Electronic Journal of Mathematical Analysis and Applications,
Vol. 3(2) July 2015, pp. 147-153.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

ON CONVERGENCE THEOREMS IN METRIC SPACES

M. A. AHMED, A. KAMAL , ASMAA M. ABD-ELAL

Abstract. In this paper, we establish some convergence theorems to a unique fixed point for any map in metric spaces. These theorems generalize and unify the results of Ahmed [1] and Ahmed et al [3, 4].

1. Introduction

In the last two decades, some convergence theorems to a unique fixed point for generalized types of quasi-nonexpansive mappings in metric spaces have appeared (see, e.g., [2, 3], [10, 11, 12]). On the other hand, in 2007, Ahmed [1] introduced a new iteration and proved a convergence theorem of this iteration to a unique fixed point for any map in metric spaces. Also, there are some remarks on convergence theorems such as Kirk [7]. Following [1, 2, 3], let (X, d) be a metric space. Assume that $T : D \subseteq X \longrightarrow X$ is any map and $F(T)$ is the set of all fixed points of T.

Definition 1.1 The mapping $T : D \longrightarrow X$ is said to be quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ if $\{x_n\} \subseteq D$ and for all $n \in N \cup \{0\}$ ($N :=$ *the set of all positive integers*) and for each $p \in F(T)$, $d(x_{n+1}), p) \leq d(x_n, p)$. is defined by [2].

As in [2, 3] the quasi-nonexpansiveness w.r.t. a sequence $\{x_n\} \Rightarrow$ the weak quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ but the converse of the last implication may not be true.

Definition 2.1 A subset D of a normed space X is called balanced (or circled) if $x \in D$ and $| \gamma | \leq 1$ implies $\gamma x \in D$ is defined by [13] .

Following [8], we assume that $L_c := \{x \in X : F(x) \leq c\}$, where $F : X \to R$. We use the symbol μ to denote the usual Kuratowski measure of noncompactness. For some properties of μ, see Zeidler ([14], pages 493-495).

The following definitions is given by Angrisani and Clavelli [6].
Definition 3.1 Let D be a topological space. The function $F : X \to R$ is said to be a regular-global-inf (r.g.i) at $x \in X$ if $F(x) > \inf_X(F)$ implies that there

---

2010 *Mathematics Subject Classification.* 47H10,54H25.
Key words and phrases. Fixed point, Metric space.
Submitted May 29, 2014.

exists $\epsilon > 0$ such that $\epsilon < F(x) - \inf_X(F)$ and a neighborhood N_X of x such that $F(y) > F(x) - \epsilon$ for each $y \in N_x$. If this condition holds for each $x \in X$, then F is said to be an r.g.i on X.

Definition 4.1 Let D be a convex subset of a normed space X. A mapping $T : D \to D$ is called directionally nonexpansive if $\|T(x) - T(m)\| \leq \|x - m\|$ for each $x \in D$ and for all $m \in [x, T(x)]$ where $[x, y]$ denotes the segment joining x and y; that is, $[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$.

2. Main Results

First we state and prove our main results as follows.

Theorem 2.1 Let $\{x_n\}$ be a sequence in a subset D of a metric space (X, d) and $T : D \longrightarrow X$ any map such that $F(T) \neq \emptyset$. Assume that $F(T)$ is a closed set. Then $\{x_n\}$ converges to a unique point in $F(T)$ if and only if $\lim\limits_{n\to\infty} ?d(x_n, F(T)) = 0$.

Proof. $(\Rightarrow)$ Suppose that $\{x_n\}$ converges to a unique point in $F(T)$. In this case, $\lim\limits_{n\to\infty} ?x_n$ exists in $F(T)$. From the closedness of $F(T)$, we find that $\lim\limits_{n\to\infty} ?x_n \in F(T) = \overline{F(T)}$.

Hence, we obtain that $d(\lim\limits_{n\to\infty} ?x_n, F(T)) = 0$. Since $d : X \times 2^X \longrightarrow [0, \infty)$ is a uniformly continuous (see, [5], page 49), we get that

$$\lim_{n\to\infty} ?d(x_n, F(T)) = d(\lim_{n\to\infty} ?x_n, F(T)) = 0.$$

$(\Leftarrow)$ Suppose that $\lim\limits_{n\to\infty} ?d(x_n, F(T)) = 0$. Since d is uniformly continuous, then

$$d(\lim_{n\to\infty} ?x_n, F(T)) = \lim_{n\to\infty} ?d(x_n, F(T)) = 0$$

Therefore, we have that $\lim\limits_{n\to\infty} ?x_n \in \overline{F(T)}$. The closedness of $F(T)$ leads to $\lim\limits_{n\to\infty} ?x_n \in F(T)$.

Remark 2.1 Theorem 2.1 generalizes and improves each of Theorem 2.1 [1], Theorem 2.1 [3] and Theorem 2.1 [4] for certain reasons. These reasons are the following:

(1) The completeness of X is superfluous in Theorem 2.1 [3, 4];

(2) The existence of $\lim\limits_{n\to\infty} (\gamma T)^n(x_0), |\gamma| \leq 1$, is superfluous in Theorem 2.1 [1];

(3) The quasi-nonexpansiveness of T w.r.t. a sequence $\{x_n\}$ (resp., The weak quasi-nonexpansiveness of T w.r.t. a sequence $\{x_n\}$) in Theorem 2.1 [3] (resp., Theorem 2.1 [4]) is superfluous.

Corollary 2.1 Let $F(T)$ be nonempty closed set. Then

(i) $\lim\limits_{n\to\infty} ?d(x_n, F(T)) = 0$ if $\{x_n\}$ converges to a point in $F(T)$.

(ii) $\{x_n\}$ converges to a point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0$, T is quasi nonexpansive w.r.t $\{x_n\}$ and X is complete.

Corollary 2.2 Let $F(\gamma T), |\gamma| \leq 1$, be a nonempty set. Then

(i) $\lim_{n\to\infty} ?d((\gamma T)^n(x_0), F(\gamma T)) = 0$ if $\{(\gamma T)^n(x_0)\}$ converges to a unique point in $F(\gamma T)$,

(ii) $\{(\gamma T)^n(x_0)\}$ converges to a unique point in $F(\gamma T)$ if $\lim_{n\to\infty} ?d((\gamma T)^n(x_0), F(\gamma T)) = 0$, $T((\gamma T)^{n-1}(x_0))$ for all $n \in N$ and for some $x_0 \in D$, $F(\gamma T)$ is a closed set and $\lim_{n\to\infty} ?(\gamma T)^n(x_0)$ exists.

Corollary 2.3 Let $\{x_n\}$ be sequence in a subset D of a metric space (X, d) and let $T : D \longrightarrow X$ be a map such that $F(T) \neq \emptyset$. then

(i) $\lim_{n\to\infty} ?d(x_n, F(T)) = 0$ if $\{x_n\}$ converges to a point in $F(T)$;

(ii) $\{x_n\}$ converges to a point in $F(T)$ if $\lim_{n\to\infty} ?d(x_n, F(T)) = 0$ is closed set, T is weakly quasi-nonexpansive with respect to $\{x_n\}$, and X is complete.

As a consequence of Theorem 2.1, We state and prove the following theorem

Theorem 2.2 Let $\{x_n\}$ be a sequence in a subset D of a metric space (X, d) and $T : D \longrightarrow X$ any map such that $F(T) \neq \emptyset$. Assume that

(i) $F(T)$ is closed set;

(ii) $d(x_n, F(T))$ is monotonically decreasing sequence in $[0, \infty)$;

(iii) $\lim_{n\to\infty} ?d(x_n, x_{n+1}) = 0$;

(iv)If the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} ?d(y_n, y_{n+1}) = 0$, then

$$\liminf_{n\to\infty} ?d(y_n, F(T)) = 0 \ \text{ or } \ \limsup_{n\to\infty} ?d(y_n, F(T)) = 0.$$

Then $\{x_n\}$ converges to a unique point in $F(T)$.

proof. From (ii) and the boundedness from below by zero of the sequence $d(x_n, F(T))$, we find that $\lim_{n\to\infty} ?d(x_n, F(T))$ exists and equals say, y. Therefore,

$$\liminf_{n\to\infty} ?d(y_n, F(T)) = \limsup_{n\to\infty} ?d(y_n, F(T)) = y.$$

The conditions (iii) and (iv) asserts that $\liminf_{n\to\infty} ?d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} ?d(x_n, F(T)) = 0$.

From the uniqueness of y, then $\lim_{n\to\infty} ?d(x_n, F(T)) = 0$. Applying Theorem 2.1, we conclude that $\{x_n\}$ converges to unique point in $F(T)$.

Corollary 2.4 Let $\{x_n\}$ be a complete metric space and let $F(T)$ be nonempty closed set. Assume that

(i) T is quasi-nonexpansive with respect to $\{x_n\}$;

(ii) $\lim_{n\to\infty} ?d(x_n, x_{n+1}) = 0$, equivalently, $\{x_n\}$ is cauchy sequence;

(iii) if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} ?d(y_n, y_{n+1}) = 0$, then

$$\liminf_{n\to\infty} ?d(y_n, F(T)) = 0 \text{ or } \limsup_{n\to\infty} ?d(y_n, F(T)) = 0.$$

Then $T^n(x_0)$ converges to a point in $F(T)$.

Corollary 2.5 Let D be balanced subset of normed space X and let $F(\gamma T), |\gamma| \leq 1$, be a nonempty closed set. Assume that

(i) $T((\gamma T)^{n-1}(x_0)) \in D$ for all $n \in N$ and (γT) is quasi-nonexpansive w.r.t. $\{(\gamma T)^n(x_0)\}$;

(ii) (γT) is asymptotically regular at $x_0 \in D$;

(iii) if the sequence $\{y_n\}$ in D satisfies $\lim_{n\to\infty} \|(I - \gamma T)(y_n)\| = 0, then$

$$\liminf_n d(y_n, F(\gamma T)) = 0 \quad \text{or} \quad \limsup_n d(y_n, F(\gamma T)) = 0.$$

If $\lim_{n\to\infty} (\gamma T)^n(x_0)$ exists, then $\{(\gamma T)^n(x_0)\}$ converges to a unique point in $F(\gamma T)$.

Corollary 2.6 Let $\{x_n\}$ be a sequence in a subset D of a complete metric space (X, d) and $T : D \longrightarrow X$ be a map such that $F(T) \neq \emptyset$ is a closed set. Assume that

(i) T is weakly quasi-nonexpansive with respect to $\{x_n\}$;

(ii) $\{d(x_n, F(T))\}$ is monotonically decreasing sequence in $[0, \infty)$;

(iii) $\lim_{n\to\infty} ?d(x_n, x_{n+1}) = 0$;

(iv) If the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} ?d(y_n, y_{n+1}) = 0$, then

$$\liminf_{n\to\infty} ?d(y_n, F(T)) = 0 \quad \text{or} \quad \limsup_{n\to\infty} ?d(y_n, F(T)) = 0.$$

Then $\{x_n\}$ converges to a point in $F(T)$.

From ([8], Corollary 2.4) and Theorem 2.1, we state and prove the following theorem.

Theorem 2.3 Let T be self map on a complete metric space (X, d) satisfy:

$$\begin{array}{ll} (i) & d(T(x), T^2(x)) \le hd(x, T(x)) \text{ for some } h \in (0,1) \text{and for all } x \in X; \\ (ii) & \mu(T(L_c)) \le k\mu(L_c) \text{ for some } k < 1 \text{ and for all } c > 0; \\ (iii) & \text{F is an r.g.i. on } X; \\ (iv) & \{x_n\} \text{ is a sequence in} X \text{such that} \lim_{n\to\infty} d(x_n, Tx_n) = 0. \\ & Then\ \{x_n\} \text{ converges to a unique point in } F(T). \end{array}$$

Proof. Using ([8], Corollary 2.4) and the conditions (i), (ii) and (iii) lead to the nonemptyness and closdness of $F(T)$. Since the condition (iv) holds, then $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Applying Theorem 2.1, we obtain that $\{x_n\}$ converges to a unique point in $F(T)$.

Corollary 2.7 Let $T : X \longrightarrow X$ be a mapping of a complete metric space (X, d) satisfying

(i) $d(T(x), T^2(x)) \le hd(x, T(x))$ for some $h \in (0, 1)$ and for all $x \in X$;

(ii) $\mu(T(L_c)) \le k\mu(L_c)$ for some $k < 1$ and for all $c > 0$;

(iii) F is an r.g.i. on X;

(iv) $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and T is weakly quasi-nonexpansive with respect to $\{x_n\}$.
Then $\{x_n\}$ converges to a point $F(T)$.

From ([8], Theorem 3.3) and Theorem 2.2, we state the following theorem without proof.
Theorem 2.4 Let D be a bounded closed subset of a Banach space X. Suppose $T : D \to D$ satisfies

$$\begin{array}{ll} (i) & \|T(x) - T^2(x)\| \le h\|x - T(x)\| \text{ for some } h \in (0,1) \text{ and for all } x \in X; \\ (ii) & \mu(T(L_c)) \le k\mu(L_c) \text{ for some } k < 1 \text{ and for all } c > 0; \\ (iii) & \text{F is an r.g.i. on } X; \\ (iv) & \{x_n\} \subseteq D \text{ satisfies} \lim_{n\to\infty} \|x_n - Tx_n\| = 0. \\ & Then\ \{x_n\} \text{ converges to a unique point in } F(T). \end{array}$$

3. Applications

Motivated by the paper of Ahmed [2], we apply Theorem 2.1 and Theorem 2.2 for obtaining convergence theorems in metric spaces.
Definition 3.1 Let D be a nonempty subset of a metric space (X, d). A point $q \in D \subseteq X$ is closed fixed point of $T : D \times D \longrightarrow X$ if $q = T(q, u)$ for some $u \in D$ is defined by [2].

Theorem 3.1 Let $\{x_n\}$ be a sequence in a subset D of a metric space (X,d) and let $T : D \times D \longrightarrow X$ be any map such that $F(T) \neq \emptyset$. Assume that $F(T)$ is closed set $\{x_n\}$ converges to a unique point in $F(T)$ if and only if

$$\liminf_{n\to\infty} ?d(x_n, F(T)) = 0.$$

Theorem 3.2 Let $\{x_n\}$ be a sequence in a subset D of a metric space (X,d) and $T : D \times D \longrightarrow X$ be any map such that $F(T) \neq \emptyset$. Assume that

(i) $F(T)$ is closed set;

(ii) $d(x_n, F(T))$ is monotonically decreasing sequence in $[0,\infty)$;

(iii) $\lim_{n\to\infty} ?d(x_n, x_{n+1}) = 0$;

(iv) If the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} ?d(y_n, y_{n+1}) = 0$, then

$$\liminf_{n\to\infty} ?d(y_n, F(T)) = 0 \quad \text{or} \quad \limsup_{n\to\infty} ?d(y_n, F(T)) = 0.$$

Then $\{x_n\}$ converges to a point in $F(T)$.

References

[1] M. A. Ahmed, A convergence theorem of a sequence of a new iterate to fixed points for any mapping and its applications, Proc. Math. Phys. Soc. Egypt (2007), 1-6.

[2] M. A. Ahmed, Common fixed points of hybrid maps and applications, Computers and Math. Appl. 60(2010), 1888- 1895.

[3] M. A. Ahmed and F. M. Zeyada, On convergence of a sequence in complete metric spaces and its applications to some iterates of quasi-nonexpansive mappings, J. Math. Anal. Appl. 274(1)(2002), 458 - 465.

[4] M. A. Ahmed and F. M. Zeyada, Some convergence theorems of a sequence in complete metric spaces and its applictions, J. Fixed Point Theory and applictions, Volume 2010, Article ID 647085, 10 pages.

[5] J.-P. Aubin, Applied Abstract Analysis, Wiley, 1977.

[6] M. Angrisani and M. Clavelli, Synthetic approaches to problems of fixed points in metric space, Annali di Matematica pure ed Applicata. Serie Quarta, 170(1996), 1-12.

[7] W. A. Kirk, Remarks on approximation and approximate fixed points in metric fixed point theory, Ann. Univ. Mariae Curie-Sklodowska Sect. ALI.2 15(1997), 167-178.

[8] W. A. Kirk and L. M. Saliga, Some results on existence and approximation in metric fixed point theory, J. Computational and Applied Math., 113(1-2)(2000), 141-152.

[9] E. Kreyszig, Introdctory functional analysis with applications, John Wiley and Sons, 1989.

[10] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mappings, J.Math. Anal. Appl., 259(2001) 1-7.

[11] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mappings with error member, J.Math. Anal. Appl., 259(2001) 18-24.

[12] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mappings with error member with uniform convex Banach space, J.Math. Anal. Appl., 266(2002) 468-471.

[13] M. Reed and B. Simon, Methods of modern mathematical physics, I: Functional analysis, Academic Press, Inc. 1972.

[14] E. Zeidler, Nonlinear functional analysis and its applications. I: Fixed-point theorems, Springer, New York, NY, USA, 1986.

M. A. Ahmed
Faculty of Science, Assiut University, Assiut, Egypt
E-mail address: mahmed68@yahoo.com

A. Kamal
Faculty of Science, portsaid University, portsaid, Egypt
E-mail address: alaa_mohamed1@yahoo.com

Asmaa M. Abd-Elal
Faculty of Science, portsaid University, portsaid, Egypt
E-mail address: asmaamoh1221@yahoo.com