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INVERSE THEOREMS OF APPROXIMATION THEORY IN $L_{p,\alpha}(\mathbb{R}^+)$

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Abstract. In this paper, We prove analogues of some inverse theorems of Stechkin ,for the Bessel harmonic analysis, using the entire functions of exponential type.

1. Introduction And Preliminaries

Yet by the year 1912, S. Bernstein obtained the estimate inverse to Jakson's inequality in the space of continuous functions for some special cases [2], later S.B.Stechkin [4], M.Timan [7], etc, proved such inverse estimates , including the case of the space L^p, $1 < p < \infty$.
In this paper, we obtain the estimate inverse to Jakson's inequality in the space $L_{p,\alpha}(\mathbb{R}^+)$ (see [1], Theorem 1.1) ,where the modulus of smoothness is constructed on the basis of the Bessel generalized translation.
$L_{p,\alpha}(\mathbb{R}^+)$,$1 \leq p \leq \infty$, is the Banach space of measurable functions $f(x)$ on $\mathbb{R}^+$ with the finite norm

$$\|f\|_{p,\alpha} = \begin{cases} \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \sup_{x\in\mathbb{R}^+} |f(x)| & \text{if } p = \infty \end{cases}$$

Everywhere α is a real number, $\alpha > \frac{-1}{2}$, Let

$$B = \frac{d^2}{dx^2} + \frac{(2\alpha+1)}{x}\frac{d}{dx},$$

be the Bessel differential operator. By $j_\alpha(x)$ we denote the Bessel normalized function of the first kind, i.e,

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{x}{2}\right)^{2n},$$

where $\Gamma(x)$ is the gamma-function (see [3]). The function $y = j_\alpha(x)$ satisfies the differential equation $By + y = 0$ with the condition initial $y(0) = 1$ and $y'(0) = 0$.

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The function $j_\alpha(x)$ is the infinitely differentiable, even.
The Bessel transform is defined by formula (see [3])

$$\widehat{f}(t) = \int_0^\infty f(x) j_\alpha(tx) x^{2\alpha+1} dx, \quad t \in \mathbb{R}^+.$$

The inverse Bessel transform is given by the formula

$$f(x) = (2^\alpha \Gamma(\alpha+1))^{-2} \int_0^\infty \widehat{f}(t) j_\alpha(tx) t^{2\alpha+1} dt.$$

We note the important property of the Bessel transform

$$\widehat{Bf}(t) = -t^2 \widehat{f}(t).$$

In $L_{p,\alpha}(\mathbb{R}^+)$,$1 \leq p \leq \infty$, we consider the Bessel generalized translation

$$T_h f(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f(\sqrt{x^2+h^2-2xh\cos x})(\sin x)^{2\alpha} x dx.$$

Some of its properties are as follows (see [3] -[8]):
(a) $T_h j_\alpha(\lambda x) = j_\alpha(\lambda x) j_\alpha(\lambda h); x, h, \lambda \in \mathbb{R}^+$,
(b) $T_h f(x) = T_x f(h)$,
(c) $\widehat{T_h f}(t) = j_\alpha(th)\widehat{f}(t)$,
(d) $B(T_h f) = T_h(Bf)$,
(e) $\|T_h f\|_{p,\alpha} \leq \|f\|_{p,\alpha}$,
(f) $\|T_h f - f\|_{p,\alpha} \to 0, h \to 0$.
The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = T_h f(x) - f(x) = (T_h - I) f(x),$$

where I is the identity operator in $L_{p,\alpha}(\mathbb{R}^+)$, and

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (T_h - 1)^k f(x).$$

The k^{th} order generalized modulus of continuity of a function $f \in L_{p,\alpha}(\mathbb{R}^+)$ is defined by

$$\omega_k(f,t)_{p,\alpha} = \sup_{0<h\leq t} \|\Delta_h^k f\|_{p,\alpha}.$$

For $\nu > 0$, we denote by $\mathfrak{M}(\nu, p, \alpha)$ the set of even entire functions of exponential $\leq \nu$ whose restrictions to $\mathbb{R}^+$ belong to $L_{p,\alpha}(\mathbb{R}^+)$. Then the functions in the space $\mathfrak{M}(\nu, p, \alpha)$ make a natural approximation tool in $L_{p,\alpha}(\mathbb{R}^+)$. The best approximation of an $f \in L_{p,\alpha}(\mathbb{R}^+)$ by functions belonging to $\mathfrak{M}(\nu, p, \alpha)$ is defined as follows

$$E_\nu(f)_{p,\alpha} := \inf\{\|f - \Phi\|_{p,\alpha} : \Phi \in \mathfrak{M}(\nu, p, \alpha)\}$$

Let $W_{p,\alpha}^m$ be the Sobolev space of order $m \in \{1, 2, ...\}$ constructed from the differential operator B, that is,

$$W_{p,\alpha}^m = \{f \in L_{p,\alpha}(\mathbb{R}^+) : B^j f \in L_{p,\alpha}(\mathbb{R}^+), j = 1, 2, ..., m\},$$

where $B^j f = B(B^{j-1} f)$, and $B^0 f = f$.
Lemma 1.1 The modulus of smoothness $\omega_k(f,t)$ has the following properties.
i) $\omega_k(f+g,t)_{p,\alpha} \leq \omega_k(f,t)_{p,\alpha} + \omega_k(g,t)_{p,\alpha}$,
ii) $\omega_k(f,t)_{p,\alpha} \leq 2^k \|f\|_{p,\alpha}$,
iii) if $f \in W_{p,\alpha}^m$, then

$$\omega_k(f,t)_{p,\alpha} \leq c_1 t^{2m} \|B^m f\|_{p,\alpha},$$

where $c_1 = c(\alpha, m)$ is a constant.
Proof .(see Proposition 4.1 and Lemma 4.6 in [1])
Lemma 1.2 For every function $f \in \mathfrak{M}(\nu, p, \alpha)$, and any numbers $m \in \mathbb{N}$ we have

$$\|B^m f\|_{p,\alpha} \leq c_2 \nu^{2m} \|f\|_{p,\alpha},$$

where $c_2 = c(\alpha, m)$ is a constant.
Proof. (see Theorem 3.4 in [1])
Lemma 1.3 If $1 \leq p < p' \leq \infty$, then

$$\|f\|_{p',\alpha} \leq c_3 \nu^{(2\alpha+2)(\frac{1}{p}-\frac{1}{p'})} \|f\|_{p,\alpha},$$

for all $f \in \mathfrak{M}(\nu, p, \alpha)$, where $c_3 = c(\alpha, p, p') > 0$ is a constant.
Proof. (see [1],Theorem 3.5)

2. Main Results

Let c_1,c_2,...be positive constants possibly depending on k, m and α.
Lemma 2.1 For $j \geq 1$ we have

$$2^{2k(j-1)} E_{2^j}(f)_{p,\alpha} \leq \sum_{l=2^{j-1}+1}^{2^j} l^{2k-1} E_l(f)_{p,\alpha}.$$

Proof. Note that

$$\sum_{l=2^{j-1}+1}^{2^j} l^{2k-1} \geq (2^{j-1})^{2k-1} 2^{j-1} = 2^{2k(j-1)}.$$

Since $E_l(f)_{p,\alpha}$ is monotonically decreasing, we conclude that

$$2^{2k(j-1)} E_{2^j}(f)_{p,\alpha} \leq \sum_{l=2^{j-1}+1}^{2^j} l^{2k-1} E_l(f)_{p,\alpha}.$$

Lemma 2.2 For $n \in \mathbb{N}$ we have

$$2^k E_n(f)_{p,\alpha} \leq \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{p,\alpha}.$$

Proof. Note that

$$\sum_{j=0}^{n} (j+1)^{2k-1} \geq \sum_{j \geq \frac{n}{2}-1}^{n} (j+1)^{2k-1} \geq \left(\frac{n}{2}\right)^{2k-1} \frac{n}{2} = 2^{-2k} n^{2k}.$$

Since $E_l(f)_{p,\alpha}$ is monotonically decreasing, we conclude that

$$2^k E_n(f)_{p,\alpha} \leq \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{p,\alpha}.$$

Lemma 2.3 If $\Phi_\nu \in \mathfrak{M}(\nu, p, \alpha)$ such that $\|f - \Phi_\nu\|_{p,\alpha} = E_\nu(f)_{p,\alpha}$ For every $\nu \in \mathbb{N}$, then

$$\|B^k \Phi_{2^{j+1}} - B^k \Phi_{2^j}\|_{p,\alpha} \leq c_2 2^{2k(j+1)+1} E_{2^j}(f)_{p,\alpha}.$$

In particular

$$\|B^k \Phi_1\|_{p,\alpha} = \|B^k \Phi_1 - B^k \Phi_0\|_{p,\alpha} \leq c_2 2^{4k+1} E_0(f)_{p,\alpha}.$$

Proof. By lemma 1.2 and the fact that $E_\nu(f)_{p,\alpha}$ is monotone decreasing with respect to ν, we obtain

$$\begin{aligned}\|B^k\Phi_{2^{j+1}} - B^k\Phi_{2^j}\|_{p,\alpha} &\leq c_2 2^{2k(j+1)}\|\Phi_{2^{j+1}} - \Phi_{2^j}\|_{p,\alpha}\\ &= c_2 2^{2k(j+1)}\|(f-\Phi_{2^j}) - (f-\Phi_{2^{j+1}})\|_{p,\alpha}\\ &\leq c_2 2^{2k(j+1)}(E_{2^j}(f)_{p,\alpha} + E_{2^{j+1}}(f)_{p,\alpha})_{p,\alpha}\\ &\leq c_2 2^{2k(j+1)+1}E_{2^j}(f)_{p,\alpha}\end{aligned}$$

and

$$\begin{aligned}\|B^k\Phi_1 - B^k\Phi_0\|_{p,\alpha} &\leq c_2\|\Phi_1 - \Phi_0\|_{p,\alpha} = c_2\|(f-\Phi_1) - (f-\Phi_0)\|_{p,\alpha}\\ &\leq c_2(E_1(f)_{p,\alpha} + E_0(f)_{p,\alpha})\\ &\leq 2c_2E_0(f)_{p,\alpha} \leq c_2 2^{4k+1}E_0(f)_{p,\alpha}\end{aligned}$$

Theorem 2.4 For any $f \in L_{p,\alpha}(\mathbb{R}^+)$, $1 \leq p \leq \infty$, $\nu \geq 0$, there exists $f_\nu \in \mathfrak{M}(\nu, p, \alpha)$, such that $\|f - f_\nu\|_{p,\alpha} = E_\nu(f)_{p,\alpha}$

Proof .

- If $p = \infty$, it is the same as the case without weight (see [5], 2.6.2).
- If $p < \infty$. Let $\{f_n\}$ be a sequence in $\mathfrak{M}(\nu, p, \alpha)$ such that $\|f - f_n\|_{p,\alpha} \to E_\nu(f)_{p,\alpha}$ as $n \to \infty$. Then

$$(\exists A > 0)(\forall n \in \mathbb{N}) : \|f_n\|_{p,\alpha} \leq A$$

Using lemma 1.3, we obtain that

$$\|f_n\|_{\infty,\alpha} \leq c_3 A\nu^{\frac{2\alpha+2}{p}}$$

By the compactness theorem (see[6], 3.3.6), there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to f_\nu(x)$ as $k \to \infty$, holds uniformly on any bounded set, for some $f_\nu \in \mathfrak{M}(\nu, \infty, \alpha)$. Hence, for any $K > 0$,

$$\|f_\nu\chi_{\{0\leq x\leq K\}}\|_{p,\alpha} = \lim_{k\to\infty}\|f_{n_k}\chi_{\{0\leq x\leq K\}}\|_{p,\alpha} \leq A$$

Passing to the limit as $K \to \infty$, we obtain that $\|f_\nu\|_{p,\alpha} \leq A$, i.e. $f_\nu \in \mathfrak{M}(\nu, p, \alpha)$ Furthermore

$$\|(f-f_\nu)\chi_{\{0\leq x\leq K\}}\|_{p,\alpha} = \lim_{k\to\infty}\|(f-f_{n_k})\chi_{\{0\leq x\leq K\}}\|_{p,\alpha} \leq \lim_{k\to\infty}\|f-f_{n_k}\|_{p,\alpha} = E_\nu(f)_{p,\alpha}$$

Passing to the limit as $K \to \infty$, we obtain that $\|f - f_\nu\|_{p,\alpha} \leq E_\nu(f)_{p,\alpha}$
In view of the definition of $E_\nu(f)_{p,\alpha}$, we have $\|f - f_\nu\|_{p,\alpha} = E_\nu(f)_{p,\alpha}$.

The following theorems are analogues of the classical inverse theorems of approximation theory due to Stechkin in the case $p = \infty$ and A.F.Timan in the case $1 \leq p < \infty$ (see [4], [5]).

Theorem 2.5 For every function $f \in L_{p,\alpha}(\mathbb{R}^+)$ and every positive integer n we have

$$\omega_k(f, \frac{1}{n})_{p,\alpha} \leq \frac{c}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2k-1}E_j(f)_{p,\alpha},$$

where $c = c(k, \alpha)$ is a positive constant.

Proof .

Let $2^m \leq n < 2^{m+1}$ for any integer $m \geq 0$.
According to theorem 2.4, for $\nu \geq 0$, there exists $\Phi_\nu \in \mathfrak{M}(\nu, p, \alpha)$, such that

$\|f-\Phi_\nu\|_{p,\alpha}=E_\nu(f)_{p,\alpha}$.
By formulas (i) and (ii) of lemma 1.1, we obtain

$$\omega_k(f,\frac{1}{n})_{p,\alpha} \leq \omega_k(f-\Phi_{2^{m+1}},\frac{1}{n})_{p,\alpha}+\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{p,\alpha}\leq 2^k\|f-\Phi_{2^{m+1}}\|_{p,\alpha}+\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{p,\alpha}.$$

Therefore

$$\omega_k(f,\frac{1}{n})_{p,\alpha}\leq 2^kE_n(f)_{p,\alpha}+\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{p,\alpha}. \quad (1)$$

Now with the aid of lemmas 2.1, 2.3 and formula (iii) of lemma 1.1, we conclude that

$$\begin{aligned}
\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{p,\alpha} &\leq \frac{c_1}{n^{2k}}\|B^k\Phi_{2^{m+1}}\|_{p,\alpha}\\
&\leq \frac{c_1}{n^{2k}}\left(\|B^k\Phi_1-B^k\Phi_0\|_{p,\alpha}+\sum_{j=0}^{m}\|B^k\Phi_{2^{j+1}}-B^k\Phi_{2^j}\|_{p,\alpha}\right)\\
&\leq \frac{c_1c_2}{n^{2k}}\left(2^{4k+1}E_0(f)_{p,\alpha}+\sum_{j=0}^{m}2^{2k(j+1)+1}E_{2^j}(f)_{p,\alpha}\right)\\
&\leq \frac{c_1c_2}{n^{2k}}2^{4k+1}\left(E_0(f)_{p,\alpha}+\sum_{j=0}^{m}2^{2k(j-1)}E_{2^j}(f)_{p,\alpha}\right)\\
&\leq \frac{c_1c_2}{n^{2k}}2^{4k+1}\left(E_0(f)_{p,\alpha}+E_1(f)_{p,\alpha}+\sum_{j=1}^{m}\sum_{l=2^{j-1}+1}^{2^j}l^{2k-1}E_l(f)_{p,\alpha}\right)\\
&\leq \frac{c_1c_2}{n^{2k}}2^{4k+1}\left(E_0(f)_{p,\alpha}+E_1(f)_{p,\alpha}+\sum_{j=2}^{2^m}(j+1)^{2k-1}E_j(f)_{p,\alpha}\right)
\end{aligned}$$

Whence

$$\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{p,\alpha}\leq\frac{c_5}{n^{2k}}\sum_{j=0}^{2^m}(j+1)^{2k-1}E_j(f)_{p,\alpha}. \quad (2)$$

Thus from (1) and (2) we derive the estimate

$$\omega_k(f,\frac{1}{n})_{p,\alpha}\leq 2^kE_n(f)_{p,\alpha}+\frac{c_5}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2k-1}E_j(f)_{p,\alpha}. \quad (3)$$

By lemma 2.2 and formula (3), we have

$$\omega_k(f,\frac{1}{n})_{p,\alpha}\leq\frac{c}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2k-1}E_j(f)_{p,\alpha}.$$

Theorem 3.6 Suppose that $f\in L_{p,\alpha}(\mathbb{R}^+)$ and

$$\sum_{j=1}^{\infty}j^{2m-1}E_j(f)_{p,\alpha}<\infty.$$

Then $f \in W^m_{p,\alpha}$ and ,for every positive integer n, we have

$$\omega_k(B^m f, \frac{1}{n})_{p,\alpha} \leq C \left(\frac{1}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2(k+m)-1} E_j(f)_{p,\alpha} + \sum_{j=n+1}^{\infty} j^{2m-1} E_j(f)_{p,\alpha} \right),$$

where $C = c(k, m, \alpha)$ is a positive constant.
Proof.
Let $2^m \leq n < 2^{m+1}$ for any integer $m \geq 0$. For every positive integer $r \leq m$, we consider the series

$$B^r \Phi_1 + \sum_{j=0}^{\infty} (B^r \Phi_{2^{j+1}} - B^r \Phi_{2^j}). \quad (4)$$

It follows from lemmas 2.3 and 2.1 that the series (4) converges in the norm of $L_{p,\alpha}(\mathbb{R}^+)$ because

$$\begin{aligned} \sum_{j=0}^{\infty} \| B^r \Phi_{2^{j+1}} - B^r \Phi_{2^j} \|_{p,\alpha} &\leq c_2 \sum_{j=0}^{\infty} 2^{2r(j+1)+1} E_{2^j}(f)_{p,\alpha} \\ &= c_2 2^{2r+1} E_1(f)_{p,\alpha} + c_2 2^{4r+1} \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2^j}(f)_{p,\alpha} \\ &\leq c_2 2^{4r+1} \left(E_1(f)_{p,\alpha} + \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2^j}(f)_{p,\alpha} \right) \\ &\leq c_2 2^{4r+1} \left(E_1(f)_{p,\alpha} + \sum_{j=1}^{\infty} \sum_{l=2^{j-1}+1}^{2^j} l^{2r-1} E_l(f)_{p,\alpha} \right) \\ &\leq c_2 2^{4r+1} \sum_{j=1}^{\infty} j^{2r-1} E_j(f)_{p,\alpha} < \infty \end{aligned}$$

Note that

$$f = \Phi_1 + \sum_{j=0}^{\infty} (\Phi_{2^{j+1}} - \Phi_{2^j}).$$

Since B is a linear continuous operator (see [1]), we have

$$B^r f = B^r \Phi_1 + \sum_{j=0}^{\infty} (B^r \Phi_{2^{j+1}} - B^r \Phi_{2^j})$$

Whence $B^r f \in L_{p,\alpha}(\mathbb{R}^+)$ for $r \leq m$ and $f \in W^m_{p,\alpha}$.
By formula (i) of lemma 1.1, we obtain

$$\omega_k(B^m f, \frac{1}{n})_{p,\alpha} \leq \omega_k(B^m f - B^m \Phi_{2^{s+1}}, \frac{1}{n})_{p,\alpha} + \omega_k(B^m \Phi_{2^{s+1}}, \frac{1}{n})_{p,\alpha}.$$

Using lemmas 1.1 , 2.1 and 2.3, we get

$$\begin{aligned}
\omega_k(B^m f - B^m\Phi_{2^{s+1}}, \frac{1}{n})_{p,\alpha} &\leq 2^k \|B^m f - B^m\Phi_{2^{s+1}}\|_{p,\alpha} \\
&\leq 2^k \sum_{j=s+1}^{\infty} \|B^m\Phi_{2^{j+1}} - B^m\Phi_{2^j}\|_{p,\alpha} \\
&\leq 2^k c_2 \sum_{j=s+1}^{\infty} 2^{2m(j+1)+1} E_{2^j}(f)_{p,\alpha} \\
&\leq c_2 2^{k+4m+1} \sum_{j=s+1}^{\infty} 2^{2m(j-1)} E_{2^j}(f)_{p,\alpha} \\
&\leq c_2 2^{k+4m+1} \sum_{j=s+1}^{\infty} \sum_{l=2^{j-1}+1}^{2^j} l^{2m-1} E_l(f)_{p,\alpha} \\
&\leq c_2 2^{k+4m+1} \sum_{j=2^s+1}^{\infty} j^{2m-1} E_j(f)_{p,\alpha}
\end{aligned}$$

Whence

$$\omega_k(B^m f - B^m\Phi_{2^{s+1}}, \frac{1}{n})_{p,\alpha} \leq c_6 \sum_{j=2^s+1}^{\infty} j^{2m-1} E_j(f)_{p,\alpha}. \tag{5}$$

Now with the aid of lemmas 2.1 , 2.3 and by formula (iii) of lemma 1.1, we conclude that

$$\begin{aligned}
\omega_k(B^m\Phi_{2^{s+1}}, \frac{1}{n})_{p,\alpha} &\leq \frac{c_1}{n^{2k}} \|B^{m+k}\Phi_{2^{s+1}}\|_{p,\alpha} \\
&\leq \frac{c_1}{n^{2k}} \left(\|B^{m+k}\Phi_1 - B^{m+k}\Phi_0\|_{p,\alpha} + \sum_{j=0}^{s} \|B^{m+k}\Phi_{2^{j+1}} - B^{m+k}\Phi_{2^j}\|_{p,\alpha} \right) \\
&\leq \frac{c_1 c_2}{n^{2k}} \left(2^{4(k+m)+1} E_0(f)_{p,\alpha} + \sum_{j=0}^{s} 2^{2(k+m)(j+1)+1} E_{2^j}(f)_{p,\alpha} \right) \\
&\leq \frac{c_1 c_2}{n^{2k}} 2^{4(k+m)+1} \left(E_0(f)_{p,\alpha} + \sum_{j=0}^{s} 2^{2(k+m)(j-1)} E_{2^j}(f)_{p,\alpha} \right) \\
&\leq \frac{c_1 c_2}{n^{2k}} 2^{4(k+m)+1} \left(E_0(f)_{p,\alpha} + E_1(f)_{p,\alpha} + \sum_{j=1}^{s} \sum_{l=2^{j-1}+1}^{2^j} l^{2(k+m)-1} E_l(f)_{p,\alpha} \right) \\
&\leq \frac{c_1 c_2}{n^{2k}} 2^{4(k+m)+1} \left(E_0(f)_{p,\alpha} + E_1(f)_{p,\alpha} + \sum_{j=2}^{2^s} (j+1)^{2(k+m)-1} E_j(f)_{p,\alpha} \right)
\end{aligned}$$

Whence

$$\omega_k(B^m\Phi_{2^{s+1}}, \frac{1}{n})_{p,\alpha} \leq \frac{c_7}{n^{2k}} \sum_{j=0}^{2^s} (j+1)^{2(k+m)-1} E_j(f)_{p,\alpha}. \tag{6}$$

Thus from (5) and (6) we derive the estimate

$$\omega_k(B^m f, \frac{1}{n})_{p,\alpha} \leq C \left(\sum_{j=n+1}^{\infty} j^{2m-1} E_j(f)_{p,\alpha} + \frac{1}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2(k+m)-1} E_j(f)_{p,\alpha} \right).$$

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