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ON A MIXED MAX-TYPE RATIONAL SYSTEM OF DIFFERENCE EQUATIONS

NOURESSADAT TOUAFEK, NABILA HADDAD

Abstract. This note deals with the form and the periodicity of positive solutions of the mixed max-type rational system of difference equations

$$x_{n+1} = \frac{x_n y_n}{y_{n-1}}, \quad y_{n+1} = \max\left\{\frac{A_n}{x_n}, y_{n-1}\right\}, \quad n \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(A_n)_{n\in\mathbb{N}_0}$ a positive two-periodic sequence, and initial values x_0, y_0 , $y_{-1} \in (0, +\infty)$.

1. Introduction

Recently, there has been a great interest in solving and studding properties of rational difference equations (see, e.g., [1], [2], [3], [7], [9], [10], [11], [12], [16], [17]). Also numerous papers are devoted to the investigation of some max-type difference equations, see, for instance, [4], [5], [6], [8], [13], [14], [15] and references cited therein.

In this work, we study the behavior of positive solutions of the mixed max-type rational system of difference equations

$$x_{n+1} = \frac{x_n y_n}{y_{n-1}}, \quad y_{n+1} = \max\left\{\frac{A_n}{x_n}, y_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{1}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(A_n)_{n\in\mathbb{N}_0}$ a positive two-periodic sequence, and initial values x_0, y_0 , $y_{-1} \in (0, +\infty)$.

Definition 1 A sequence $(x_n)_{n=-k}^{\infty}$ is said to be eventually periodic with period $p \in \mathbb{N}$ if there is an $n_0 \geq -k$, such that $x_{n+p} = x_n$ for $n \geq n_0$. If $n_0 = -k$, then we say that the sequence $(x_n)_{n=-k}^{\infty}$ is periodic with period p.

2. Closed Form and Periodicity of the Solutions

Now we give the solutions of system (1) in a closed form. For the sake of easier presentation, we formulate and prove our results depending of the relation between the quantities $\frac{A_0}{x_0}$ and y_{-1}.

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Theorem 1 Suppose that $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=-1}^{\infty}$ is a solution of system (1) such that $\frac{A_0}{x_0} \le y_{-1}$. Then the following statements hold:

(1) If $\frac{A_1 y_{-1}}{x_0 y_0} \le y_0$, then

$$x_{2n} = x_0,\ x_{2n+1} = \frac{x_0 y_0}{y_{-1}},\ y_{2n} = y_0,\ y_{2n+1} = y_{-1},\ n \in \mathbb{N}_0. \tag{2}$$

(2) If $\frac{A_1 y_{-1}}{x_0 y_0} \ge y_0$, then

$$x_{2n} = x_0,\ x_1 = \frac{x_0 y_0}{y_{-1}},\ x_{2n+3} = \frac{A_1}{y_0},\ y_{2n+2} = \frac{A_1 y_{-1}}{x_0 y_0},\ y_{2n+1} = y_{-1},\ n \in \mathbb{N}_0. \tag{3}$$

Proof. From the condition $\frac{A_0}{x_0} \le y_{-1}$, we have that

$$x_1 = \frac{x_0 y_0}{y_{-1}}, \tag{4}$$

$$y_1 = \max\left\{\frac{A_0}{x_0}, y_{-1}\right\} = y_{-1}. \tag{5}$$

From (4) and (5) we get

$$x_2 = \frac{x_1 y_1}{y_0} = \frac{\left(\frac{x_0 y_0}{y_{-1}}\right) y_{-1}}{y_0} = x_0 \tag{6}$$

and

$$y_2 = \max\left\{\frac{A_1}{x_1}, y_0\right\} = \max\left\{\frac{A_1 y_{-1}}{x_0 y_0}, y_0\right\}. \tag{7}$$

(1) If $\frac{A_1 y_{-1}}{x_0 y_0} \le y_0$, we have

$$y_2 = y_0. \tag{8}$$

From (5), (6) and (8) we obtain

$$x_3 = \frac{x_2 y_2}{y_1} = \frac{x_0 y_0}{y_{-1}}, \tag{9}$$

$$y_3 = \max\left\{\frac{A_0}{x_2}, y_1\right\} = \max\left\{\frac{A_0}{x_0}, y_{-1}\right\} = y_{-1}. \tag{10}$$

Now, from (8), (9) and (10) we get

$$x_4 = \frac{x_3 y_3}{y_2} = \frac{\left(\frac{x_0 y_0}{y_{-1}}\right) y_{-1}}{y_0} = x_0, \tag{11}$$

$$y_4 = \max\left\{\frac{A_1}{x_3}, y_2\right\} = \max\left\{\frac{A_1 y_{-1}}{x_0 y_0}, y_0\right\} = y_0. \tag{12}$$

Also, from (10), (11) and (12) we have

$$x_5 = \frac{x_4 y_4}{y_3} = \frac{x_0 y_0}{y_{-1}},$$

$$y_5 = \max\left\{\frac{A_0}{x_4}, y_3\right\} = \max\left\{\frac{A_0}{x_0}, y_{-1}\right\} = y_{-1}.$$

By induction we get

$$x_{2n} = x_0,\ x_{2n+1} = \frac{x_0 y_0}{y_{-1}},\ y_{2n} = y_0,\ y_{2n+1} = y_{-1},\ n \in \mathbb{N}_0.$$

This completes the proof of formulas in (2).

(2) Now, we prove formulas in (3). Since $\frac{A_1 y_{-1}}{x_0 y_0} \geq y_0$, from (7), we get

$$y_2 = \frac{A_1 y_{-1}}{x_0 y_0}. \tag{13}$$

From (5), (6) and (13) we have

$$x_3 = \frac{x_2 y_2}{y_1} = \frac{x_0 \left(\frac{A_1 y_{-1}}{x_0 y_0}\right)}{y_{-1}} = \frac{A_1}{y_0}, \tag{14}$$

$$y_3 = \max\left\{\frac{A_0}{x_2}, y_1\right\} = \max\left\{\frac{A_0}{x_0}, y_{-1}\right\} = y_{-1}. \tag{15}$$

Now, from (13), (14) and (15) we obtain

$$x_4 = \frac{x_3 y_3}{y_2} = \frac{\left(\frac{A_1}{y_0}\right) y_{-1}}{\frac{A_1 y_{-1}}{x_0 y_0}} = x_0, \tag{16}$$

$$y_4 = \max\left\{\frac{A_1}{x_3}, y_2\right\} = \max\left\{y_0, \frac{A_1 y_{-1}}{x_0 y_0}\right\} = \frac{A_1 y_{-1}}{x_0 y_0}. \tag{17}$$

Also from (15), (16) and (17)

$$x_5 = \frac{x_4 y_4}{y_3} = \frac{x_0 \left(\frac{A_1 y_{-1}}{x_0 y_0}\right)}{y_{-1}} = \frac{A_1}{y_0},$$

$$y_5 = \max\left\{\frac{A_0}{x_4}, y_3\right\} = \max\left\{\frac{A_0}{x_0}, y_{-1}\right\} = y_{-1}.$$

By induction we get formulas in (3), that is

$$x_{2n} = x_0,\ x_1 = \frac{x_0 y_0}{y_{-1}},\ x_{2n+3} = \frac{A_1}{y_0},\ y_{2n+2} = \frac{A_1 y_{-1}}{x_0 y_0},\ y_{2n+1} = y_{-1},\ n \in \mathbb{N}_0.$$

The proof is complete.

Theorem 2 Suppose that $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=-1}^{\infty}$ is a solution of system (1) such that $\frac{A_0}{x_0} \geq y_{-1}$. Then the following statements hold:

(1) If $\frac{A_1 y_{-1}}{x_0 y_0} \leq y_0$, then

$$x_{2n+2} = \frac{A_0}{y_{-1}},\ x_{2n+1} = \frac{x_0 y_0}{y_{-1}},\ y_{2n} = y_0,\ y_{2n+1} = \frac{A_0}{x_0},\ n \in \mathbb{N}_0. \tag{18}$$

(2) If $\frac{A_1 y_{-1}}{x_0 y_0} \geq y_0$, then

$$x_{2n+2} = \frac{A_0}{y_{-1}},\ x_1 = \frac{x_0 y_0}{y_{-1}},\ x_{2n+3} = \frac{A_1}{y_0},\ y_{2n+2} = \frac{A_1 y_{-1}}{x_0 y_0},\ y_{2n+1} = \frac{A_0}{x_0},\ n \in \mathbb{N}_0. \tag{19}$$

Proof. From the condition $\frac{A_0}{x_0} \geq y_{-1}$, we have that

$$x_1 = \frac{x_0 y_0}{y_{-1}}, \tag{20}$$

$$y_1 = \max\left\{\frac{A_0}{x_0}, y_{-1}\right\} = \frac{A_0}{x_0}. \tag{21}$$

From (20) and (21) we get

$$x_2 = \frac{x_1 y_1}{y_0} = \frac{\left(\frac{x_0 y_0}{y_{-1}}\right)\frac{A_0}{x_0}}{y_0} = \frac{A_0}{y_{-1}}, \tag{22}$$

and

$$y_2 = \max\left\{\frac{A_1}{x_1}, y_0\right\} = \max\left\{\frac{A_1 y_{-1}}{x_0 y_0}, y_0\right\}. \tag{23}$$

(1) If $\frac{A_1 y_{-1}}{x_0 y_0} \le y_0$, we get

$$y_2 = y_0. \tag{24}$$

From (21), (22) and (24) we have

$$x_3 = \frac{x_2 y_2}{y_1} = \frac{\left(\frac{A_0}{y_{-1}}\right) y_0}{\frac{A_0}{x_0}} = \frac{x_0 y_0}{y_{-1}}, \tag{25}$$

$$y_3 = \max\left\{\frac{A_0}{x_2}, y_1\right\} = \max\left\{\frac{A_0}{\frac{A_0}{y_{-1}}}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0}. \tag{26}$$

Now, from (24), (25) and (26) we obtain

$$x_4 = \frac{x_3 y_3}{y_2} = \frac{\left(\frac{x_0 y_0}{y_{-1}}\right)\left(\frac{A_0}{x_0}\right)}{y_0} = \frac{A_0}{y_{-1}}, \tag{27}$$

$$y_4 = \max\left\{\frac{A_1}{x_3}, y_2\right\} = \max\left\{\frac{A_1 y_{-1}}{x_0 y_0}, y_0\right\} = y_0. \tag{28}$$

Also from (26), (27) and (28) we get

$$x_5 = \frac{x_4 y_4}{y_3} = \frac{\left(\frac{A_0}{y_{-1}}\right) y_0}{\frac{A_0}{x_0}} = \frac{x_0 y_0}{y_{-1}},$$

$$y_5 = \max\left\{\frac{A_0}{x_4}, y_3\right\} = \max\left\{\frac{A_0}{\frac{A_0}{y_{-1}}}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0}.$$

By induction we have

$$x_{2n+2} = \frac{A_0}{y_{-1}},\ x_{2n+1} = \frac{x_0 y_0}{y_{-1}},\ y_{2n} = y_0,\ y_{2n+1} = \frac{A_0}{x_0},\ n \in \mathbb{N}_0.$$

This completes the proof of formulas in (18).

(2) Now, we prove formulas in (19). Since $\frac{A_1 y_{-1}}{x_0 y_0} \ge y_0$, from (23) we have

$$y_2 = \frac{A_1 y_{-1}}{x_0 y_0}. \tag{29}$$

From (21), (22) and (29) we get

$$x_3 = \frac{x_2 y_2}{y_1} = \frac{\left(\frac{A_0}{y_{-1}}\right)\left(\frac{A_1 y_{-1}}{x_0 y_0}\right)}{\frac{A_0}{x_0}} = \frac{A_1}{y_0}, \tag{30}$$

$$y_3 = \max\left\{\frac{A_0}{x_2}, y_1\right\} = \max\left\{\frac{A_0}{\frac{A_0}{y_{-1}}}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0}. \tag{31}$$

Now, from (29), (30) and (31) we obtain

$$x_4 = \frac{x_3 y_3}{y_2} = \frac{\left(\frac{A_1}{y_0}\right)\left(\frac{A_0}{x_0}\right)}{\frac{A_1 y_{-1}}{x_0 y_0}} = \frac{A_0}{y_{-1}}, \tag{32}$$

$$y_4 = \max\left\{\frac{A_1}{x_3}, y_2\right\} = \max\left\{\frac{A_1}{\frac{A_1}{y_0}}, \frac{A_1 y_{-1}}{x_0 y_0}\right\} = \frac{A_1 y_{-1}}{x_0 y_0}. \tag{33}$$

Also from (31), (32) and (33) we get

$$x_5 = \frac{x_4 y_4}{y_3} = \frac{\left(\frac{A_0}{y_{-1}}\right)\left(\frac{A_1 y_{-1}}{x_0 y_0}\right)}{\frac{A_0}{x_0}} = \frac{A_1}{y_0},$$

$$y_5 = \max\left\{\frac{A_0}{x_4}, y_3\right\} = \max\left\{\frac{A_0}{\frac{A_0}{y_{-1}}}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0}.$$

By induction we get the formulas in (19), that is

$$x_{2n+2} = \frac{A_0}{y_{-1}},\ x_1 = \frac{x_0 y_0}{y_{-1}},\ x_{2n+3} = \frac{A_1}{y_0},\ y_{2n+2} = \frac{A_1 y_{-1}}{x_0 y_0},\ y_{2n+1} = \frac{A_0}{x_0},\ n \in \mathbb{N}_0.$$

The proof is complete.

Corollary 1 Let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq -1}$ be a solution of system (1). Then $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq -1}$are either periodic (or eventually periodic) of period two.

Proof. It follows from Theorem 1 and Theorem 2.

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References

[1] C. Çinar, On the positive solutions of the difference equation $x_{n+1} = x_{n-1}/(1 + x_n x_{n-1})$, Appl. Math. Comput., 150, 21-24, 2004.

[2] Q. Din, On a system of rational difference equation, Demonstr. Math., 47, 324-335, 2014.

[3] E. M. Elsayed, On the solutions of higher order rational system of recursive sequences, Math. Balk., New Ser., 21, 287-296, 2008.

[4] E. M. Elsayed and B. D. Iričanin, On a max-type and a min-type difference equation, Appl. Math. Comput., 215, 608–614, 2009.

[5] E. M. Elsayed and S. Stević, On the max-type equation $x_{n+1} = \max\{A/x_n, x_{n-2}\}$, Nonlinear Anal. TMA., 71, 910-922, 2009.

[6] E. M. Elsayed, B. D. Iričanin and S. Stević, On the max-type equation $x_{n+1} = \max\{A_n/x_n, x_{n-1}\}$, Ars. Combin., 95, 187-192, 2010.

[7] T. F. Ibrahim and N. Touafek, On a third order rational difference equation with variable coefficients, Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms, 20, 251-264, 2013.

[8] B. D. Iricănin and N. Touafek, On a second order max-type system of difference equations, Indian J. Math., 54, 119-142, 2012.

[9] A.S. Kurbanli, On the behavior of solutions of the system of rational difference equations: $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}$, $y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}$, and $z_{n+1} = z_{n-1}/(y_n z_{n-1} - 1)$, Discrete Dyn. Nat. Soc., Vol. 2011, Article ID 932362, 12 pages, 2011.

[10] A.S. Kurbanli, C. Çinar and I. Yalçinkaya, On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}$, $y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}$, Math. Comput. Modelling, 53, 1261-1267, 2011.

[11] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, Math. Comput. Modelling, 55, 1987-1997, 2012.

[12] N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sr., 55(103), 217-224, 2012.

[13] N. Touafek and Y. Halim, On max type difference equations: expressions of solutions, Int. J. Nonlinear Sci., 4, 396-402, 2011.
[14] I. Yalçinkaya, On the max-type equation $x_{n+1} = max\{1/x_n, A_n x_{n-1}\}$ with a period-two parameter, Discrete Dyn. Nat. Soc., Vol. 2012, Article ID 327437, 9 pages, 2012.
[15] I. Yalçinkaya, C. Çinar and A. Gelisken, On the recursive sequence $x_{n+1} = max\{x_n, A\}/x_n^2 x_{n-1}$, Discrete Dyn. Nat. Soc., Vol. 2010, Article ID 583230, 13 pages, 2010.
[16] Y. Yazlik, On the solutions and behavior of rational difference equations, J. Comput. Anal. Appl., 17, 584-594, 2014.
[17] Q. Zhang, L. Yang and J. Liu, Dynamics of a system of rational third-order difference equation, Adv. Difference Equ., Vol. 2012, Article ID 136, 6 pages, 2012.

N. Touafek
LMAM Laboratory, Department of Mathematics, University of Jijel, Jijel, Algeria
E-mail address: touafek@univ-jijel.dz, ntouafek@gmail.com

N. Haddad
LMAM Laboratory, University of Jijel, Jijel, Algeria
E-mail address: nabilahaddadt@yahoo.com