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OSCILLATION THEOREMS OF SECOND ORDER NONLINEAR DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we study the oscillation of solutions of the second order delay dynamic equation

$$(r(t)g(x(t), x^{\Delta}(t)))^{\Delta} + p(t)f(x(\tau(t))) = 0,$$

on a time scale $\mathbb{T}$. Oscillation behavior of this equation is not studied before. Several new oscillation criteria are established for such a dynamic differential equations under quite general assumptions. Some examples are also given to illustrate our main results. These examples are not discussed before.

1. Introduction

In this paper, we discuss the oscillation of second-order nonlinear delay dynamic equation

$$(r(t)g(x(t), x^{\Delta}(t)))^{\Delta} + p(t)f(x(\tau(t))) = 0,\ t \in \mathbb{T},\ t \geq t_0, \tag{1}$$

subject to the hypotheses

(H_1) $\mathbb{T}$ is a time scale which is unbounded above and $t_0 \in \mathbb{T}$ with $t_0 > 0$. The time scale interval $[t_0, \infty)_{\mathbb{T}}$ is defined by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \bigcap \mathbb{T}$.
(H_2) $r(t)$ and $p(t)$ are positive right dense continuous functions on $\mathbb{T}$ such that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty, \tag{2}$$

or,

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} < \infty. \tag{3}$$

(H_3) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ for all $x \neq 0$ and there exists a positive constant L such that $\frac{f(x)}{x} \geq L$.

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(H_4) $g \in C^1(\mathbb{R}^2, \mathbb{R})$, $vg(u,v) > 0$ for all $v \neq 0$ and there exist positive constants K_1, K_2 such that

$$K_1 \leq \frac{g(u,v)}{v} \leq K_2.$$

(H_5) $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing and differentiable function such that

$$\tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty.$$

By a solution of (1), we mean a nontrivial real valued function x satisfies (1) for $t \in \mathbb{T}$. A solution x of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Eq. (1) is said to be oscillatory if all of its solutions are oscillatory. In this work, we study the solutions of (1) which are not identically vanishing eventually.

Recent attention has been given to dynamic equations on time scales. We refer the reader to Hilger [14] for a comprehensive treatment of the subject. Several authors have expounded on various aspects of this new theory (see Agarwal et al. [1] and the references cited therein). The book by Bohner and Peterson [6] summarizes and organizes much of time scale calculus. We refer also to Bohner and Peterson [7] for advances in dynamic equations on time scales.

In recent years, many results have been obtained on the oscillation and nonoscillation of dynamic equations on time scales (see for example the papers [2, 3, 4], [8, 9], [11, 12, 13], [15] and [17, 18]).

In this work, we give some new oscillation criteria of Eq. (1) by using the generalized Riccati transformation. Our results not only unify the oscillation of second order nonlinear delay differential and difference equations but also can be applied on different types of time scales.

This paper is organized as follows: In section 2, we present some preliminaries on time scales. In section 3, we give basic Lammas. In section 4, we establish some new sufficient conditions for oscillation of (1). Finally, in section 5, we present some examples to illustrate our results.

2. Some Preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) = inf\{s \in \mathbb{T}, s > t\} \text{ and } \rho(t) = sup\{s \in \mathbb{T}, s < t\}.$$

A point $t \in \mathbb{T}$, $t > inf\mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $t < sup\mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^k$ is derived from the time scale $\mathbb{T}$ as follows:
If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.
A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided that it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. By $C^1_{rd}(\mathbb{T}, \mathbb{R})$, we

mean the set of functions whose delta derivative belong to $C_{rd}(\mathbb{T},\mathbb{R})$.
For a function $f:\mathbb{T}\to\mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space), the delta derivative f^{Δ} is defined by

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\sigma(t)-t},$$

provided f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^{\Delta}(t)=\lim_{s\to t^{+}}\frac{f(\sigma(t))-f(t)}{t-s}=\lim_{s\to t^{+}}\frac{f(t)-f(s)}{t-s}$$

provided this limit exists.
A function $f:[a,b]\to\mathbb{R}$ is said to be differentiable if its derivative exists. The derivative f^{Δ} and the shift f^{σ} of a function f are related by the equation

$$f^{\sigma}=f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t).$$

The derivative rules of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$) of two differentiable functions f and g are given by

$$(f.g)^{\Delta}(t)=f^{\Delta}(t)g(t)+f^{\sigma}(t)g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g^{\sigma}(t)$$
$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t=[f(t)g(t)]_a^b-\int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t$$

or,

$$\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t=[f(t)g(t)]_a^b-\int_a^b f^{\Delta}(t)g(t)\Delta t$$

and the infinite integral is defined by

$$\int_b^{\infty} f(s)\Delta s=\lim_{t\to\infty}\int_b^{t} f(s)\Delta s.$$

Throughout this paper, we use
$d_{-}(t):=\max\{0,-d(t)\}$ and $\alpha(t):=\frac{R(t)}{R(t)+\mu(t)}$ where $R(t)=kr(t)\int_{t_0}^{t}\frac{\Delta s}{r(s)}$, for $k>0$, $t\geq t_0$.

3. Basic lemmas

In this section, we present some lemmas that we need in the proofs of our results in section 4.
Lemma 1 (Hardy et al. [[12], Theorem 41]). If A and B are nonnegative real numbers, then

$$\lambda AB^{\lambda-1}-A^{\lambda}\leq(\lambda-1)B^{\lambda},\ \ \lambda>1,$$

where the equality holds if and only if $A=B$.
Lemma 2 If (H_1)-(H_5), (2) hold and (1) has a positive solution x on $[t_0,\infty)_{\mathbb{T}}$, then $(r(t)g(x,x^{\Delta}))^{\Delta}<0$, $x^{\Delta}(t)>0$, and $x(t)>\alpha(t)x(\sigma(t))$ for $t\in[t_0,\infty)_{\mathbb{T}}$.

Proof. Since x is a positive solution of (1) on $[t_0,\infty)_{\mathbb{T}}$, then we have

$(r(t)g(x(t), x^{\Delta}(t)))^{\Delta} = -p(t)f(x(\tau(t))) < 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$.

Therefore $r(t)g(x(t), x^{\Delta}(t))$ is strictly decreasing on $[t_0, \infty)_{\mathbb{T}}$. We claim that $x^{\Delta}(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. If not, then there is $t \geq t_1$ such that

$r(t)g(x(t), x^{\Delta}(t)) \leq r(t_1)g(x(t_1), x^{\Delta}(t_1)) = c_1 < 0.$
Then,

$g(x(t), x^{\Delta}(t)) \leq \frac{c_1}{r(t)}.$

Using (H_4), we get

$K_1 x^{\Delta}(t) < g(x(t), x^{\Delta}(t)) < \frac{c_1}{r(t)}$, $K_1 > 0$.
Hence,

$x^{\Delta}(t) < \frac{c_1}{K_1}\frac{1}{r(t)}.$
Integrating from t_1 to t, we get

$x(t) < x(t_1) + \frac{c_1}{K_1}\int_{t_1}^{t} \frac{\Delta s}{r(s)} \to -\infty$ as $t \to \infty$,

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $x^{\Delta}(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. Therefore,

$$\begin{aligned} x(t) > x(t) - x(t_1) &= \int_{t_1}^{t} x^{\Delta}(s)\Delta s \\ &> \int_{t_1}^{t} \frac{1}{K_2} g(x(s), x^{\Delta}(s))\Delta s \\ &> \frac{1}{K_2}\int_{t_1}^{t} \frac{r(s)g(x(s), x^{\Delta}(s))}{r(s)}\Delta s, K_2 > 0. \end{aligned}$$

Using the fact that $r(t)g(x(t), x^{\Delta}(t))$ is strictly decreasing, we get

$$\begin{aligned} x(t) &> \frac{r(t)g(x(t), x^{\Delta}(t))}{K_2}\int_{t_1}^{t} \frac{\Delta s}{r(s)} \\ &> \frac{K_1}{K_2} r(t)x^{\Delta}(t)\int_{t_1}^{t} \frac{\Delta s}{r(s)}, K_1 > 0 \\ &> kr(t)x^{\Delta}(t)\int_{t_1}^{t} \frac{\Delta s}{r(s)} = R(t)x^{\Delta}(t), \text{for } k > 0, \end{aligned}$$

where $R(t) = kr(t)\int_{t_1}^{t} \frac{\Delta s}{r(s)}$.
And so,

$\frac{x(t)}{x^\sigma(t)} = \frac{x(t)}{x(t)+\mu(t)x^\Delta(t)} > \frac{R(t)}{R(t)+\mu(t)} = \alpha(t)$ on $[t_0,\infty)_\mathbb{T}$.

4. Main Results

Theorem 1 Assume that (H_1)-(H_5), (2) and Lemma 2 hold, and let $\tau \in C^1_{rd}([t_0,\infty)_\mathbb{T},\mathbb{T})$, $\tau([t_0,\infty)_\mathbb{T}) = [t_0,\infty)_\mathbb{T}$. If there exists a positive Δ -differentiable function $\delta(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}[L\alpha(\tau(s))p(s)\delta^\sigma(s) - \frac{r(\tau(s))(\delta^\Delta(s))^2}{4K\delta^\sigma(s)\alpha(\tau(s))\tau^\Delta(s)}]\Delta s = \infty, \tag{4}$$

where $L, K > 0$, then every solution of Eq. (1) is oscillatory on $[t_0,\infty)_\mathbb{T}$.

Proof. Assume that Eq. (1) has a nonoscillatory solution on $[t_0,\infty)_\mathbb{T}$. Also, assume that $x(t) > 0, x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_\mathbb{T}$, $t_1 \in [t_0,\infty)_\mathbb{T}$ and there is $T \in [t_0,\infty)_\mathbb{T}$ such that $x(t)$ satisfies the conclusion of Lemma 2 on $[T,\infty)_\mathbb{T}$. Consider the generalized Riccati substitution

$$w(t) = \delta(t)\frac{r(t)g(x(t),x^\Delta(t))}{x(\tau(t))}.$$

Using the delta derivative rules of the product and quotient of two functions, we have

$$\begin{aligned} w^\Delta(t) &= \delta^\Delta(t)\frac{r(t)g(x(t),x^\Delta(t))}{x(\tau(t))} + \delta^\sigma(t)(\frac{r(t)g(x(t),x^\Delta(t))}{x(\tau(t))})^\Delta \\ &= \frac{\delta^\Delta(t)}{\delta(t)}w(t) + \delta^\sigma(t)(\frac{x(\tau(t))(r(t)g(x(t),x^\Delta(t)))^\Delta - (x(\tau(t)))^\Delta(r(t)g(x(t),x^\Delta(t)))}{(x(\tau(t))x^\sigma(\tau(t))}) \\ &\le \frac{\delta^\Delta(t)}{\delta(t)}w(t) + \delta^\sigma(t)(\frac{-p(t)f(x(\tau(t)))}{x^\sigma(\tau(t))} - \frac{x^\Delta(\tau(t))\tau^\Delta(t)}{\delta(t)x^\sigma(\tau(t))}w(t)). \end{aligned}$$

Using the fact $\frac{f(x)}{x} \ge L$ and $\frac{x(t)}{x^\sigma(t)} > \alpha(t)$, we have

$$w^\Delta(t) \le \frac{\delta^\Delta(t)}{\delta(t)}w(t) + \delta^\sigma(t)(-Lp(t)\alpha(\tau(t)) - \frac{x^\Delta(\tau(t))\tau^\Delta(t)}{\delta(t)x^\sigma(\tau(t))}w(t)). \tag{5}$$

Since $(r(t)g(x,x^\Delta))^\Delta < 0$, then by integrating from $\tau(t)$ to t and using the definition of $w(t)$ we get,

$$x^\Delta(\tau(t)) > K\frac{x(\tau(t))}{\delta(t)r(\tau(t))}w(t), \quad K = \frac{1}{K_2}. \tag{6}$$

Now, from (6) in (5) we have

$$w^\Delta(t) \le \frac{\delta^\Delta(t)}{\delta(t)}w(t) - Lp(t)\alpha(\tau(t))\delta^\sigma(t) - K\frac{\delta^\sigma(t)\alpha(\tau(t))\tau^\Delta(t)}{\delta^2(t)r(\tau(t))}w^2(t). \tag{7}$$

If $A \ge 0$ and $B \ge 0$ are defined by:

$$A = \frac{(K\delta^\sigma(t)\alpha(\tau(t))\tau^\Delta(t))^{\frac12}}{\delta(t)(r(\tau(t)))^{\frac12}}w(t) \text{ and } B = \frac{\delta(t)(r(\tau(t)))^{\frac12}}{2(K\delta^\sigma(t)\alpha(\tau(t))\tau^\Delta(t))^{\frac12}},$$

then by using Lemma 1 for $\lambda = 2$, we get

$$\frac{\delta^\Delta(t)}{\delta(t)}w(t) - K\frac{\delta^\sigma(t)\alpha(\tau(t))\tau^\Delta(t)}{\delta^2(t)r(\tau(t))}w^2(t) \le \frac{r(\tau(t))(\delta^\Delta(t))^2}{4K\delta^\sigma(t)\alpha(\tau(t))\tau^\Delta(t)}. \tag{8}$$

Hence, from (7) and (8), we get

$$w^{\Delta}(t) \leq -Lp(t)\alpha(\tau(t))\delta^{\sigma}(t) + \frac{r(\tau(t))(\delta^{\Delta}(t))^2}{4K\delta^{\sigma}(t)\alpha(\tau(t))\tau^{\Delta}(t)}. \tag{9}$$

Integrating the above inequality from t_0 to t, we obtain

$$\int_{t_0}^{t}[L\alpha(\tau(s))p(s)\delta^{\sigma}(s) - \frac{r(\tau(s))(\delta^{\Delta}(s))^2}{4K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}]\Delta s \leq w(t_0) - w(t) \leq w(t_0).$$

Taking the limit supremum as $t \to \infty$, we obtain a contradiction to condition (4). Therefore, every solution of Eq. (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Theorem 2 Assume that (H_1)-(H_5), (2) and Lemma 2 hold, and let $\tau \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\tau([t_0, \infty)_{\mathbb{T}}) = [t_0, \infty)_{\mathbb{T}}$. If there exist functions $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$ (where $\mathbb{D} \equiv \{(t, s) : t \geq s \geq t_0\}$) such that

$$H(t,t) = 0,\ t \geq t_0, H(t,s) > 0,\ t > s \geq t_0 \tag{10}$$

and H has a nonpositive continuous Δ- partial derivative with respect to the second variable $H^{\Delta_s}(t, s)$ which satisfies

$$H^{\Delta_s}(\sigma(t), s) + H(\sigma(t), \sigma(s))\frac{\delta^{\Delta}(t)}{\delta(t)} = -\frac{h(t,s)}{\delta(t)}(H(\sigma(t), \sigma(s)))^{\frac{1}{2}} \tag{11}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} \theta(t,s)\Delta s = \infty, \tag{12}$$

where $\delta(t)$ is a positive Δ- differentiable function and

$$\theta(t,s) = LH(\sigma(t), \sigma(s))\alpha(\tau(s))p(s)\delta^{\sigma}(s) - \tfrac{r(\tau(s))(h_-(t,s))^2}{4K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}, \text{ for } L, K > 0,$$

then every solution of Eq. (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that Eq. (1) has a nonoscillatory solution on $[t_0, \infty)_{\mathbb{T}}$. Also, assume that $x(t) > 0,\ x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}},\ t_1 \in [t_0, \infty)_{\mathbb{T}}$ and there is $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusion of lemma 2 on $[T, \infty)_{\mathbb{T}}$. We proceed as in the proof of Theorem 1 to obtain (7).

$$Lp(t)\alpha(\tau(t))\delta^{\sigma}(t) \leq -w^{\Delta}(t) + \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) - K\frac{\delta^{\sigma}(t)\alpha(\tau(t))\tau^{\Delta}(t)}{\delta^2(t)r(\tau(t))}w^2(t). \tag{13}$$

Multiplying (13) by $H(\sigma(t),\sigma(s))$ and integrating with respect to s from t_0 to $\sigma(t)$, we obtain

$$\begin{aligned}
&\int_{t_0}^{\sigma(t)} LH(\sigma(t),\sigma(s))\delta^{\sigma}(s)p(s)\alpha(\tau(s))\Delta s\\
&\leq -\int_{t_0}^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s+\int_{t_0}^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)}w(s)\Delta s\\
&-\int_{t_0}^{\sigma(t)} KH(\sigma(t),\sigma(s))\frac{\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}{\delta^2(s)r(\tau(s))}w^2(s)\Delta s.
\end{aligned}$$

Integrating by parts and using (10) and (11), we have

$$\int_{t_0}^{\sigma(t)} LH(\sigma(t),\sigma(s))\delta^{\sigma}(s)p(s)\alpha(\tau(s))\Delta s\leq$$

$$H(\sigma(t),t_0)w(t_0)+\int_{t_0}^{\sigma(t)}[\frac{h_-(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{\frac{1}{2}}w(s)-KH(\sigma(t),\sigma(s))\frac{\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}{\delta^2(s)r(\tau(s))}w^2(s)]\Delta s.$$

If $A\geq 0$ and $B\geq 0$ are defined by:

$$A=(H(\sigma(t),\sigma(s)))^{\frac{1}{2}}\frac{(K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s))^{\frac{1}{2}}}{\delta(s)(r(\tau(s)))^{\frac{1}{2}}}w(s) \text{ and } B=\frac{h_-(t,s)(r(\tau(s)))^{\frac{1}{2}}}{2(K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s))^{\frac{1}{2}}},$$

then by using Lemma 1 for $\lambda=2$, we obtain

$$\frac{h_-(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{\frac{1}{2}}w(s)-KH(\sigma(t),\sigma(s))\frac{\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}{\delta^2(s)r(\tau(s))}w^2(s)\leq\frac{r(\tau(s))(h_-(t,s))^2}{4K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}.$$

Therefore,

$$\int_{t_0}^{\sigma(t)} LH(\sigma(t),\sigma(s))\delta^{\sigma}(s)p(s)\alpha(\tau(s))\Delta s\leq H(\sigma(t),t_0)w(t_0)+\int_{t_0}^{\sigma(t)}\frac{r(\tau(s))(h_-(t,s))^2}{4K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}\Delta s.$$

By the definition of $\theta(t,s)$, we get

$$\int_{t_0}^{\sigma(t)}\theta(t,s)\Delta s\leq H(\sigma(t),t_0)w(t_0).$$

Hence.

$$\frac{1}{H(\sigma(t),t_0)}\int_{t_0}^{\sigma(t)}\theta(t,s)\Delta s\leq w(t_0).$$

Which contradicts the assumption (12). This contradiction completes the proof.

Corollary 1 Assume that (H_1)-(H_5), (2) and Lemma 2 hold, and let $\tau\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau([t_0,\infty)_{\mathbb{T}})=[t_0,\infty)_{\mathbb{T}}$. If there exists a positive Δ -differentiable function $\delta(t)$ such that for $m \geq 1$

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m [L\alpha(\tau(s))p(s)\delta^\sigma(s) - \frac{r(\tau(s))(\delta^\Delta(s))^2}{4K\delta^\sigma(s)\alpha(\tau(s))\tau^\Delta(s)}]\Delta s = \infty, \quad (14)$$

where $L, K > 0$, then every solution of Eq. (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that Eq. (1) has a nonoscillatory solution on $[t_0, \infty)_{\mathbb{T}}$. Also, assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and there is $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusion of lemma 1 on $[T, \infty)_{\mathbb{T}}$. we proceed as in the proof of Theorem 1 to obtain (9), i.e.,

$$Lp(t)\alpha(\tau(t))\delta^\sigma(t) + \tfrac{r(\tau(t))(\delta^\Delta(t))^2}{4K\delta^\sigma(t)\alpha(\tau(t))\tau^\Delta(t)} \leq -w^\Delta(t),$$

multiplying the above inequality by $(t-s)^m$ and integrating from t_1 to t, we obtain

$$\begin{aligned}&\int_{t_1}^{t} (t-s)^m [Lp(s)\alpha(\tau(s))\delta^\sigma(s) + \frac{r(\tau(s))(\delta^\Delta(s))^2}{4K\delta^\sigma(s)\alpha(\tau(s))\tau^\Delta(s)}]\Delta s \\ &\leq -[(t-s)^m w(s)]_{t_1}^{t} + \int_{t_1}^{t} ((t-s)^m)^{\Delta_s} w^\sigma(s)\Delta s \\ &\leq (t-t_1)^m w(t_1) + \int_{t_1}^{t} ((t-s)^m)^{\Delta_s} w^\sigma(s)\Delta s.\end{aligned}$$

Since $((t-s)^m)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1} \leq 0$ for $m \geq 1$, then we have

$$\frac{1}{t^m}\int_{t_1}^{t} (t-s)^m [Lp(s)\alpha(\tau(s))\delta^\sigma(s) + \frac{r(\tau(s))(\delta^\Delta(s))^2}{4K\delta^\sigma(s)\alpha(\tau(s))\tau^\Delta(s)}]\Delta s \leq (\frac{t-t_1}{t})^m w(t_1) < \infty,$$

which contradicts the assumption (14). This contradiction completes the proof.
Theorem 3 Assume that (H_1)-(H_5) and Lemma 2 hold. If

$$\limsup_{t\to\infty} \frac{CR(\tau(t))}{r(\tau(t))} \int_{t}^{\infty} p(s)\Delta s > 1, \ C > 0, \quad (15)$$

then every solution of Eq. (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that Eq. (1) has a nonoscillatory solution on $[t_0, \infty)_{\mathbb{T}}$. Also, assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and there is $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusion of lemma 2 on $[T, \infty)_{\mathbb{T}}$. From (1), we have

$$-(r(t)g(x(t), x^\Delta(t)))^\Delta = p(t)f(x(\tau(t))) \geq Lp(t)x(\tau(t)), \text{ for } L > 0.$$

Integrating the above inequality from $\tau(t)$ to u and letting $u \to \infty$, we get

$$\int\limits_{\tau(t)}^{\infty} Lp(s)x(\tau(s))\Delta s \leq r(\tau(t))g(x(\tau(t)), x^{\Delta}(\tau(t))) - \lim_{u\to\infty} r(u)g(x(u), x^{\Delta}(u)).$$

Since $r(s)g(x(s), x^{\Delta}(s))$ is decreasing and $r(s)g(x(s), x^{\Delta}(s)) > 0$, then we obtain

$$\int\limits_{\tau(t)}^{\infty} Lp(s)x(\tau(s))\Delta s \leq r(\tau(t))g(x(\tau(t)), x^{\Delta}(\tau(t))).$$

Using (H_4), we get

$$\int\limits_{\tau(t)}^{\infty} Lp(s)x(\tau(s))\Delta s \leq K_2 r(\tau(t))x(\tau(t)).$$

Since $x(t) > R(t)x^{\Delta}(t)$, then $x(\tau(t)) > R(\tau(t))x^{\Delta}(\tau(t))$ and consequently

$$\frac{LR(\tau(t))}{K_2 r(\tau(t))} \int_{\tau(t)}^{\infty} p(s)x(\tau(s))\Delta s \leq x(\tau(t)).$$

Hence,

$$\frac{LR(\tau(t))}{K_2 r(\tau(t))} \int_{t}^{\infty} p(s)x(\tau(s))\Delta s \leq \frac{LR(\tau(t))}{K_2 r(\tau(t))} \int_{\tau(t)}^{\infty} p(s)x(\tau(s))\Delta s \leq x(\tau(t)).$$

Since $x(t)$ and $\tau(t)$ are strictly increasing, then we get

$$\frac{CR(\tau(t)}{r(\tau(t))} \int\limits_{t}^{\infty} p(s)\Delta s \leq 1, \text{ where } C = \frac{L}{K_2},$$

which contradicts the assumption (15). This contradiction completes the proof.

Theorem 4 Assume that (H_1)-(H_5), (3) hold and $\tau \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $\tau([t_0,\infty)_{\mathbb{T}}) = [t_0,\infty)_{\mathbb{T}}$. Also, assume that there exists a positive Δ -differentiable function $\delta(t)$ such that (4) holds. If

$$\int_{t_0}^{\infty} [\frac{1}{r(t)} \int_{t_0}^{t} p(s)\Delta s]\Delta t = \infty, \tag{16}$$

then every solution of Eq. (1) is oscillatory or converges to zero on $[t_0,\infty)_{\mathbb{T}}$.

Proof. We proceed as in Theorem 1, we assume that (1) has a nonoscillatory solution such that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}, t_1 \in [t_0,\infty)_{\mathbb{T}}$. As in the proof of Lemma 2, we see that there exist two possible cases for the sign of $x^{\Delta}(t)$. When $x^{\Delta}(t)$ is an eventually positive, the proof is similar to the proof of Theorem 1.

Next, suppose that $x^{\Delta}(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Then $x(t)$ is decreasing and $\lim_{t\to\infty} x(t) = b \geq 0$. We assert that $b = 0$. If not, then $x(\tau(t)) > x(t) > x(\sigma(t)) > b > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Since $f(x(\tau(t))) \geq Lb$, then there exists a number $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $f(x(\tau(t))) \geq Lx(\tau(t))$ for $t \geq t_2$.

Defining the function $u(t) = r(t)g(x(t), x^{\Delta}(t))$ and using (1), we get:

$$u^{\Delta}(t) = -p(t)f(x(\tau(t))) \leq -Lp(t)x(\tau(t)) \leq -Lbp(t) \text{ for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Hence, for $t \in [t_2,\infty)_{\mathbb{T}}$ we have

$$u(t) \leq u(t_2) - Lb \int_{t_2}^{t} p(s)\Delta s.$$

Since $u(t_2) = r(t_2)g(x(t_2), x^{\Delta}(t_2)) < 0$, then

$$u(t) \leq -Lb \int_{t_2}^{t} p(s)\Delta s.$$

Therefore,

$g(x(t), x^{\Delta}(t)) \leq -Lb\frac{1}{r(t)}\int_{t_2}^{t} p(s)\Delta s.$
Using (H_4), we get

$x^{\Delta}(t) \leq -\frac{Lb}{K_2}\frac{1}{r(t)}\int_{t_2}^{t} p(s)\Delta s$, for $K_2 > 0$.
Hence,

$\int_{t_2}^{t} x^{\Delta}(s)\Delta s \leq -\frac{Lb}{K_2}\int_{t_2}^{t}[\frac{1}{r(s)}\int_{t_2}^{s} p(\xi)\Delta\xi]\Delta s.$
From condition (16), we get $x(t) \to -\infty$, and this is a contradiction to the fact that $x(t) > 0$ for $t \geq t_1$. thus $b = 0$ and then $x(t) \to 0$ as $t \to \infty$. The proof is completed.

5. Examples

In this section, we give some examples to illustrate our main results. In fact these examples are not studied before and there is no previous Theorems determine the oscillatory behavior of such equations.
Example 1 Consider the second order nonlinear delay dynamic equation

$$(t\frac{x^{\Delta}(t)}{x^2(t)+1})^{\Delta} + \frac{\nu}{t\alpha(\tau(t))}x(\tau(t))(x^2(\tau(t))+1) = 0 \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, t_0 \geq 0, \quad (17)$$

where ν is positive parameter.
This equation has the form (1) with

$r(t) = t$, $g(x(t), x^{\Delta}(t)) = \frac{x^{\Delta}(t)}{x^2(t)+1}$, $p(t) = \frac{\nu}{t\alpha(\tau(t))}$, $f(x) = x(x^2+1)$.
Therefore, we take $L = K = 1$.

Theorem 1 can be applied for this example. In this case, we assume that

$\delta(t) = 1$.

Therefore,

$\int_{t_0}^{\infty}\frac{\Delta t}{r(t)} = \int_{t_0}^{\infty}\frac{\Delta t}{t} = \infty$ (i.e., eq. (2) holds),

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}[L\alpha(\tau(s))p(s)\delta^{\sigma}(s) - \frac{r(\tau(s))(\delta^{\Delta}(s))^2}{4K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}]\Delta s = \limsup_{t\to\infty}\int_{t_0}^{t}\frac{\nu}{s}\Delta s = \infty.$$

Hence by Theorem 1, every solution of (17) is oscillatory.
Example 2 Consider the second order nonlinear delay dynamic equation

$$(\frac{\cos^2(x(t))}{1+x^2(t)}x^{\Delta}(t))^{\Delta} + \frac{\lambda}{t^2}x(t)(2+\sin x(t)) = 0 \text{for} t \in [t_0, \infty)_{\mathbb{T}}, t_0 \geq 0, \quad (18)$$

where λ is positive constant.
This equation is of the form (1) with

$r(t) = 1$, $g(x(t), x^{\Delta}(t)) = \frac{\cos^2(x(t))}{1+x^2(t)} x^{\Delta}(t)$, $p(t) = \frac{\lambda}{t^2}$, $f(x) = x(2 + \sin x)$, $\tau(t) = t$. Therefore, we take $L = K = 1$.

It is clear that

$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \int_{t_0}^{\infty} \Delta t = \infty$ (i.e., eq. (2) holds) and $R(\tau(t)) = \tau(t) - t_0$ for $k = 1$.

Therefore, we can find $0 < b < 1$ such that

$\alpha(\tau(t)) = \frac{R(\tau(t))}{R(\tau(t))+\mu(\tau(t))} = \frac{\tau(t)-t_0}{\tau(t)-t_0+\sigma(\tau(t))-\tau(t)} = \frac{\tau(t)-t_0}{\sigma(\tau(t))-t_0} > \frac{b\tau(t)}{\sigma(\tau(t))}$ for $t \geq t_b > t_0$.

Let $\delta(t) = t$. Then by Theorem 1, we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} [L\alpha(\tau(s))p(s)\delta^{\sigma}(s) - \frac{r(\tau(s))(\delta^{\Delta}(s))^2}{4K\delta^{\sigma}(s)\alpha(\tau(s))\tau^{\Delta}(s)}]\Delta s = (\lambda b - \frac{1}{4b}) \limsup_{t\to\infty} \int_{t_0}^{t} \frac{\Delta s}{s} \Delta s = \infty,$$

for $\lambda > \frac{1}{4b^2}$. Then by Theorem 1, every solution of (18) is oscillatory if $\lambda > \frac{1}{4b^2}$.

Example 3 Consider the second order nonlinear delay dynamic equation

$$(t^{1-\gamma} \frac{x^2(t)(x^{\Delta}(t))^3}{1 + x^2(t)(x^{\Delta}(t))^2})^{\Delta} + \frac{\nu}{t\sigma(t)} x(x^{2\gamma} + 3) = 0 for\ t \in [t_0, \infty)_{\mathbb{T}},\ t_0 \geq 0, \quad (19)$$

where ν is a positive constant and $\gamma \geq 1$ is the quotient of odd positive integers. Here,

$r(t) = t^{1-\gamma}$, $g(x(t), x^{\Delta}(t)) = \frac{x^2(t)(x^{\Delta}(t))^3}{1+x^2(t)(x^{\Delta}(t))^2}$, $p(t) = \frac{\nu}{t\sigma(t)}$, $f(x) = x(x^{2\gamma} + 3)$ and $\tau(t) = t^{\frac{1}{\gamma}}$.
It is clear that
$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\gamma}} = \infty$ for $\gamma \geq 1$ (i.e., eq. (2) holds) and $R(\tau(t)) > \tau(t) - t_0 > n\tau(t)$ for $k = 1$, $0 < n < 1$.

Let $C = 1$. Then by Theorem 3, we have

$$\limsup_{t\to\infty} \frac{CR(\tau(t))}{r(\tau(t))} \int_{t}^{\infty} p(s)\Delta s > \limsup_{t\to\infty} \frac{n\tau(t)}{\tau^{1-\gamma}(t)} \int_{t}^{\infty} \frac{\nu}{s\sigma(s)} \Delta s > n\nu > 1 \text{ for } \nu > \frac{1}{n}.$$

Then by Theorem 3 every solution of Eq. (19) is oscillatory if $\nu > \frac{1}{n}$.

Remark 1

(1) The important point to note here is that the current work is a generalization of the recent results which were established in [5].
(2) The results are obtained in the given examples can not be obtained by using either Theorem 1 in [10] or the results in [16].

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