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OSCILLATION THEOREMS FOR FOURTH-ORDER NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

F. Z. LADRANI, A.HAMMOUDI, A. BENAISSA CHERIF

Abstract. In this paper, we will establish some oscillation criteria for the fourth-order nonlinear dynamic equation on time scales

$$(p(x^{\Delta})^{\alpha})^{\Delta^3}(t) + q(t)(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) + f(t, x(\tau(t))) = 0$$

on a time scales, where α is a quotient of odd positive integer and $\alpha > 0$.

1. Introduction

Consider the fourth-order nonlinear delay dynamic equation with damping

$$(p(x^{\Delta})^{\alpha})^{\Delta^3}(t) + q(t)(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) + f(t, x(\tau(t))) = 0 \qquad (1)$$

on a time scale $\mathbb{T}$, where α is a quotient of odd positive integer and $\alpha > 0$. Since we are interested in oscillation, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$.

The equation (1) will be studied under the following assumptions:

$\mathcal{A}_1$: $f : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function verifying

$$xf(t, x) > 0, \qquad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}, \ x \in \mathbb{R} \setminus \{0\}.$$

$\mathcal{A}_2$: There exist a function $r : \mathbb{T} \longrightarrow \mathbb{R}$ which is a positive and rd-continuous, such that

$$\frac{f(t, x)}{x^{\alpha}} \geq r(t), \qquad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}, \ x \in \mathbb{R} \setminus \{0\}.$$

$\mathcal{A}_3$: p and q are positive real-valued and rd-continuous functions defined on $\mathbb{T}$, $1 - q(t)\mu(t) \neq 0$, $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$.

By a solution of (1) we mean a nontrivial real-valued function $x \in C_{rd}^4([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$, $T_x \in [t_0, \infty)_{\mathbb{T}}$ which satisfies (1) on $[T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration.

A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

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The theory of time scales was introduced by Hilger [1] in order to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [2],[3], summarize and organize much of time scale calculus.

The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology, natural and social sciences. In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales.

Recently, there has been an increasing interest in obtaining sufficient conditions for oscillation and nonoscillation of solutions of various equations on time scales, we refer the reader to the articles [6],[7],[1],[12],[4],[11] and the references cited therein. S.H. Saker [4] studied a class of second-order delay dynamic equation with a half-linear damping

$$(p(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta}+q(t)\,x^{\gamma}(t)=0.$$

Erbe et al [7] studied a class of second-order delay dynamic equations with a nonlinear damping

$$(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta}+p(t)\left(x^{\Delta^{\sigma}}(t)\right)^{\gamma}(t)+q(t)\,f(t,x(\tau(t)))=0.$$

Erbe et al [6] investigated a third-order dynamic equation with a half-linear damping

$$x^{\Delta^{3}}(t)+q(t)\,x(t)=0.$$

Said R. Grace et al [8] studied oscillation of the fourth-order dynamic equations

$$x^{\Delta^{4}}(t)+q(t)\,x^{\lambda}(t)=0.$$

Yunsong et al [11] studied a fourth-order dynamic equation with a half-linear damping

$$x^{\Delta^{4}}(t)+p(t)\,x^{\gamma}(\tau(t))=0.$$

Tongxing Li et al [9] studied oscillation for the fourth-order delay dynamic equation on time scales

$$(rx^{\Delta^{3}})^{\Delta}(t)+p(t)\,x(\tau(t))=0.$$

Ravi P et al [12] studied oscillation of unbounded solutions to a fourth-order delay dynamic equation with a half-linear damping

$$(r(x^{\Delta^{3}})^{\gamma})^{\Delta}(t)+p(t)\,(x^{\Delta^{3}})^{\gamma}(t)+q(t)\,x^{\gamma}(\tau(t))=0.$$

So far, there are any results on oscillatory of (1). Hence the aim of this paper is to give some oscillation criteria for this equation.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale is not bounded above and is of the form $[t_0,\infty)_{\mathbb{T}}$. It follows that the jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ defined by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\},\qquad \rho(t):=\sup\{s\in\mathbb{T}:s<t\},$$

where $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$ denotes the empty set. The point $t\in\mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t)=t,\ \rho(t)<t,\ \sigma(t)=$

$t,\ \sigma(t) > t$, respectively. The graininess function μ, for a time scale, is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \longrightarrow \mathbb{R}$, the function $f^{\sigma}(t)$ denotes $f(\sigma(t))$. The $\Delta-$derivative of $f : \mathbb{T} \longrightarrow \mathbb{R}$ at a right dense point t is defined by

$$f^{\Delta}(t) = \lim_{s\longrightarrow t} \frac{f(t) - f(s)}{t - s}.$$

If t is not right scattered, then the derivative is defined by

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}.$$

A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

A function f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

The $\Delta-$derivative f^{Δ} and the shift f^{σ} of a function f are related by the equation

$$f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t).$$

We will use the following product and quotient rules for the derivative of the product fg and the quotient $\dfrac{f}{g}$ (where $g^{\sigma}(t)g(t) \neq 0$) of two differentiable functions f and g,

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta},$$

and

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - g^{\Delta}f}{gg^{\sigma}}.$$

For $a, b \in \mathbb{T}$, and for a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t,$$

and the improper integrals are defined in the usual way by

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\longrightarrow\infty} \int_a^b f(t)\Delta t.$$

The following result is used frequently in the remainder of this paper.

Lemma 2.1. [2] *Assume* $1+\mu(t)p(t) \neq 0$ *and fix* $t_0 \in \mathbb{T}$. *Then* $e_p(\cdot, t_0)$ *is a solution of the initial value problem*

$$y^{\Delta}(t) = p(t)\, y(t), \qquad y(t_0) = y_0.$$

3. Main results

In this section, we establish some sufficient conditions which guarantee that every solution x of (1) oscillates on $[t_0, \infty)$.
Before stating the main results, we begin with the following lemma.

Lemma 3.1. *Suppose that x is an eventually positive solution of (1) and*

$$\int_{t_0}^{\infty} e_{-q(t)}(t, t_0)\,\Delta t = \int_{t_0}^{\infty} \left\{ \frac{1}{p(t)} \right\}^{\frac{1}{\alpha}} \Delta t = \infty, \tag{2}$$

then there are only the following two possible cases for $t \in [t_1, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large:

(1) $x^{\Delta}(t) > 0$, $(p(x^{\Delta})^{\alpha})^{\Delta}(t) > 0$, $(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) > 0$,
(2) $x^{\Delta}(t) > 0$, $(p(x^{\Delta})^{\alpha})^{\Delta}(t) < 0$, $(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) > 0$.

Proof. Let x be an eventually positive solution of (1). Then there exists a $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, +\infty)_{\mathbb{T}}$. From (1), we have

$$(p(x^{\Delta})^{\alpha})^{\Delta^3}(t) + q(t)\,(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) = -f(t, x(\tau(t))) < 0, \tag{3}$$

for $t \in [t_1, +\infty)_{\mathbb{T}}$. Hence, we obtain by (3) that

$$\left(\frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}}{e_{-q}(., t_0)} \right)^{\Delta}(t) = \frac{(p(x^{\Delta})^{\alpha})^{\Delta^3}(t) + q(t)\,(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)}{e^{\sigma}_{-q}(t, t_0)} < 0.$$

Thus, $\dfrac{(p(x^{\Delta})^{\alpha})^{\Delta^2}}{e_{-q}(., t_0)}$ is decreasing on $[t_1, +\infty)_{\mathbb{T}}$. Then $(p(x^{\Delta})^{\alpha})^{\Delta^2}, (p(x^{\Delta})^{\alpha})^{\Delta}$, and x^{Δ} are of constant sign eventually. We claim that $(p(x^{\Delta})^{\alpha})^{\Delta^2} > 0$ for $t \in [t_1, +\infty)_{\mathbb{T}}$. If not, then there exist a $t_2 \in [t_1, +\infty)_{\mathbb{T}}$ such that

$$(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) \leq -e_{-q}(t, t_0), \qquad t \in [t_2, +\infty)_{\mathbb{T}}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\lim_{t \longrightarrow \infty} (p(x^{\Delta})^{\alpha})^{\Delta}(t) \leq -\int_{t_2}^{+\infty} e_{-q}(s, t_0)\,\Delta s = -\infty,$$

which implies that

$$\lim_{t \longrightarrow \infty} p(t)\left(x^{\Delta}\right)^{\alpha}(t) = -\infty,$$

and so there exist a $t_3 \in [t_2, +\infty)_{\mathbb{T}}$ such that

$$x^{\Delta}(t) \leq -\left(\frac{1}{p(t)} \right)^{\frac{1}{\alpha}}, \qquad t \in [t_3, +\infty)_{\mathbb{T}}.$$

Integrating the above inequality from t_3 to t, we obtain

$$x(t) \leq x(t_3) - \int_{t_3}^{t} \left(\frac{1}{p(s)} \right)^{\frac{1}{\alpha}} \Delta s.$$

This gives $\lim_{t\longrightarrow\infty} x(t) = -\infty$, which is a contradiction.
If $(p(x^{\Delta})^{\alpha})^{\Delta}(t) > 0$, then $x^{\Delta}(t) > 0$ due to $(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) > 0$. If $(p(x^{\Delta})^{\alpha})^{\Delta}(t) < 0$, then $x^{\Delta}(t) > 0$ due to $x(t) > 0$. The proof is complete. □

Lemma 3.2. *Assume that x is a solution of (1) which satisfies case (1) of Lemma 3.1. Then*

$$(p(x^{\Delta})^{\alpha})^{\Delta}(t) \geq (t-t_1)(p(x^{\Delta})^{\alpha})^{\Delta^2}(t), \qquad t \in [t_1,\infty)_{\mathbb{T}}. \tag{4}$$

If there exist a function $\phi \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$\phi(t) - \phi^{\Delta}(t)(t-t_1) \leq 0, \qquad t \in [t_2,\infty)_{\mathbb{T}}, \tag{5}$$

then $\dfrac{(p(x^{\Delta})^{\alpha})^{\Delta}}{\phi}$ is a nonincreasing function on $[t_2,\infty)_{\mathbb{T}}$, and

$$\phi(t)\, p(t) (x^{\Delta})^{\alpha}(t) \geq (p(x^{\Delta})^{\alpha})^{\Delta}(t) \int_{t_2}^{t} \phi(s)\,\Delta s, \qquad t \in [t_2,\infty)_{\mathbb{T}}. \tag{6}$$

Further, if there exist a function $\psi \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that

$$\phi(t)\,\psi(t) - \psi^{\Delta}(t) \int_{t_2}^{t} \phi(s)\,\Delta s \leq 0, \qquad t \in [t_3,\infty)_{\mathbb{T}}, \tag{7}$$

then $\dfrac{p(x^{\Delta})^{\alpha}}{\psi}$ is a nonincreasing function on $[t_3,\infty)_{\mathbb{T}}$, and

$$x(t) \geq x^{\Delta}(t) \left\{\frac{p(t)}{\psi(t)}\right\}^{\frac{1}{\alpha}} \int_{t_3}^{t} \left\{\frac{\psi(s)}{p(s)}\right\}^{\frac{1}{\alpha}} \Delta s := R(t,t_3)\, x^{\Delta}(t), \quad t \in [t_3,\infty)_{\mathbb{T}}. \tag{8}$$

Proof. From $(p(x^{\Delta})^{\alpha})^{\Delta}(t) > 0$, and $(p(x^{\Delta})^{\alpha})^{\Delta^2}(t) > 0$, for $t \in [t_1,\infty)_{\mathbb{T}}$, we have

$$(p(x^{\Delta})^{\alpha})^{\Delta}(t) \geq \int_{t_1}^{t} (p(x^{\Delta})^{\alpha})^{\Delta^2}(s)\Delta s \geq (p(x^{\Delta})^{\alpha})^{\Delta^2}(t)(t-t_1).$$

Thus,

$$\begin{aligned}\left(\frac{(p(x^{\Delta})^{\alpha})^{\Delta}}{\phi}\right)^{\Delta}(t) &= \frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)\phi(t) - (p(x^{\Delta})^{\alpha})^{\Delta}(t)\,\phi^{\Delta}(t)}{\phi(t)\,\phi^{\sigma}(t)} \\ &\leq \frac{(p(x^{\Delta})^{\alpha})^{\Delta}(t)}{\phi(t)\,\phi^{\sigma}(t)} \left\{\frac{\phi(t)}{t-t_1} - \phi^{\Delta}(t)\right\} \leq 0.\end{aligned}$$

Therefore, $\dfrac{(p(x^{\Delta})^{\alpha})^{\Delta}}{\phi}$ is a nonincreasing function on $[t_2,\infty)_{\mathbb{T}}$. Then, we obtain

$$\begin{aligned}p(t)(x^{\Delta})^{\alpha}(t) &= p(t_2)(x^{\Delta})^{\alpha}(t_2) + \int_{t_2}^{t} \frac{(p(x^{\Delta})^{\alpha})^{\Delta}(s)}{\phi(s)}\phi(s)\,\Delta s \\ &\geq \frac{(p(x^{\Delta})^{\alpha})^{\Delta}(t)}{\phi(t)} \int_{t_2}^{t} \phi(s)\,\Delta s.\end{aligned}$$

Hence,

$$\left(\frac{p\left(x^{\Delta}\right)^{\alpha}}{\psi}\right)^{\Delta}(t) = \frac{(p\left(x^{\Delta}\right)^{\alpha})^{\Delta}(t)\psi\left(t\right)-p\left(x^{\Delta}\right)^{\alpha}\left(t\right)\psi^{\Delta}\left(t\right)}{\psi\left(t\right)\psi^{\sigma}\left(t\right)}$$
$$\leq \frac{p\left(t\right)\left(x^{\Delta}\right)^{\alpha}\left(t\right)}{\psi\left(t\right)\psi^{\sigma}\left(t\right)}\left\{\frac{\phi\left(t\right)\psi\left(t\right)}{\int_{t_2}^{t}\phi\left(s\right)\Delta s}-\psi^{\Delta}\left(t\right)\right\}\leq 0.$$

Thus, $\dfrac{p\left(x^{\Delta}\right)^{\alpha}}{\psi}$ is a nonincreasing function on $[t_3,\infty)_{\mathbb{T}}$, we have

$$x\left(t\right)\geq\int_{t_3}^{t}\left\{\frac{p\left(s\right)\left(x^{\Delta}\right)^{\alpha}\left(s\right)}{\psi\left(s\right)}\right\}^{\frac{1}{\alpha}}\left(\frac{\psi\left(s\right)}{p\left(s\right)}\right)^{\frac{1}{\alpha}}\Delta s=R\left(t,t_3\right)x^{\Delta}\left(t\right).$$

This completes the proof. □

Remark 3.3. *We can take for example* $\phi\left(t\right):=\left(t-t_1\right)$ *and* $\psi\left(t\right):=\int_{t_2}^{t}\left(s-t_1\right)\Delta s$.

We give the main results and for simplification, we note $Td\left(t\right):=\dfrac{d^{\sigma}\left(t\right)}{d\left(t\right)}$, $d_{+}\left(t\right):=\max\left\{0,d\left(t\right)\right\}$ and

$$Q_{\alpha}^{d}\left(t\right):=\begin{cases}1 & \text{if } \alpha\geq 1,\\ Td^{\alpha-1}\left(t\right) & \text{if } \alpha<1.\end{cases}$$

Theorem 3.4. *Let (2) holds. Assume that there exist a positive function* $\delta\in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ *such that for all sufficiently large* $t_1\in[t_0,\infty)_{\mathbb{T}}$, *for some* $t_1\in[t_0,\infty)_{\mathbb{T}}$, $t_2\in[t_1,\infty)_{\mathbb{T}}$, $t_3\in[t_2,\infty)_{\mathbb{T}}$, *and* $t_4\in[t_3,\infty)_{\mathbb{T}}$

$$\limsup_{t\longrightarrow\infty}\int_{t_4}^{t}\delta^{\sigma}\left(s\right)r\left(s\right)k\left(s,t_3\right)-\frac{1}{4}\left\{\frac{\delta^{\Delta}\left(s\right)}{\delta\left(s\right)}-q\left(s\right)\frac{T\delta\left(s\right)}{T\phi\left(s\right)}\right\}^{2}\frac{\delta\left(s\right)T\phi\left(s\right)}{T\delta\left(s\right)}\Delta s=\infty, \tag{9}$$

where ϕ *and* ψ *are defined as in Lemma 3.2, and*

$$k\left(t,t_3\right):=\frac{1}{\psi\left(\tau\left(t\right)\right)\phi^{\sigma}\left(t\right)}\left\{\int_{t_3}^{\tau(t)}\left\{\frac{\psi\left(s\right)}{p\left(s\right)}\right\}^{\frac{1}{\alpha}}\Delta s\right\}^{\alpha}\int_{t_3}^{\tau(t)}\phi\left(s\right)\Delta s.$$

If there exist a positive functions $\theta,\lambda\in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ *such that*

$$\frac{\lambda\left(t\right)}{\xi\left(t,t_1\right)}-\lambda^{\Delta}\left(t\right)\leq 0,\qquad t\in[t_2,\infty)_{\mathbb{T}}, \tag{10}$$

where

$$\xi\left(t,t_1\right):=\left(p\left(t\right)\right)^{\frac{1}{\alpha}}\int_{t_1}^{t}\left\{\frac{1}{p\left(s\right)}\right\}^{\frac{1}{\alpha}}\Delta s$$

and

$$\limsup_{t\longrightarrow\infty}\int_{t_3}^{t}\theta^{\sigma}\left(s\right)c\left(s\right)-\frac{\left[\theta_{+}^{\Delta}\right]^{\alpha+1}\left(s\right)T\lambda^{\alpha^{2}}\left(s\right)p\left(s\right)}{\left(\alpha+1\right)^{\alpha+1}\theta^{\alpha}\left(s\right)\left(Q_{\alpha}^{\lambda}\right)^{\alpha}\left(s\right)T\theta^{\alpha}\left(s\right)}\Delta s=\infty, \tag{11}$$

where

$$c(t) := \frac{1}{T\lambda^{\alpha}(t)} \int_t^{\infty} \left\{ \int_v^{\infty} r(u) \frac{\lambda^{\alpha}(\tau(u))}{\lambda^{\alpha}(u)} \Delta u \right\} \Delta v.$$

Then (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Suppose first that x satisfies (1) of lemma 3.1. Let

$$\omega_1(t) := \delta(t) \frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)}{(p(x^{\Delta})^{\alpha})^{\Delta}(t)}, \qquad t \in [t_1, \infty)_{\mathbb{T}}. \tag{12}$$

Then $\omega_1(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and

$$\omega_1^{\Delta}(t) = \delta^{\Delta}(t) \frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)}{(p(x^{\Delta})^{\alpha})^{\Delta}(t)} + \delta^{\sigma}(t) \left\{ \frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}}{(p(x^{\Delta})^{\alpha})^{\Delta}} \right\}^{\Delta}(t),$$

which implies that

$$\omega_1^{\Delta}(t) = \delta^{\Delta}(t) \frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)}{(p(x^{\Delta})^{\alpha})^{\Delta}(t)} + \delta^{\sigma}(t) \frac{(p(x^{\Delta})^{\alpha})^{\Delta^3}(t)}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} - \frac{\delta^{\sigma}(t) \left((p(x^{\Delta})^{\alpha})^{\Delta^2}(t) \right)^2}{(p(x^{\Delta})^{\alpha})^{\Delta}(t)\, (p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)}. \tag{13}$$

Since $\dfrac{(p(x^{\Delta})^{\alpha})^{\Delta}(t)}{\phi(t)}$ is a nondecreasing function, we have

$$\frac{(p(x^{\Delta})^{\alpha})^{\Delta}(t)}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} \geq \frac{\phi(t)}{\phi^{\sigma}(t)}. \tag{14}$$

From (12), (1) and the above inequality, we obtain

$$\begin{aligned} \omega_1^{\Delta}(t) &\leq \frac{\delta^{\Delta}(t)}{\delta(t)} \omega_1(t) - \delta^{\sigma}(t) \frac{q(t) (p(x^{\Delta})^{\alpha})^{\Delta^2}(t)}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} - \frac{\delta^{\sigma}(t) r(t) x^{\alpha}(\tau(t))}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} - \frac{T\delta(t)}{\delta(t) T\phi(t)} \omega_1^2(t) \\ &\leq \frac{\delta^{\Delta}(t)}{\delta(t)} \omega_1(t) - q(t) T\delta(t) \frac{(p(x^{\Delta})^{\alpha})^{\Delta}(t)}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} \omega_1(t) - \frac{\delta^{\sigma}(t) r(t) x^{\alpha}(\tau(t))}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} - \frac{T\delta(t)}{\delta(t) T\phi(t)} \omega_1^2(t). \end{aligned}$$

Substituting (14) in (13), we find

$$\omega_1^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} \omega_1(t) - q(t) \frac{T\delta(t)}{T\phi(t)} \omega_1(t) - \frac{\delta^{\sigma}(t) r(t) x^{\alpha}(\tau(t))}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} - \frac{T\delta(t)}{\delta(t) T\phi(t)} \omega_1^2(t). \tag{15}$$

In view of (6), (8) and the fact that $\frac{(p(x^{\Delta})^{\alpha})^{\Delta}(t)}{\phi(t)}$ is a nonincreasing function, we have

$$\begin{aligned} \frac{x^{\alpha}(\tau(t))}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} &= \frac{x^{\alpha}(\tau(t))}{(p(x^{\Delta})^{\alpha})^{\Delta}(\tau(t))} \frac{(p(x^{\Delta})^{\alpha})^{\Delta}(\tau(t))}{(p(x^{\Delta})^{\alpha})^{\Delta\sigma}(t)} \\ &\geq \frac{1}{\psi(\tau(t)) \phi^{\sigma}(t)} \left\{ \int_{t_3}^{\tau(t)} \left\{ \frac{\psi(s)}{p(s)} \right\}^{\frac{1}{\alpha}} \Delta s \right\}^{\alpha} \int_{t_3}^{\tau(t)} \phi(s) \Delta s \\ &= k(t, t_3). \end{aligned} \tag{16}$$

Substituting (16) in (15), we get

$$\omega_1^{\Delta}(t) \leq -\delta^{\sigma}(t)\, r(t)\, k(t,t_3) + \left\{ \frac{\delta^{\Delta}(t)}{\delta(t)} - q(t)\frac{T\delta(t)}{T\phi(t)} \right\} \omega_1(t) - \frac{T\delta(t)}{\delta(t)\, T\phi(t)} \omega_1^2(t).$$

Set

$$B(t) := \left\{ \frac{\delta^{\Delta}(t)}{\delta(t)} - q(t)\frac{T\delta(t)}{T\phi(t)} \right\}, \text{ and } A(t) := \frac{T\delta(t)}{\delta(t)\, T\phi(t)}.$$

Using the inequality [5]

$$By - Ay^2 \leq \frac{B^2}{4A}, \qquad A > 0,\ B \in \mathbb{R}, \tag{17}$$

we get

$$\omega_1^{\Delta}(t) \leq -\delta^{\sigma}(t)\, r(t)\, k(t,t_3) + \frac{1}{4}\left\{ \frac{\delta^{\Delta}(t)}{\delta(t)} - q(t)\frac{T\delta(t)}{T\phi(t)} \right\}^2 \frac{\delta(t)\, T\phi(t)}{T\delta(t)}.$$

Integrating the above inequality from t_4 to t, we obtain

$$\int_{t_4}^{t} \delta^{\sigma}(s)\, r(s)\, k(s,t_3) - \frac{1}{4}\left\{ \frac{\delta^{\Delta}(s)}{\delta(s)} - q(s)\frac{T\delta(s)}{T\phi(s)} \right\}^2 \frac{\delta(s)\, T\phi(s)}{T\delta(s)} \Delta s \leq \omega_1(t_4) - \omega_1(t) \leq \omega_1(t_4),$$

which contradicts (9).

Secondly suppose that x satisfies (2) of lemma 3.1. Let

$$\omega_2(t) := \theta(t)\frac{p(t)\left(x^{\Delta}\right)^{\alpha}(t)}{x^{\alpha}(t)}, \qquad t \in [t_1,\infty)_{\mathbb{T}}. \tag{18}$$

Then $\omega_2(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$ and

$$\omega_2^{\Delta}(t) = \theta^{\Delta}(t)\frac{p(t)\left(x^{\Delta}\right)^{\alpha}(t)}{x^{\alpha}(t)} + \theta^{\sigma}(t)\frac{\left(p\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta}(t)}{x^{\alpha}(\sigma(t))} - \theta^{\sigma}(t)\frac{p(t)\left(x^{\Delta}\right)^{\alpha}(t)\,(x^{\alpha})^{\Delta}(t)}{x^{\alpha}(\sigma(t))\, x^{\alpha}(t)}. \tag{19}$$

Since $x^{\Delta}(t) > 0$ and $(p\left(x^{\Delta}\right)^{\alpha})^{\Delta}(t) < 0$, we obtain

$$\begin{aligned} x(t) &\geq \int_{t_1}^{t} \left(p(s)\left(x^{\Delta}(s)\right)^{\alpha}\right)^{\frac{1}{\alpha}} \left\{\frac{1}{p(s)}\right\}^{\frac{1}{\alpha}} \Delta s \\ &\geq x^{\Delta}(t)\,(p(t))^{\frac{1}{\alpha}} \int_{t_1}^{t} \left\{\frac{1}{p(s)}\right\}^{\frac{1}{\alpha}} \Delta s = \xi(t,t_1)\, x^{\Delta}(t). \end{aligned} \tag{20}$$

Thus,

$$\begin{aligned} \left(\frac{x}{\lambda}\right)^{\Delta}(t) &= \frac{x^{\Delta}(t)\,\lambda(t) - x(t)\,\lambda^{\Delta}(t)}{\lambda(t)\,\lambda^{\sigma}(t)} \\ &\leq \frac{x(t)}{\lambda(t)\,\lambda^{\sigma}(t)}\left\{\frac{\lambda(t)}{\xi(t,t_1)} - \lambda^{\Delta}(t)\right\} \leq 0. \end{aligned} \tag{21}$$

Hence $\dfrac{x}{\lambda}$ is a nonincreasing function on $[t_2,\infty)_{\mathbb{T}}$ and so

$$\frac{x(t)}{x^{\sigma}(t)} \geq \frac{\lambda(t)}{\lambda^{\sigma}(t)}, \qquad \frac{x(\tau(t))}{x(t)} \geq \frac{\lambda(\tau(t))}{\lambda(t)}. \tag{22}$$

By Pötzsche's chain rule [2, Theorem 1.90], we see that

$$\begin{aligned} (x^{\alpha})^{\Delta}(t) &= \alpha x^{\Delta}(t)\int_0^1 [hx(t)+(1-h)x^{\sigma}(t)]^{\alpha-1}dh \\ &\geq \alpha Q_{\alpha}^{\lambda}(t)\,x^{\Delta}(t)\,x^{\alpha-1}(t). \end{aligned} \tag{23}$$

Substituting (23) in (19), we have

$$\begin{aligned} \omega_2^{\Delta}(t) &\leq \theta^{\Delta}(t)\frac{p(t)\left(x^{\Delta}\right)^{\alpha}(t)}{x^{\alpha}(t)}+\theta^{\sigma}(t)\frac{\left(p\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta}(t)}{x^{\alpha}(\sigma(t))} \\ &\quad -\alpha Q_{\alpha}^{\lambda}(t)\,\theta^{\sigma}(t)\frac{p(t)\left(x^{\Delta}\right)^{\alpha+1}(t)}{x^{\alpha}(\sigma(t))\,x(t)}. \end{aligned} \tag{24}$$

On the other hand, by (1), we get

$$(p\left(x^{\Delta}\right)^{\alpha})^{\Delta^2}(s)-(p\left(x^{\Delta}\right)^{\alpha})^{\Delta^2}(t)+\int_t^s r(u)\,(x(\tau(u)))^{\alpha}\,\Delta u\leq 0.$$

It follows from $x^{\Delta}>0$ and (22) that

$$(p\left(x^{\Delta}\right)^{\alpha})^{\Delta^2}(s)-(p\left(x^{\Delta}\right)^{\alpha})^{\Delta^2}(t)+x^{\alpha}(t)\int_t^s r(u)\frac{\lambda^{\alpha}(\tau(u))}{\lambda^{\alpha}(u)}\Delta u\leq 0.$$

When s tends to ∞ in the above inequality, we obtain

$$(p\left(x^{\Delta}\right)^{\alpha})^{\Delta^2}(t)\geq x^{\alpha}(t)\int_t^{\infty} r(u)\frac{\lambda^{\alpha}(\tau(u))}{\lambda^{\alpha}(u)}\Delta u.$$

Therefore,

$$-(p\left(x^{\Delta}\right)^{\alpha})^{\Delta}(s)+(p\left(x^{\Delta}\right)^{\alpha})^{\Delta}(t)+x^{\alpha}(t)\int_t^s\left\{\int_v^{\infty} r(u)\frac{\lambda^{\alpha}(\tau(u))}{\lambda^{\alpha}(u)}\Delta u\right\}\Delta v\leq 0.$$

Then

$$(p\left(x^{\Delta}\right)^{\alpha})^{\Delta}(t)+x^{\alpha}(t)\int_t^{\infty}\left\{\int_v^{\infty} r(u)\frac{\lambda^{\alpha}(\tau(u))}{\lambda^{\alpha}(u)}\Delta u\right\}\Delta v\leq 0.$$

Thus, we get by (22) that

$$\begin{aligned} \frac{(p\left(x^{\Delta}\right)^{\alpha})^{\Delta}(t)}{x^{\alpha}(\sigma(t))} &\leq -\frac{x^{\alpha}(t)}{x^{\alpha}(\sigma(t))}\int_t^{\infty}\left\{\int_v^{\infty} r(u)\frac{\lambda^{\alpha}(\tau(u))}{\lambda^{\alpha}(u)}\Delta u\right\}\Delta v \\ &\leq -\left\{\frac{\lambda(t)}{\lambda^{\sigma}(t)}\right\}^{\alpha}\int_t^{\infty}\left\{\int_v^{\infty} r(u)\frac{\lambda^{\alpha}(\tau(u))}{\lambda^{\alpha}(u)}\Delta u\right\}\Delta v=-c(t) \end{aligned} \tag{25}$$

Substituting (20), (22) and (25) in (24), we have

$$\omega_2^{\Delta}(t)\leq-\theta^{\sigma}(t)\,c(t)+\theta^{\Delta}(t)\frac{p(t)\left(x^{\Delta}\right)^{\alpha}(t)}{x^{\alpha}(t)}-\alpha Q_{\alpha}^{\lambda}(t)\,\theta^{\sigma}(t)\frac{p(t)\left(x^{\Delta}\right)^{\alpha+1}(t)}{x^{\alpha}(\sigma(t))\,x(t)}.$$

Thus,

$$\omega_2^{\Delta}(t) \leq -\theta^{\sigma}(t)\, c(t) + \frac{\left[\theta^{\Delta}\right]_{+}(t)}{\theta(t)}\omega_2(t) - \frac{\alpha Q_{\alpha}^{\lambda}(t)\, T\theta(t)}{\theta^{\frac{1}{\alpha}}(t)\, p^{\frac{1}{\alpha}}(t)\, T\lambda^{\alpha}(t)}\omega_2^{1+\frac{1}{\alpha}}(t),$$

Using the inequality [5]

$$By - Ay^{1+\frac{1}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{B^{\beta+1}}{A^{\beta}}, \qquad A > 0,\ B > 0 \text{ and } \beta > 0,$$

which yields

$$\omega_2^{\Delta}(t) \leq -\theta^{\sigma}(t)\, c(t) + \frac{\left[\theta_{+}^{\Delta}\right]^{\alpha+1}(t)\, T\lambda^{\alpha^2}(t)\, p(t)}{(\alpha+1)^{\alpha+1}\theta^{\alpha}(t)\left(Q_{\alpha}^{\lambda}\right)^{\alpha}(t)\, T\theta^{\alpha}(t)}.$$

Integrating the last inequality from t_3 to t, we have

$$\int_{t_3}^{t} \theta^{\sigma}(s)\, c(s) - \frac{\left[\theta_{+}^{\Delta}\right]^{\alpha+1}(s)\, T\lambda^{\alpha^2}(s)\, p(s)}{(\alpha+1)^{\alpha+1}\theta^{\alpha}(s)\left(Q_{\alpha}^{\lambda}\right)^{\alpha}(s)\, T\theta^{\alpha}(s)}\Delta s \leq \omega_2(t_3) - \omega_2(t) \leq \omega_2(t_3),$$

which contradicts (11). The proof is complete. □

Theorem 3.5. *Let (2) holds and $\alpha \geq 1$. Assume that there exist a positive functions $\eta, m \in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}\right)$ such that for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$, $t_2 \in [t_1,\infty)_{\mathbb{T}}$, $t_3 \in [t_2,\infty)_{\mathbb{T}}$, and $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that*

$$\limsup_{t\longrightarrow\infty} \int_{t_4}^{t} \eta^{\sigma}(s)\, r(s)\frac{m^{\alpha}(\tau(s))}{m^{\alpha}(\sigma(s))} - \frac{1}{4\alpha}E^2(s)\, F(s,t_3)\,\Delta s = \infty, \tag{26}$$

and

$$\frac{m(t)}{R(t,t_3)} - m^{\Delta}(t) \leq 0, \qquad t \in [t_4,\infty)_{\mathbb{T}}, \tag{27}$$

where ϕ, ψ and $R(t,t_3)$ are defined as in Lemma 3.2, and

$$F(t,t_3) := \frac{p(t)\,\phi(t)\,\eta^2(t)\, Tm^{\alpha}(t)}{(t-t_3)\,\eta^{\sigma}(t)\, R^{\alpha-1}(t,t_3)}\left\{\int_{t_3}^{t}\phi(s)\,\Delta s\right\}^{-1},$$

$$E(t) := \frac{\eta^{\Delta}(t)}{\eta(t)} - q(t)\frac{T\eta(t)}{Tm^{\alpha}(t)}.$$

If there exist a positive functions $\theta, \lambda \in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}\right)$ such that (11) holds. Then (1) is oscillatory.

Proof. Let x be a non-oscillatory solution of (1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Suppose first that x satisfies (1) of lemma 3.1. Let

$$\omega_3(t) := \eta(t)\frac{\left(p\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta^2}(t)}{x^{\alpha}(t)}, \qquad t \in [t_1,\infty)_{\mathbb{T}}. \tag{28}$$

Then $\omega_3(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, and

$$\omega_3^{\Delta}(t) = \eta^{\Delta}(t)\frac{\left(p\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta^2}(t)}{x^{\alpha}(t)} + \eta^{\sigma}(t)\frac{\left(p\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta^3}(t)}{x^{\alpha}(\sigma(t))} - \eta^{\sigma}(t)\frac{\left(p\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta^2}(t)\left(x^{\alpha}\right)^{\Delta}(t)}{x^{\alpha}(\sigma(t))\, x^{\alpha}(t)}. \tag{29}$$

Substituting (1) in (29), we have

$$\begin{aligned}\omega_3^{\Delta}(t) &\leq \eta^{\Delta}(t)\frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)}{x^{\alpha}(t)}-\eta^{\sigma}(t)\frac{q(t)(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)}{x^{\alpha}(\sigma(t))}\\ &\quad -\eta^{\sigma}(t)r(t)\frac{x^{\alpha}(\tau(t))}{x^{\alpha}(\sigma(t))}-\eta^{\sigma}(t)\frac{(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)(x^{\alpha})^{\Delta}(t)}{x^{\alpha}(\sigma(t))x^{\alpha}(t)}. \qquad (30)\end{aligned}$$

By (8) and (27), we have

$$\left(\frac{x}{m}\right)^{\Delta}(t)\leq\frac{x(t)}{m^{\sigma}(t)m(t)}\left\{\frac{m(t)}{R(t,t_3)}-m^{\Delta}(t)\right\}\leq 0.$$

Then $\dfrac{x}{m}$ is a nonincreasing function on $t\in[t_4,\infty)_{\mathbb{T}}$, and

$$\frac{x(t)}{x^{\sigma}(t)}\geq\frac{m(t)}{m^{\sigma}(t)},\qquad \frac{x(\tau(t))}{x^{\sigma}(t)}\geq\frac{m(\tau(t))}{m^{\sigma}(t)}. \qquad (31)$$

By Pötzsche's chain rule, we have

$$(x^{\alpha})^{\Delta}(t)\geq\alpha x^{\Delta}(t)x^{\alpha-1}(t). \qquad (32)$$

By inequality (4), (6) , (8) and (32), we obtain

$$\begin{aligned}(x^{\alpha})^{\Delta}(t) &\geq \alpha R^{\alpha-1}(t,t_3)\left(x^{\Delta}(t)\right)^{\alpha}\\ &\geq \alpha\frac{R^{\alpha-1}(t,t_3)}{\phi(t)p(t)}(p(x^{\Delta})^{\alpha})^{\Delta}(t)\int_{t_3}^{t}\phi(s)\Delta s\\ &\geq \alpha(t-t_3)\frac{R^{\alpha-1}(t,t_3)}{\phi(t)p(t)}(p(x^{\Delta})^{\alpha})^{\Delta^2}(t)\int_{t_3}^{t}\phi(s)\Delta s\\ &\geq \frac{\alpha\eta^2(t)Tm^{\alpha}(t)}{\eta^{\sigma}(t)F(t,t_3)}(p(x^{\Delta})^{\alpha})^{\Delta^2}(t). \qquad (33)\end{aligned}$$

Substituting (33) and (28) in (30), we get

$$\omega_3^{\Delta}(t)\leq\frac{\eta^{\Delta}(t)}{\eta(t)}\omega_3(t)-T\eta(t)\frac{q(t)x^{\alpha}(t)}{x^{\alpha}(\sigma(t))}\omega_3(t)-\eta^{\sigma}(t)r(t)\frac{x^{\alpha}(\tau(t))}{x^{\alpha}(\sigma(t))}-\frac{\alpha Tm^{\alpha}(t)x^{\alpha}(t)\omega_3^2(t)}{F(t,t_3)x^{\alpha}(\sigma(t))}. \qquad (34)$$

With (31), we get

$$\begin{aligned}\omega_3^{\Delta}(t) &\leq \frac{\eta^{\Delta}(t)}{\eta(t)}\omega_3(t)-\frac{T\eta(t)}{Tm^{\alpha}(t)}q(t)\omega_3(t)-\eta^{\sigma}(t)r(t)\frac{m^{\alpha}(\tau(t))}{m^{\alpha}(\sigma(t))}-\frac{\alpha}{F(t,t_3)}\omega_3^2(t)\\ &\leq -\eta^{\sigma}(t)r(t)\frac{m^{\alpha}(\tau(t))}{m^{\alpha}(\sigma(t))}+E(t)\omega_3(t)-\frac{\alpha}{F(t,t_3)}\omega_3^2(t).\end{aligned}$$

By inequality (17), we obtain

$$\omega_3^{\Delta}(t)\leq-\eta^{\sigma}(t)r(t)\frac{m^{\alpha}(\tau(t))}{m^{\alpha}(\sigma(t))}+\frac{1}{4\alpha}E^2(t)F(t,t_3).$$

Integrating the latter inequality from t_4 to t, we have

$$\int_{t_4}^{t}\eta^{\sigma}(s)r(s)\frac{m^{a}(\tau(s))}{m^{\alpha}(\sigma(s))}-\frac{1}{4\alpha}E^2(s)F(s,t_3)\Delta s\leq\omega_3(t_4)-\omega_3(t)\leq\omega_3(t_4),$$

which contradicts (26). The proof of Case (2) is the same as that of Case (2) in Theorem 3.4, and so is omitted. This completes the proof. □

4. Example

As some application of the main results, we present the following example

Example 1. *Consider a fourth-order half-linear delay dynamic equation*

$$\left(\sqrt[3]{x'}\right)^{(3)}(t)+\frac{1}{t}\left(\sqrt[3]{x'}\right)^{(2)}(t)+t^{-\frac{7}{3}}\sqrt[3]{x(t)}=0,\quad t\in[0,\infty)_{\mathbb{R}}. \tag{35}$$

Here, $\alpha=\frac{1}{3}$, $p(t)=1$, $r(t)=t^{-\frac{7}{3}}$, $q(t)=1\backslash t$ *and* $\tau(t)=t$. *Set* $\phi(t)=t-t_1$, $\psi(t)=\int_{t_2}^{t}(s-t_1)\,ds$, $\lambda(t)=t-t_1$, *and* $\delta(t)=\theta(t)=1$. *Then (2), (5), and (7) holds,*

$$k(t,t_3)\geq\eta t^{\frac{4}{3}},\qquad \text{for } t \text{ large enough},$$

$$r(t)\,k(t,t_3)-\frac{1}{4}q^2(t)\geq\frac{\eta}{t}-\frac{1}{4t^2},\qquad \text{for } t \text{ large enough}.$$

where $\eta\in(0,1)$. *Thus, (9) holds, therefore, we have* $\xi(t,t_1)=t-t_1$, *then (10) holds,*

$$c(t)=\tfrac{9}{4}t^{-\frac{1}{3}}.$$

Thus, (11) holds. By Theorem 3.4, equation (35) is oscillatory.

5. conclusion

It's clear that the form of problem (1) is more general than all the problems considered in [8], [9] and [11].

In problem (1) we have considered (for example) a combination of terms of the form x^{Δ^4}, $x^{\Delta}.x^{\Delta^3}$, $x^{\Delta}.x^{\Delta^2}.x^{\Delta^3}$, $x^{\Delta^2}.x^{\Delta}$, ... and the function f is not precised, but in [12] we found only the terms x^{Δ^4}, $x^{\Delta}.x^{\Delta^3}$ and x^{Δ^3} and the function is $q(.)\,x^{\gamma}(\tau(.))$.

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Fatima Zohra Ladrani
Laboratory of Mathematics, Sidi Bel-Abbes University, Sidi Bel-Abbes, Algeria
E-mail address: fatifuture@yahoo.fr

Ahmed Hammoudi
Laboratory of Mathematics, Sidi Bel-Abbes University, Sidi Bel-Abbe, Algeria
E-mail address: hymmed@yahoo.com

Amine. Benaissa Cherif
Laboratory of Mathematics, Sidi Bel-Abbes University, Sidi Bel-Abbes, Algeria
E-mail address: amine.banche@gmail.com