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GENERALIZATIONS OF SOME INEQUALITIES FOR THE p-GAMMA, q-GAMMA AND k-GAMMA FUNCTIONS

KWARA NANTOMAH, EDWARD PREMPEH

ABSTRACT. In this paper, we present and prove some generalizations of some inequalities for the p-Gamma, q-Gamma and k-Gamma functions. Our approach makes use of the series representations of the psi, p-psi, q-psi and k-psi functions.

1. Introduction

We begin by recalling some basic definitions related to the Gamma function.

The Psi function, $\psi(t)$ is defined as,

$$\psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \qquad t > 0. \tag{1}$$

where $\Gamma(t)$ is the classical Euler's Gamma function defined by

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\,dx, \qquad t > 0. \tag{2}$$

The p-psi function, $\psi_p(t)$ is defined as,

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \qquad t > 0. \tag{3}$$

where $\Gamma_p(t)$ is the p-Gamma function defined by (see [3], [2])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0. \tag{4}$$

The q-psi function, $\psi_q(t)$ is defined as,

$$\psi_q(t) = \frac{d}{dt}\ln(\Gamma_q(t)) = \frac{\Gamma'_q(t)}{\Gamma_q(t)}, \qquad t > 0. \tag{5}$$

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where $\Gamma_q(t)$ is the q-Gamma function defined by (see [5])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0. \tag{6}$$

Similarly, the k-psi function, $\psi_k(t)$ is defined as follows.

$$\psi_k(t) = \frac{d}{dt} \ln(\Gamma_k(t)) = \frac{\Gamma'_k(t)}{\Gamma_k(t)}, \quad t > 0. \tag{7}$$

where $\Gamma_k(t)$ is the k-Gamma function defined by (see [1], [6])

$$\Gamma_k(t) = \int_0^{\infty} e^{-\frac{x^k}{k}} x^{t-1}\, dx, \quad k > 0, \quad t > 0. \tag{8}$$

In [4], Krasniqi and Shabani proved the following results.

$$\frac{p^{-t} e^{-\gamma t} \Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{p^{1-t} e^{\gamma(1-t)} \Gamma(\alpha+1)}{\Gamma_p(\alpha+1)} \tag{9}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$.

Also in [2], Krasniqi, Mansour and Shabani proved the following.

$$\frac{(1-q)^t e^{-\gamma t} \Gamma(\alpha)}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(1-q)^{t-1} e^{\gamma(1-t)} \Gamma(\alpha+1)}{\Gamma_q(\alpha+1)} \tag{10}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$ and $q \in (0,1)$.

In a recent paper [7], K. Nantomah also proved the following result.

$$\frac{k^{-\frac{t}{k}} e^{-t\left(\frac{k\gamma-\gamma}{k}\right)} \Gamma(\alpha)}{\Gamma_k(\alpha)} \leq \frac{\Gamma(\alpha+t)}{\Gamma_k(\alpha+t)} \leq \frac{k^{\frac{1-t}{k}} e^{(1-t)\left(\frac{k\gamma-\gamma}{k}\right)} \Gamma(\alpha+1)}{\Gamma_k(\alpha+1)} \tag{11}$$

for $t \in (0,1)$, where α is a positive real number.

The main objective of this paper, is to establish and prove some generalizations of the inequalities (9), (10) and (11) as previously established in [4], [2] and [7] respectively.

2. Preliminaries

We present the following auxiliary results.

Lemma 2.1. *The function $\psi(t)$ as defined in (1) has the following series representation.*

$$\psi(t) = -\gamma + (t-1) \sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)} = -\gamma - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{n(n+t)} \tag{12}$$

where γ is the Euler-Mascheroni's constant.

Proof. See [8].

Lemma 2.2. *The function $\psi_p(t)$ as defined in (3) has the following series representation.*

$$\psi_p(t) = \ln p - \sum_{n=0}^{p} \frac{1}{n+t} \tag{13}$$

Proof. See [4].

Lemma 2.3. *The function $\psi_q(t)$ as defined in (5) has the following series representation.*

$$\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}} \tag{14}$$

Proof. See [2].

Lemma 2.4. *The function $\psi_k(t)$ as defined in (7) also has the following series representation.*

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} \tag{15}$$

Proof. See [7]

3. Main Results

We now state and prove the results of this paper.

Lemma 3.1. *Let $a > 0$, $b > 0$ and $t > 1$. Then,*

$$a\gamma + b\ln p + a\psi(t) - b\psi_p(t) > 0$$

Proof. Using the series representations in equations (12) and (13) we have,

$$a\gamma + b\ln p + a\psi(t) - b\psi_p(t) = a(t-1)\sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)} + b\sum_{n=0}^{p} \frac{1}{(n+t)} > 0$$

Lemma 3.2. *Let $a > 0$, $b > 0$ and $\alpha + \beta t > 1$. Then,*

$$a\gamma + b\ln p + a\psi(\alpha + \beta t) - b\psi_p(\alpha + \beta t) > 0$$

Proof. Follows directly from Lemma 3.1

Lemma 3.3. *Let $a > 0$, $b > 0$, $q \in (0, 1)$ and $t > 1$. Then,*

$$a\gamma - b\ln(1-q) + a\psi(t) - b\psi_q(t) > 0$$

Proof. Using the series representations in equations (12) and (14) we have,

$$a\gamma - b\ln(1-q) + a\psi(t) - b\psi_q(t) = a(t-1)\sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)} - b\ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}} > 0$$

Lemma 3.4. *Let $a > 0$, $b > 0$, $q \in (0, 1)$ and $\alpha + \beta t > 1$. Then,*

$$a\gamma - b\ln(1-q) + a\psi(\alpha + \beta t) - b\psi_q(\alpha + \beta t) > 0$$

Proof. Follows directly from Lemma 3.3

Lemma 3.5. *Let $a \geq b > 0$, $k \geq 1$ and $t > 0$. Then,*

$$\frac{ka\gamma - b\gamma}{k} + \frac{b}{k}\ln k + \frac{a-b}{t} + a\psi(t) - b\psi_k(t) \geq 0$$

Proof. Using the series representations in equations (12) and (15) we have,

$$\frac{ka\gamma - b\gamma}{k} + \frac{b}{k}\ln k + \frac{a-b}{t} + a\psi(t) - b\psi_k(t) = t\left[a\sum_{n=1}^{\infty}\frac{1}{n(n+t)} - b\sum_{n=1}^{\infty}\frac{1}{nk(nk+t)}\right] \geq 0$$

Lemma 3.6. *Let $a \geq b > 0$, $k \geq 1$ and $\alpha + \beta t > 0$. Then,*

$$\frac{ka\gamma - b\gamma}{k} + \frac{b}{k}\ln k + \frac{a-b}{\alpha+\beta t} + a\psi(\alpha+\beta t) - b\psi_k(\alpha+\beta t) \geq 0$$

Proof. Follows directly from Lemma 3.5

Theorem 3.7. *Define a function Ω by*

$$\Omega(t) = \frac{p^{b\beta t}e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b}, \quad t \in (0,\infty), \quad p \in N \tag{16}$$

where a, b, α, β are positive real numbers such that $\alpha+\beta t > 1$. Then Ω is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$, the following inequalities are valid.

$$\frac{p^{-b\beta t}e^{-a\beta\gamma t}\Gamma(\alpha)^a}{\Gamma_p(\alpha)^b} < \frac{\Gamma(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b} < \frac{p^{b\beta(1-t)}e^{a\beta\gamma(1-t)}\Gamma(\alpha+\beta)^a}{\Gamma_p(\alpha+\beta)^b}. \tag{17}$$

Proof. Let $f(t) = \ln\Omega(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned} f(t) &= \ln\frac{p^{b\beta t}e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b} \\ &= b\beta t\ln p + a\beta\gamma t + a\ln\Gamma(\alpha+\beta t) - b\ln\Gamma_p(\alpha+\beta t) \end{aligned}$$

Then,

$$\begin{aligned} f'(t) &= a\beta\gamma + b\beta\ln p + a\beta\psi(\alpha+\beta t) - b\beta\psi_p(\alpha+\beta t) \\ &= \beta\left[a\gamma + b\ln p + a\psi(\alpha+\beta t) - b\psi_p(\alpha+\beta t)\right] > 0. \quad \text{(by Lemma 3.2)} \end{aligned}$$

That implies f is increasing on $t \in (0,\infty)$. Hence Ω is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$\Omega(0) < \Omega(t) < \Omega(1)$$

yielding the result.

Theorem 3.8. *Define a function ϕ by*

$$\phi(t) = \frac{(1-q)^{-b\beta t}e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b}, \quad t \in (0,\infty), \quad q \in (0,1) \tag{18}$$

where a, b, α, β are positive real numbers such that $\alpha+\beta t > 1$. Then ϕ is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$, the following inequalities are valid.

$$\frac{(1-q)^{b\beta t}e^{-a\beta\gamma t}\Gamma(\alpha)^a}{\Gamma_q(\alpha)^b} < \frac{\Gamma(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b} < \frac{(1-q)^{b\beta(t-1)}e^{a\beta\gamma(1-t)}\Gamma(\alpha+\beta)^a}{\Gamma_q(\alpha+\beta)^b}. \quad (19)$$

Proof. Let $g(t) = \ln\phi(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned} g(t) &= \ln \frac{(1-q)^{-b\beta t}e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b} \\ &= -b\beta t\ln(1-q) + a\beta\gamma t + a\ln\Gamma(\alpha+\beta t) - b\ln\Gamma_q(\alpha+\beta t) \end{aligned}$$

Then,

$$\begin{aligned} g'(t) &= -b\beta\ln(1-q) + a\beta\gamma + a\beta\psi(\alpha+\beta t) - b\beta\psi_q(\alpha+\beta t) \\ &= \beta\left[a\gamma - b\ln(1-q) + a\psi(\alpha+\beta t) - b\psi_q(\alpha+\beta t)\right] > 0. \quad \text{(by Lemma 3.4)} \end{aligned}$$

That implies g is increasing on $t \in (0,\infty)$. Hence ϕ is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$\phi(0) < \phi(t) < \phi(1)$$

yielding the result.

Theorem 3.9. *Define a function θ by*

$$\theta(t) = \frac{(\alpha+\beta t)^{(a-b)}e^{t(\frac{ka\beta\gamma-b\beta\gamma}{k})}\Gamma(\alpha+\beta t)^a}{k^{-\frac{b\beta t}{k}}\Gamma_k(\alpha+\beta t)^b}, \quad t \in (0,\infty), \quad k \geq 1 \quad (20)$$

where a, b, α, β are positive real numbers such that $a \geq b$. Then θ is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$, the following inequalities are valid.

$$\frac{\alpha^{(a-b)}e^{-t(\frac{ka\beta\gamma-b\beta\gamma}{k})}\Gamma(\alpha)^a}{(\alpha+\beta t)^{(a-b)}k^{\frac{b\beta t}{k}}\Gamma_k(\alpha)^b} \leq \frac{\Gamma(\alpha+\beta t)^a}{\Gamma_k(\alpha+\beta t)^b} \leq \frac{(\alpha+\beta)^{(a-b)}e^{(1-t)(\frac{ka\beta\gamma-b\beta\gamma}{k})}\Gamma(\alpha+\beta)^a}{(\alpha+\beta t)^{(a-b)}k^{\frac{b\beta}{k}(t-1)}\Gamma_k(\alpha+\beta)^b}. \quad (21)$$

Proof. Let $h(t) = \ln\theta(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned} h(t) &= \ln \frac{(\alpha+\beta t)^{(a-b)}e^{t(\frac{ka\beta\gamma-b\beta\gamma}{k})}\Gamma(\alpha+\beta t)^a}{k^{-\frac{b\beta t}{k}}\Gamma_k(\alpha+\beta t)^b} \\ &= (a-b)\ln(\alpha+\beta t) + \frac{b\beta t}{k}\ln k + t(\frac{ka\beta\gamma-b\beta\gamma}{k}) \\ &\quad + a\ln\Gamma(\alpha+\beta t) - b\ln\Gamma_k(\alpha+\beta t) \end{aligned}$$

Then,

$$\begin{aligned} h'(t) &= \frac{ka\beta\gamma-b\beta\gamma}{k} + \frac{b\beta}{k}\ln k + \beta\frac{a-b}{\alpha+\beta t} + a\beta\psi(\alpha+\beta t) - b\beta\psi_k(\alpha+\beta t) \\ &= \beta\left[\frac{ka\gamma-b\gamma}{k} + \frac{b}{k}\ln k + \frac{a-b}{\alpha+\beta t} + a\psi(\alpha+\beta t) - b\psi_k(\alpha+\beta t)\right] \geq 0 \end{aligned}$$

That is as a result of Lemma 3.6. That implies h is increasing on $t \in (0, \infty)$. Hence θ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$\theta(0) \leq \theta(t) \leq \theta(1)$$

yielding the result.

4. Concluding Remarks

We dedicate this section to some remarks concerning our results.

Remark 4.1. If in Theorem 3.7 we set $a = b = \beta = 1$, then the inequalities in (9) are restored.

Remark 4.2. If in Theorem 3.8 we set $a = b = \beta = 1$, then the inequalities in (10) are restored.

Remark 4.3. If in Theorem 3.9 we set $a = b = \beta = 1$, then the inequalities in (11) are restored.

With the foregoing Remarks, the results of [4], [2] and [7] have been generalized.

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References

[1] R. Díaz and E. Pariguan, *On hypergeometric functions and Pachhammer k-symbol*, Divulgaciones Matemtícas **15**(2)(2007), 179-192.
[2] V. Krasniqi, T. Mansour and A. Sh. Shabani, *Some Monotonicity Properties and Inequalities for* Γ *and* ζ *Functions*, Mathematical Communications **15**(2)(2010), 365-376.
[3] V. Krasniqi and F. Merovci, *Logarithmically completely monotonic functions involving the generalized gamma function*, Le Matematiche **LXV**(2010), 15-23.
[4] V. Krasniqi, A. Sh. Shabani, *Convexity Properties and Inequalities for a Generalized Gamma Function*, Applied Mathematics E-Notes **10**(2010), 27-35.
[5] T. Mansour, *Some inequalities for the q-Gamma Function*, J. Ineq. Pure Appl. Math. **9**(1)(2008), Art. 18.
[6] F. Merovci, *Power Product Inequalities for the* Γ_k *Function*, Int. Journal of Math. Analysis **4**(21)(2010), 1007-1012.
[7] K. Nantomah, *Some Inequalities for the k-Gamma Function*, J. Math. Inequal (under review).
[8] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Camb. Univ. Press, 1969.

Kwara Nantomah
Department of Mathematics, University for Development Studies, Navrongo Campus, P. O. Box 24, Navrongo, UE/R, Ghana.
E-mail address: mykwarasoft@yahoo.com, knantomah@uds.edu.gh

Edward Prempeh
Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana.
E-mail address: eprempeh.cos@knust.edu.gh