - \mathcal{H} + \dfrac{n}{2(t-T)} \\[2mm] \dfrac{d\tau}{dt} = 1, \end{cases} \tag{36}$$

we have the following monotonicity formula

$$\begin{aligned} &\frac{d}{dt}\mathcal{W}(g, f_+, \sigma) \\ &\quad = 2\sigma \int_M \Big(|h_{ij} + \nabla_i\nabla_j f_+ + \frac{1}{2(t-T)} g_{ij}|^2 + \theta(-\nabla f_+)\Big) \frac{e^{-f_+}}{(4\pi\sigma)^{\frac{n}{2}}} d\mu. \end{aligned} \tag{37}$$

Furthermore, if $\theta(-\nabla f) \geq 0$ then W_+ is monotonically nondecreasing and the monotonicity is strict unless

$$h_{ij} + \nabla_i\nabla_j f_+ + \frac{1}{2(t-T)} g_{ij} = 0 \quad \textit{and} \quad \theta(-\nabla f_+) = 0.$$

The implication of this monotonicity formula is that we can only have gradient expanding solitons but not expanding breathers.

6. Examples of geometric flow

Here, we give some examples of geometric-curvature flows where the energy and entropies are valid. They include **the Hamilton's Ricci flow, Ricci-harmonic map flow, List extended flow, mean curvature flow, Yamabe flows** and some others. We give highlights of the first two and remark that in these cases the error term $\mathcal{D}$ and the quantity $\beta - \Delta\mathcal{H}$ are nonnegative. We may also need some restrictions on curvatures to obtain and apply the monotonicity formulas.

6.1. **Hamilton's Ricci flow.** Let $(M, g(t))$ be a solution to the Hamilton's Ricci flow (Richard Hamilton [5].)

$$\partial_t g_{ij}(t,x) = -2R_{ij}. \tag{38}$$

This is the case where $h_{ij} = R_{ij}$ is the Ricci tensor and $\mathcal{H} = R$ is the scalar curvature on M. Here, the scalar curvature evolves by

$$\partial_t R = \Delta R + 2|R_{ij}|^2.$$

By twice contracted second Bianchi identity $g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R$, which implies

$$2\langle\ div\ h, \nabla f\rangle - \langle \nabla \mathcal{H}, \nabla f\rangle = 0,$$

the quantity $\mathcal{D}(X)$ vanishes identically and

$$\beta - \Delta R \equiv 0.$$

Note that the positivity of curvature is preserved along the Ricci flow. A ground-breaking result in geometric analysis and indeed mathematics in general is the introduction and analytic and geometric applications of Perelman's $\mathcal{F}$-energy and $\mathcal{W}$-entropy [13] to the theory of Ricci flow, which ultimately led to the complete proof of Poincaré conjecture. For more details see [1, 6, 9, 14]

6.2. **Ricci-harmonic map flow.** Let (M, g) and (N, ξ) be compact (without boundary) Riemannian manifolds of dimensions m and n respectively. Let a smooth map $\varphi : M \to N$ be a critical point of the Dirichlet energy integral $E(\varphi) = \int_M |\nabla\varphi|^2 d\mu_g$, where N is isometrically embedded in $\mathbb{R}^d$, $d \geq n$, by the Nash embedding theorem. The configuration $(g(x,t), \varphi(x,t)), t \in [0,T)$ of a one parameter family of Riemannian metrics $g(x,t)$ and a family of smooth maps $\varphi(x,t)$ is defined to be Ricci-harmonic map flow (Reto Müller [10],) if it satisfies the coupled system of nonlinear parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t} g(x,t) = -2Rc(x,t) + 2\alpha \nabla\varphi(x,t) \otimes \nabla\varphi(x,t) \\ \\ \dfrac{\partial}{\partial t}\varphi(x,t) = \tau_g \varphi(x,t), \end{cases} \tag{39}$$

where $Rc(x,t)$ is the Ricci curvature tensor for the metric g, $\alpha(t) \equiv \alpha > 0$ is a time-dependent coupling constant, $\tau_g\varphi$ is the intrinsic Laplacian of φ, which denotes the tension field of map φ and $\nabla\varphi \otimes \nabla\varphi = \varphi^*\xi$ is the pullback of the metric ξ on N via the map φ. See List [8] when the target manifold is one dimensional. Here $h_{ij} = R_{ij} - \alpha\partial_i\varphi\partial_j\varphi =: S_{ij}$, $\mathcal{H} = R_\alpha|\nabla\varphi|^2 =: S$ and

$$\partial_t S = \Delta S + 2|S_{ij}|^2 + 2\alpha|\tau_g\varphi|^2 - 2\dot{\alpha}|\nabla\varphi|^2. \tag{40}$$

Using the twice contracted second Bianchi identity, we have

$$(gij\nabla_i S_{jk} - \frac{1}{2}\nabla_k S)X_j = -\alpha\tau_g\varphi\nabla_j\varphi X_j. \tag{41}$$

Then, $\mathcal{D}(S_{ij}, X) = 2\alpha|\tau_g\varphi - \nabla_X\varphi|^2 - 2\dot{\alpha}|\nabla\varphi|^2$ and $\beta - \Delta S = 2\alpha|\tau_g\varphi|^2 - \dot{\alpha}|\nabla\varphi|^2$ for all X on M. Thus both $\mathcal{D}$ and $\beta - \Delta S$ are nonnegative as long as $\alpha(t)$ is non-increasing in time.

References

[1] B. Chow, S. Chu, D. Glickenstein, C. Guenther, J. Idenberd. T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, The Ricci Flow: Techniques and Applications. Part I, Geometric Aspect, AMS, Providence, RI, 2007.

[2] M. Feldman, T. Ilmanen, D, Knopf, Rotationally symmetric shrinking and expanding gradient Käler-Ricci solitons, J. Diff. Geom. 65, 169-209, 2003

[3] M. Felman, T. Ilmanen, L. Ni, Entropy and reduced distance for Ricci expanders, J. Geom. Anal., 15(1), 49-62, 2005.

[4] H. Guo, R. Philipowski, A. Thalmaier, Entropy and lowest eigenvalue on evolving manifolds, Pacific J. Math. Vol. 264, 61-82, 2013.

[5] R. Hamilton, Three-Manifolds with Positive Ricci Curvature, J. Differential Geometry, 17(2), 253-306, 1982.

[6] B. Kleiner, J. Lott, Note on Perelman's paper. Geometry and Topology, 12, 2587-2858, 2008

[7] J-F. Li, Eigenvalues and energy functionals with monotonicity formulae under Ricci flow, Math. Ann. 338(4), 927-946, 2007.

[8] B. List, Evolution of an extended Ricci flow system, Comm. Anal. Geom., 16(5), 007-1048, 2008.

[9] R. Müller, Monotone volume formulas for geometric flow, J. Reine Angew. Math. 643, 39-57, 2010.

[10] R. Müller, Ricci flow coupled with harmonic map flow, Ann. Sci. Ec. Norm. Sup., 4(45), 101-142, 2012

[11] L. Ni, The entropy formula for linear heat equation, Journal of Geom. Analysis 14(1), 87-100, 2004.

[12] L. Ni, A note on Perelman's Li-Yau-Hamilton inequality, Comm. Anal. Geom., 14, 883-905, 2006.

[13] G. Perelman, The entropy formula for the Ricci flow and its geometric application, arXiv:math.DG/0211159v1 (2002).

[14] Qi S. Zhang, Sobolev Inequalities, Heat Kernels under Ricci Flow and the Poincaré Conjecture, CRC Press, Boca Raton, FL, 2011.

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