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NATURAL METRICS AND BOUNDEDNESS OF SUPERPOSITION OPERATOR ACTING BETWEEN $\mathcal{B}^*_\alpha$ AND $F^*(p,q,s)$

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Abstract. In this paper, we study Lipschitz continuous and boundedness of the superposition operator S_ϕ acting between the hyperbolic $\mathcal{B}^*_\alpha$ and the hyperbolic $F^*(p,q,s)$ spaces. We characterize all entire functions that transform hyperbolic Bloch-type spaces into another by superposition operator. We prove that all superposition operators induced by such entire functions are bounded.

1. Introduction

Let X and Y be two metric spaces of analytic functions on the unit disk and ϕ denotes a complex-valued function in the plane $\mathbb{C}$ such that $\phi \circ f \in Y$ whenever $f \in X$ we say that ϕ acts by superposition from X into Y. If X and Y contain the linear function, then ϕ must be entire function. We denote the unit disc of the complex plane by $\mathbb{D}$. The superposition operator S_ϕ on X is defined by

$$S_\phi(f) = (\phi \circ f), \quad f \in X.$$

If $S_\phi f \in Y$ for $f \in X$, note that if X and Y are also linear spaces, the operator S_ϕ is linear if and only if ϕ is a linear function that fixes the origin. Let $H(\mathbb{D})$ denote the classes of functions holomorphic in the unit disc $\mathbb{D}$. A function $f \in H(\mathbb{D})$ belongs to α-Bloch space $\mathcal{B}^\alpha, 0 < \alpha < \infty$ if

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f'(z)| < \infty.$$

The little α-Bloch space $\mathcal{B}_{\alpha,0}$ consisting of all $f \in \mathcal{B}_\alpha$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

If (X,d) is a metric space, we denote the open and closed balls with center x and radius $r > 0$ by $B(x,r) := \{y \in X : d(y,x) < r\}$ and $\bar{B}(x,r) := \{y \in X : d(x,y) \leq r\}$, respectively.

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Superposition operators in Bergman space A^p studied in [5, 6]. Later, Buckley and Vukotic considered superposition operators from Besov spaces into Bergman spaces in [3], univalent interpolation in Besov spaces and superposition into Bergman spaces in [4], superposition operators between the Bloch space and Bergman spaces were characterised in [1], and those between the conformally invariant Q_p spaces and Bloch-type spaces in [15].

2. Basic concepts and propositions

The class of hyperbolic functions is a subset of the class $B(\mathbb{D})$ of all analytic functions f in the unit disc $\mathbb{D}$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}$.

They are usually defined by using either the hyperbolic derivative $f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ of $f \in B(\mathbb{D})$, and $\phi^*(z) = \frac{1-|z|}{1-|\phi(z)|^2}\phi'(z)$ (cf. [9]).

The hyperbolic Bloch space is defined as follows:

Definition 2.1. *(see [11]) For $0 < \alpha \leq 1$, a function $f \in B(\mathbb{D})$ is said to belong to the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ if*

$$\|f\|_{\mathcal{B}^*_\alpha} = \sup_{z\in\mathbb{D}} f^*(z)(1-|z|^2)^\alpha < \infty.$$

The little hyperbolic Bloch-type class $\mathcal{B}^*_{\alpha,0}$ consists of all $f \in \mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1} f^*(z)(1-|z|^2)^\alpha = 0.$$

The Schwarz-Pick lemma implies $\mathcal{B}^*_\alpha = B(\mathbb{D})$ for all $\alpha \geq 1$ with $\|f\|_{\mathcal{B}^*_\alpha} \leq 1$, and therefore the hyperbolic α-Bloch-classes are of interest only when $0 < \alpha < 1$. The usual α-Bloch-spaces and their norms are denoted by the same symbols but without $*$.

Pérez-Gonzálezet al. defined a natural metric on the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ in [11] as

$$d(f, g; \mathcal{B}^*_\alpha) := d_{\mathcal{B}^*_\alpha}(f, g) + \|f - g\|_{\mathcal{B}_\alpha} + |f(0) - g(0)|,$$

where

$$d_{\mathcal{B}^*_\alpha}(f, g) := \sup_{a\in\mathbb{D}} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| (1-|z|^2)^\alpha$$

for $f, g \in \mathcal{B}^*_\alpha$.

Definition 2.2. *(see [10]) For $0 < p, s < \infty$, $-2 < q < \infty$, the hyperbolic class $F^*(p, q, s)$ consists of those functions $f \in B(\mathbb{D})$ for which*

$$\|f\|^p_{F^*(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q g^s(z, a) dA(z) < \infty.$$

Moreover, we say that $f \in F^*(p, q, s)$ belongs to the class $F^*_0(p, q, s)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q g^s(z, a) dA(z) = 0.$$

Where dA is the normalized 2-dimensional Lebesgue measure on $\mathbb{D}$, $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$ is the Green's function of $\mathbb{D}$ with $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation related to the point $a \in \mathbb{D}$. Note that hyperbolic classes are not linear spaces, since they consist of functions that are self-maps of $\mathbb{D}$.

For $f, g \in F^*(p,q,s)$, defined their distance by

$$d(f,g;F^*(p,q,s)) := d_{F^*(p,q,s)}(f,g) + \|f-g\|_{F(p,q,s)} + |f(0)-g(0)|,$$

where

$$d_{F^*(p,q,s)}(f,g) := \left(\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f^*(z)-g^*(z)|^p (1-|z|^2)^q g^s(z,a) dA(z)\right)^{\frac{1}{p}},\ with\ p \geq 1.$$

The following result on the complete metric spaces $d(.,.;\mathcal{B}^*_\alpha)$ was proved in [11].

Proposition 2.1. *The class $\mathcal{B}^*_\alpha$ equipped with the metric $d(.,.;\mathcal{B}^*_\alpha)$ is a complete metric space. Moreover,$\mathcal{B}^*_{\alpha,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_\alpha$.*

The following result on the complete metric spaces $d(.,.;F^*(p,q,s))$ was proved in [8].

Proposition 2.2. *The class $F^*(p,q,s)$ equipped with the metric $d(.,.;F^*(p,q,s))$ is a complete metric space. Moreover, $F^*_0(p,q,s)$ is a closed (and therefore complete) subspace of $F^*(p,q,s)$.*

Definition 2.3. *The superposition operator $S_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is said to be bounded, if there is a positive constant C such that $\|S_\phi f\|_{F^*(p,q,s)} \leq C\|f\|_{\mathcal{B}^*_\alpha}$ for all $f \in \mathcal{B}^*_\alpha$.*

Definition 2.4. *The superposition operator $S_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is said to be compact, if it maps any ball in $\mathcal{B}^*_{p,\alpha}$ onto a pre-compact set in $F^*(p,q,s)$.*

Lemma 2.1. *For each positive number δ and for every sequence (γ_n) of complex numbers such that $\gamma_0 = 0,\ |\gamma_1| \geq 5\delta,\ |arg\gamma_1 - \theta_0| < \frac{\pi}{4},\ arg\gamma_n \to \theta_0,\ or\ arg\gamma_n \uparrow \theta_0$ and*

$$|\gamma_n| \geq \max\left\{3|\gamma_{n-1}|, \sum_{k=1}^{n-1} |\gamma_k - \gamma_{k-1}|\right\} \quad for\ all \quad n \geq 2, \tag{1}$$

there exists a domain Ω with the following properties:

(i) Ω is simply connected;

(ii) Ω contains the infinite polygonal line $L = \bigcup_{n=1}^{\infty}[\gamma_{n-1},\gamma_n]$, where $[\gamma_{n-1},\gamma_n]$ denotes the line segment from γ_{n-1} to γ_n;

(iii) There exists a conformal mapping f of Δ onto Ω which takes the origin to a prescribed point belongs to $\mathcal{B}^*$;

(iv) $dist(\gamma, \partial\Delta) = \delta$ for each point γ on L.

Proof. The proof is very similar to the proof of lemma 3.3 in [2].

3. Lipschitz continuous and boundedness of superposition operators S_ϕ from $\mathcal{B}^*_\alpha$ to $F^*(p,q,s)$

Lemma 3.1. *(see [13]) For $0 < \alpha < 1$, there exist two functions $f, g \in \mathcal{B}^*_\alpha$ such that for some constant ϵ,*

$$(|f'(z)| + |g'(z)|) \geq \frac{\epsilon}{(1-|z|^2)^\alpha} > 0, \qquad \forall\ z\ \in\ \mathbb{D}. \tag{2}$$

Throughout this paper we assume that

$$(\phi^*(f(z))+\phi^*(g(z)))\geq \frac{\epsilon}{(1-|f(z)|^2)^{\alpha}}>0, \qquad \forall\, z\ \in\ \mathbb{D}. \tag{3}$$

Now, we give the following result.

Theorem 3.1. *Assume ϕ is non-constant analytic mapping from $\mathbb{D}$ into itself and let $0<\alpha\leq 1$, $0\leq p<\infty$, $-1<q<\infty$ and $0\leq s\leq 1$. Suppose that (3) is satisfied. Then the following statements are equivalent:*
(i) $S_\phi:\mathcal{B}^*_\alpha\to F^*(p,q,s)$ is bounded;

(ii) $S_\phi:\mathcal{B}^*_\alpha\to F^*(p,q,s)$ is Lipschitz continuous;

(iii)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}}(1-|z|^2)^q g^s(z,a)dA(z)<\infty.$$

Proof. To prove (i)$\Leftrightarrow$ (iii), first assume that (iii) holds, for any $f\in\mathcal{B}^*_\alpha$, and $|f(z)|$ is bounded. Then, we obtain

$$\begin{aligned}\|S_\phi f\|_{F^*(p,q,s)} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}((\phi\circ f)^*(z))^p(1-|z|^2)^q g^s(z,a)dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(\phi^*(f(z))^p|f'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\\ &\leq \|\phi(f(z))\|^p_{\mathcal{B}^*_\alpha}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}}(1-|z|^2)^q g^s(z,a)dA(z)<\infty.\end{aligned}$$

Hence, it follows that (i) holds.
Conversely, by assuming that (i) holds and (3), there exists a constant $\epsilon>0$ such that $(\phi^*(f(z))+\phi^*(g(z)))\geq\frac{\epsilon}{(1-|f(z)|^2)^{\alpha}}>0$, where $f,\ g\in\mathcal{B}^*_\alpha$, and $\|S_\phi f\|_{F^*(p,q,s)}\leq C\|\phi(f(z))\|_{\mathcal{B}^*_\alpha}$.
We can assume $|f'(z)|\leq|g'(z)|$. Then, we have

$$\begin{aligned}&\|S_\phi f\|_{F^*(p,q,s)}+\|S_\phi g\|_{F^*(p,q,s)}\\ &\geq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left[((\phi\circ f)^*(z))^p+((\phi\circ g)^*(z))^p\right]\, g^p(z,a)dA(z)\\ &\geq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left[(\phi^*(f(z)))^p|f'(z)|^2+(\phi^*(g(z)))^p|g'(z)|^p\right]\, g^p(z,a)dA(z)\\ &\geq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left[(\phi^*(f(z)))^p+(\phi^*(g(z)))^p\right]|f'(z)|^p\ (1-|z|^2)^q g^s(z,a)dA(z)\\ &\geq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left[\phi^*(f(z))+\phi^*(g(z))\right]^p|f'(z)|^p\ (1-|z|^2)^q g^s(z,a)dA(z)\\ &\geq \epsilon^p\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^p}{(1-|f(z)|^p)^{p\alpha}}(1-|z|^2)^q g^s(z,a)dA(z).\end{aligned}$$

Then, we have

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}}(1-|z|^2)^q g^s(z,a)dA(z)\leq\|S_\phi f\|^p_{F^*(p,q,s)}+\|S_\phi g\|^p_{F^*(p,q,s)}<\infty.$$

So (iii) is satisfied.

To prove (ii)$\Leftrightarrow$ (iii), assume first that $S_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(\phi\circ f, \phi\circ g; F^*(p,q,s)) \le Cd(\phi(f(z)), \phi(g(z)); \mathcal{B}^*_\alpha), \quad \text{for all } f,g\in \mathcal{B}^*_\alpha.$$

Taking $\phi(g) = 0$, this implies

$$\|\phi\circ f\|_{F^*(p,q,s)} \le C\big(\|\phi(f(z))\|_{\mathcal{B}^*_\alpha} + \|\phi(f(z))\|_{\mathcal{B}_\alpha} + |\phi(f(0))|\big), \quad \text{for all } f\in \mathcal{B}^*_\alpha. \tag{4}$$

The assertion (iii) for $\alpha = 1$, follows by choosing $f(z) = z$ in (4). Moreover, from (3), for $f, g \in \mathcal{B}^*_\alpha$, we deduce that

$$\big(\phi^*(f(z)) + \phi^*(g(z))\big)(1-|z|^2)^\alpha \ge \epsilon > 0, \quad \text{for all } z\in\mathbb{D}. \tag{5}$$

Therefore, combining (4) and (5), we have

$$\begin{aligned}
&\|\phi(f(z))\|_{\mathcal{B}^*_\alpha} + \|\phi(g(z))\|_{\mathcal{B}^*_\alpha} + \|\phi(f(z))\|_{\mathcal{B}_\alpha}\\
&+\|\phi(g(z))\|_{\mathcal{B}_\alpha} + |\phi(f(0))| + |\phi(g(0))|\\
&\ge \|\phi\circ f\|_{F^*(p,q,s)} + \|\phi\circ g\|_{F^*(p,q,s)}\\
&\ge \epsilon^2 \int_{\mathbb{D}} \frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}}(1-|z|^2)^q g^s(z,a)\,dA(z).
\end{aligned}$$

For which the assertion (iii) follows.

Assume now that (iii) is satisfied, we have

$$\begin{aligned}
&d(\phi\circ f, \phi\circ g; F^*(p,q,s))\\
&= d_{F^*(p,q,s)}(\phi\circ f, \phi\circ g) + \|\phi\circ f - \phi\circ g\|_{F(p,q,s)}\\
&+\big|\phi(f(0)) - \phi(g(0))\big|\\
&\le d_{\mathcal{B}^*_\alpha}(\phi(f(z)), \phi(g(z)))\left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|f'(z)|^p}{(1-|f(z)|^2)^{2\alpha}} F^*(p,q,s)\,dA(z)\right)^{\frac{1}{p}}\\
&+\|\phi(f(z)) - \phi(g(z))\|_{\mathcal{B}_\alpha}\left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}} F^*(p,q,s)\,dA(z)\right)^{\frac{1}{p}}\\
&+|\phi(f(0)) - \phi(g(0))| \le C\, d(\phi(f(z)), \phi(g(z)); \mathcal{B}^*_\alpha).
\end{aligned}$$

Thus $S_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is Lipschitz continuous. This completes the proof.

Remark 3.1. *We know that a superposition operator $S_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is said to be bounded if there is a positive constant C such that $\|S_\phi f\|_{F^*(p,q,s)} \le C\|\phi(f(z))\|_{\mathcal{B}^*_\alpha}$; for all $f\in\mathcal{B}^*_\alpha$. Theorem 3.1 shows that $S_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is bounded if and only if it is Lipschitz continuous, that is, if there exists a positive constant C such that $d(\phi\circ f, \phi\circ g; F^*(p,q,s)) \le Cd(\phi(f(z)), \phi(g(z)); \mathcal{B}^*_\alpha)$, for all $f, g \in \mathcal{B}^*_\alpha$.*

4. Compactness of superposition operator

The compactness of the superposition operator S_ϕ has been defined by Definition 2.4. The following proposition is based on the compactness of the operator S_ϕ.

Proposition 4.1. *Assume ϕ is analytic mapping from $\mathbb{D}$ into itself. Let $0 < \alpha \le 1$, $-1 < q < \infty$ and $0 \le p, s < \infty$. If $S_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is compact, it maps closed balls onto compact sets.*

Proof. If $B \subset \mathcal{B}_\alpha^*$ is a closed ball and $g \in F^*(p,q,s)$ belongs to the closure of $S_\phi(B)$, we can find a sequence $(f_n)_{n=1}^\infty \subset B$ such that $\phi \circ f_n$ converges to $g \in F^*(p,q,s)$ as $n \to \infty$. But $(f_n)_{n=1}^\infty$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^\infty$ converging uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. As in earlier arguments of proposition 2.1 in [11], we get a positive estimate which shows that f must belong to the closed ball B. On the other hand, also the sequence $\phi \circ (f_{n_j})_{j=1}^\infty$ converges uniformly on compact subsets to an analytic function, which is $g \in F^*(p,q,s)$. We get $g = \phi \circ f$, i.e. g belongs to $S_\phi(B)$. Thus, this set is closed and also compact.

Compactness of superposition operators acting between $\mathcal{B}_\alpha^*$ and $F^*(p,q,s)$ classes can be given in the following result.

Theorem 4.1. *Assume ϕ is analytic mapping from $\mathbb{D}$ into itself. $0 \le p < \infty$, $-1 < q < \infty$ and $0 \le s \le 1$. Then $S_\phi : \mathcal{B}_\alpha^* \to F^*(p,q,s)$ is compact if*

$$\lim_{r \to 1^-} \sup_{a \in \mathbb{D}} \int_{|f(z)|>r} \frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}} (1-|z|^2)^q g^s(z,a)\, dA(z) = 0. \tag{6}$$

Proof. We first assume that (6) holds. Let $B := \bar{B}(g,\delta) \subset \mathcal{B}_\alpha^*$, $g \in \mathcal{B}_\alpha^*$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^\infty \subset B$ be any sequence. We show that its image has a convergent subsequence in $F^*(p,q,s)$, which proves the compactness of S_ϕ by definition.

Again, $(f_n)_{n=1}^\infty \subset B(\mathbb{D})$ is normal, hence, there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. By Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^\infty$ converges uniformly on the compact subsets of $\mathbb{D}$ to f_{n_j}. It follows that also the sequences $(\phi \circ f_{n_j})_{j=1}^\infty$ and $(\phi \circ f'_{n_j})_{j=1}^\infty$ converge uniformly on the compact subsets of $\mathbb{D}$ to $\phi \circ f$ and $\phi \circ f'$, respectively. Moreover, $f \in B \subset \mathcal{B}_\alpha^*$ since for any fixed $R, 0 < R < 1$, the uniform convergence yield

$$\begin{aligned}
&\sup_{|z|\le R} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| (1-|z|^2)^\alpha \\
&+ \sup_{|z|\le R} |f'(z) - g'(z)|(1-|z|^2)^\alpha + |f(0) - g(0)| \\
&= \lim_{j\to\infty} \sup_{|z|\le R} \left| \frac{f'_{n_j}(z)}{1-|f_{n_j}(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| (1-|z|^2)^\alpha \\
&+ \lim_{j\to\infty} \big(\sup_{|z|\le R} |f'_{n_j}(z) - g'(z)|(1-|z|^2)^\alpha + |f_{n_j}(0) - g(0)| \big) < \delta.
\end{aligned}$$

Hence, $d(f,g;\mathcal{B}_\alpha^*) \le \delta$.

Let $\varepsilon > 0$. Since (6) is satisfied, we may fix $r, 0 < r < 1$, such that

$$\sup_{a \in \mathbb{D}} \int_{|f(z)|>r} \frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}} (1-|z|^2)^q g^s(z,a)\, dA(z) \le \varepsilon.$$

By the uniform convergence, we may fix $N_1 \in \mathbb{N}$ such that

$$|\phi(0) \circ f_{n_j} - \phi(0) \circ f| \le \varepsilon, \qquad \text{for all } j \ge N_1. \tag{7}$$

The condition (6) is known to imply the compactness of $S_\phi : \mathcal{B}_\alpha \to F(p,q,s)$, hence possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$\|\phi\circ f_{n_j}-\phi\circ f\|_{F(p,q,s)}\leq\varepsilon,\quad \text{for all } j\geq N_2;\;\; N_2\in\mathbb{N}. \tag{8}$$

Since $(f_{n_j})_{j=1}^{\infty}\subset B$, $f\in B$ and $|f'_{n_j}(z)|\leq|f'(z)|$ it follows that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{|f(z)|\geq r}\big[(\phi\circ f_{n_j})^*(z)-(\phi\circ f)^*(z)\big]^p(1-|z|^2)^q g^s(z,a)dA(z)\\
&\leq\sup_{a\in\mathbb{D}}\int_{|f(z)|\geq r}\big[(\phi^*(f_{n_j}(z))|f'_{n_j}(z)|-(\phi^*(f(z))|f'(z)|\big]^p(1-|z|^2)^q g^s(z,a)dA(z)\\
&\leq\sup_{a\in\mathbb{D}}\int_{|f(z)|\geq r}\big[(\phi^*(f_{n_j}(z))-(\phi^*(f(z))\big]^p|f'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\\
&\leq d_{\mathcal{B}^*_\alpha}(\phi(f_{n_j}(z)),\phi(f(z)))\sup_{a\in\mathbb{D}}\int_{|f(z)|>r}\frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}}(1-|z|^2)^q g^s(z,a)dA(z),
\end{aligned}$$

hence,

$$\sup_{a\in\mathbb{D}}\int_{|f(z)|\geq r}\big[(\phi\circ f_{n_j})^*(z)-(\phi\circ f)^*(z)\big]^p(1-|z|^2)^q g^s(z,a)dA(z)\leq C\varepsilon. \tag{9}$$

On the other hand, by the uniform convergence on the compact disc $\mathbb{D}$, we can find an $N_3\in\mathbb{N}$ such that for all $j\geq N_3$,

$$\left|\frac{\phi'(f_{n_j})(z)}{1-|\phi(f_{n_j}(z))|^2}-\frac{\phi'(f(z))}{1-|\phi(f(z))|^2}\right|\leq\varepsilon.$$

For all z with $|f(z)|\leq r$. Hence, for such j,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{|f(z)|\leq r}\big[(\phi\circ f_{n_j})^*(z)-(\phi\circ f)^*(z)\big]^p(1-|z|^2)^q g^s(z,a)dA(z)\\
&\leq\sup_{a\in\mathbb{D}}\int_{|f(z)|\leq r}\big[\phi^*(f_{n_j}(z))-\phi^*(f(z))\big]^p|f'(z)|^p(1-|z|^2)^q g^s(z,a)\big)dA(z)\\
&\leq\varepsilon\bigg(\sup_{a\in\mathbb{D}}\int_{|f(z)|\leq r}\frac{|f'(z)|^p}{(1-|f(z)|^2)^{p\alpha}}(1-|z|^2)^q g^s(z,a)dA(z)\bigg)^{\frac{1}{p}}\leq C\varepsilon,
\end{aligned}$$

hence,

$$\sup_{a\in\mathbb{D}}\int_{|f(z)|\leq r}\big[(\phi\circ f_{n_j})^*(z)-(\phi\circ f)^*(z)\big]^p(1-|z|^2)^q g^s(z,a)dA(z)\leq C\,\varepsilon. \tag{10}$$

where C is a positive constant which is obtained from (iii) of Theorem 3.1. Combining (7), (8), (9) and (10) we deduce that $f_{n_j}\to f$ in $F^*(p,q,s)$.
The proof is therefore completed.

5. Superposition operators from $\mathcal{B}^*_\alpha$ to $\mathcal{B}^*_\beta$

First we show that S_ϕ maps $\mathcal{B}^*_\alpha$ into $\mathcal{B}^*_\beta$ unless ϕ is a constant, where $0<\beta<\alpha$.

Theorem 5.1. *Let $0<\beta<\alpha$ and ϕ be an entire function.*
Suppose that $|f(z)|\leq\lambda$, where λ is a positive constant and $f\in\mathcal{B}^*_\alpha$. Then the superposition operator S_ϕ maps $\mathcal{B}^*_\alpha$ into $\mathcal{B}^*_\beta$ if and only if ϕ is a constant function.

Proof. If ϕ is a constant, it is obvious that $S_\phi(\mathcal{B}^*_\alpha) \subset \mathcal{B}^*_\beta$. Now assume that ϕ is not a constant. We distinguish two cases to prove that $S_\phi(\mathcal{B}^*_\alpha) \not\subset \mathcal{B}^*_\beta$.

(i) If $0 < \alpha < 1$. Since ϕ is not a constant, there exists a disk $|\gamma - \gamma_0| < r$ and $0 < \gamma_0 < 1$, on which $\phi^*(f(z)) > \delta > 0$.
Let $f(z) = \gamma_0 + r(1-z)^{1-\alpha} \in \mathcal{B}^*_\alpha$. Then, for $z \in \mathbb{D}$,

$$\begin{aligned}(1-|z|^2)^\beta ((\phi \circ f)^*(z)) &= (1-|z|^2)^\beta (\phi^*(f(z)))|f'(z)| \\ &\geq \frac{\delta r(1-\alpha)(1-|z|^2)^\beta}{|1-z|^\alpha}.\end{aligned}$$

The right side of the above inequality tends to infinity as $z \to 1$ along with the positive radius. This shows that $S_\phi(f) = (\phi \circ f) \notin \mathcal{B}^*_\beta$ and $S_\phi(\mathcal{B}^*_\alpha) \not\subset \mathcal{B}^*_\beta$.

(ii) If $\alpha = 1$. Since ϕ is unbounded, there exists a complex sequence (γ_n) with $\gamma_n \to \infty$, as $n \to \infty$ such that $|\phi(\gamma_n)| \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that γ_n satisfies the conditions in Lemma 2.1 with some $\delta > 0$ by adding $\gamma_0 = 0$ and choosing a subsequence if necessary. By Lemma 2.1, there exists a domain Ω and a conformal mapping f of $\mathbb{D}$ onto Ω such that $\gamma_n \in \Omega$ for $n = 0, 1, \ldots$ and $f \in \mathcal{B}^*$. Since any function in $\mathcal{B}^*_\beta$ is bounded and, hence, $S_\phi(f) = (\phi \circ f) \notin \mathcal{B}^*_\beta$ and $S_\phi(\mathcal{B}^*_\alpha) \not\subset \mathcal{B}^*_\beta$, since $(\phi \circ f)$ is unbounded. The proof is completed.

In the next theorem we study superposition operator from $\mathcal{B}^*_\alpha$ to $\mathcal{B}^*_\beta$, *where* $\alpha \leq \beta$.

Theorem 5.2. *Let* $0 < \alpha < 1,\ \alpha \leq \beta$. *Suppose that* $|f(z)| \leq \lambda$, *where* λ *is a positive constant and* $f \in \mathcal{B}^*_\alpha$.*Then for any entire function* ϕ, S_ϕ *is a bounded operator from* $\mathcal{B}^*_\alpha$ *into* $\mathcal{B}^*_\beta$.

Proof. Let $\alpha < 1,\ \alpha \leq \beta,$ and ϕ be an entire function. Let $M > 0$. For a function f with $\|f\|_{\mathcal{B}_\alpha} \leq M$ we have,

$$\phi^*(f(0)) \leq M_1 = \max_{|\gamma|=\lambda} |\phi(\gamma)|,$$
$$\phi^*(f(z)) \leq M_2 = \max_{|\gamma|=\lambda} \phi^*(\gamma), \quad \text{for } z \in \mathbb{D}.$$

Thus

$$\begin{aligned}\|\phi \circ f\|_{\mathcal{B}^*_\beta} &\leq |(\phi \circ f)^*(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta (\phi^*(f(z)))|f'(z)| \\ &\leq M_1 + M_2 \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| \\ &= M_1 + M_2 |f\|_{\mathcal{B}_\alpha} \\ &\leq M_1 + MM_2.\end{aligned}$$

This completes the proof.

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