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A GENERALIZED STATISTICAL CONVERGENCE VIA MODULI

M. GÜRDAL, M. O. ÖZGÜR

Abstract. By using modulus functions we study $\mathcal{I}^f$-statistical convergence and $\mathcal{I}^f$-statistical Cauchy in normed spaces where $\mathcal{I}$ is an ideal.

1. Introduction

The concept of the statistical convergence of a sequence of reals $S = \{s_n\}$ was first introduced by Fast [4] (see also [14]) as follows: let A be a subset of $\mathbb{N}$. Then the asymptotic density of A denoted by $\delta(A) := \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in A\}|$, where the vertical bars denote the cardinality of the enclosed set. A sequence $S = \{s_n\}_{k\in\mathbb{N}}$ is said to converge statistically to s and we write $\lim_{n\to\infty} s_n = s$ $(stat)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |s_k - s| \ge \varepsilon\}| = 0.$$

Properties of statistically convergent sequences were studied in [3, 5, 6, 8, 11]. In [8], Kolk begins to study the applications of statistical convergence to Banach spaces. In [3] there are important results that relate the statistical convergence to classical properties of Banach spaces.

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers, $\mathcal{P}(X)$ stands for the power set of X.

In [9] an interesting generalization of this notion was proposed. Namely, it is easy to check that the family $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$ forms an ideal of subsets of $\mathbb{N}$. Thus, one may consider an arbitrary ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ (assumed non-trivial, i.e. $\emptyset \ne \mathcal{I} \ne \mathcal{P}(\mathbb{N})$) to modify the definition of statistical convergence as follows. The sequence $\{s_n\}_{n\in\mathbb{N}}$ of elements of real numbers is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if, for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |s_n - s| \ge \varepsilon\} \in \mathcal{I}$. The article [9] contains many examples and properties of $\mathcal{I}$-convergence. Statistical convergence can be generalized by using ideals. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$. The sequence (s_k) of X is said to be $\mathcal{I}$-statistically convergent to s, if for each $\varepsilon > 0$, $\delta > 0$ the set

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \le n : \|s_k - s\| \ge \varepsilon\}| \ge \delta \right\} \in \mathcal{I}.$$

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This idea was first mentioned by Savaş and Das [13] in 2011 and has been further studied by Gürdal and Sarı [7] and Yamancı and Gürdal [15].

Following [9] $\mathcal{I}$ is called admissible if it contains all singletons. The ideal $\mathcal{I}^{fin}$ of all finite subsets of $\mathbb{N}$ is the smallest admissible ideal in $\mathcal{P}(\mathbb{N})$. Observe that the usual convergence, in a given X normed space coincides with $\mathcal{I}^{fin}$-convergence, and that the usual convergence implies $\mathcal{I}$-convergence, for any admissible ideal $\mathcal{I}$.

We recall that $f:\mathbb{R}^+\to\mathbb{R}^+$ is called modulus function, or simply modulus, if it is satisfies:

(1) $f(s)=0$ if and only if $s=0$.
(2) $f(s+p)\leq f(s)+f(p)$ for every $s,p\in\mathbb{R}^+$
(3) f is increasing.
(4) f is continuous from the right at 0.

From these properties it is clear that a modulus function must be continuous on $\mathbb{R}^+$. Examples of modulus functions are $f(s)=\dfrac{s}{1+s}$ and $f(s)=s^p$ $(0<p\leq 1)$.

The notion of a modulus function was introduced by Nakano [12], Maddox [10] have introduced and discussed some properties of sequence space defined by using modulus function.

In this note we intend to unify these two approaches and use ideals and modulus functions to introduce the concept of $\mathcal{I}^f$-statistical convergence in line of recent works of Savaş and Das [13] and Aizpuru et al. [2], and investigate some of their consequences.

2. Definitions and notations

First we recall some of the basic concepts, which will be used in this paper. All the concepts listed below are studied by Aizpuru et al. [2].

Let f be an unbounded modulus function. The f-density of a set $A\subseteq\mathbb{N}$ is defined by

$$\delta_f(A)=\lim_n\frac{f(|A(n)|)}{f(n)}$$

in case this limit exists.

Let X be a normed space and let $(s_n)_n$ be a sequence in X. We will say that the f-statistical limit of $(s_n)_n$ is $s\in X$, and write $f\text{-}st\lim s_n=s$, if for each $\varepsilon>0$ we have $\delta_f(\{i\in\mathbb{N}:\|s_i-s\|>\varepsilon\})=0$.

Note that if $A\subseteq\mathbb{N}$ is finite we have that there exist $n_0,p\in\mathbb{N}$ such that $|A(n)|=p$ if $n\geq n_0$ and it will be $\delta_f(A)=0$ for each unbounded f. Therefore, if $\lim s_n=s$ and f is an unbounded modulus function then $f\text{-}st\lim s_n=s$.

It is straightforward to see that $f\text{-}st\lim(s_n+p_n)=f\text{-}st\lim s_n+f\text{-}st\lim p_n$ and $\alpha f\text{-}st\lim s_n=f\text{-}st\lim \alpha s_n$, whenever $\alpha\in\mathbb{K}$ and the limits on the right sides exist. Also, it is easy to prove that for $X=\mathbb{K}$ we have $f\text{-}st\lim s_ip_i=f\text{-}st\lim s_i f\text{-}st\lim p_i$.

Although it is quite clear that $\delta(A)=1-\delta(\mathbb{N}\backslash A)$ whenever one of the sides exist, the situation is a bit different for unbounded moduli. First, assume $A\subseteq\mathbb{N}$ and $\delta_f(A)=0$. For every $n\in\mathbb{N}$

$$f(n)\leq f(|A(n)|)+f(|(\mathbb{N}\backslash A)(n)|)$$

and so

$$1\leq\frac{f(|A(n)|)}{f(n)}+\frac{f(|(\mathbb{N}\backslash A)(n)|)}{f(n)}\leq\frac{f(|A(n)|)}{f(n)}+1.$$

By taking limits we deduce that $\delta_f(\mathbb{N}\backslash A)=1$. On the other hand, the naturally expected reciprocal is false:

Example 1 Let $f(x)=\log(x+1)$. If $E=\{n^2:n\in\mathbb{N}\}$ and $O=\mathbb{N}\backslash A$ then we have $\delta_f(E)=\delta_f(O)=1$. Moreover, if $S=\{n^2:n\in\mathbb{N}\}$ then $\delta_f(S)=\frac{1}{2}$, $\delta_f(\mathbb{N}\backslash S)=1$ and so f-$st\lim\chi_{(n)}$ does not even exist, whereas $st\lim\chi_{(n)}=0$.

Let us note that for any unbounded modulus f and any $A\subseteq\mathbb{N}$ we have that $\delta_f(A)=0$ implies $\delta(A)=0$. Indeed, if $\delta_f(A)=0$ then for every $p\in\mathbb{N}$ there exists $n_0\in\mathbb{N}$ such that if $n\geq n_0$ then $f(|A(n)|)\leq\frac{1}{p}f(n)\leq\frac{1}{p}pf\left(\frac{1}{p}n\right)=f\left(\frac{1}{p}n\right)$, which implies $|A(n)|\leq\frac{1}{p}n$ and so $\delta(A)=0$.

3. Main results

In this section we study the concepts of ideal f-statistical convergence and ideal f-statistical Cauchy convergence on normed spaces and prove some important results. The results are analogues to those given by Aizpuru et al. [2]. These notions generalize the notions of f-statistical convergence.

Following the line of Savaş and Das [13] we now introduce the following definition using ideals and modulus functions.

Definition 1 Let $\mathcal{I}\subset\mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$. The sequence (s_k) of X is said to be $\mathcal{I}^f$-statistically convergent to L (in short $\mathcal{I}^f$-$stat$), if for each $\varepsilon>0$, $\delta>0$ the set

$$\left\{n\in\mathbb{N}:\frac{f(|\{k\leq n:\|s_k-L\|\geq\varepsilon\}|)}{f(n)}\geq\delta\right\}\in\mathcal{I}$$

or equivalently if for each $\varepsilon>0$

$$\delta_{\mathcal{I}^f}(A_n(\varepsilon))=\mathcal{I}\text{-}\lim_n\frac{f(|A_n(\varepsilon)|)}{f(n)}=0$$

where $A_n(\varepsilon)=\{k\leq n:\|s_k-L\|\geq\varepsilon\}$.

For $\mathcal{I}=\mathcal{I}^{fin}$, $\mathcal{I}^f$-statistical convergence coincides with f-statistical convergence.

Corollary 1 Let f,g be unbounded moduli, X a normed space, $S=(s_n)_n$ a sequence in X and $s,p\in X$. We have

(1) The $\mathcal{I}^f$-statistical convergence implies the $\mathcal{I}$-statistical convergence (to the same limit)

(2) The $\mathcal{I}^f$-statistical limit is unique whenever it exists.

(3) Moreover, two different methods of statistical convergence are always compatible, which means that if $\mathcal{I}^f$-$st\lim s_n=s$ and $\mathcal{I}^g$-$st\lim s_n=p$ then $s=p$.

We now introduce our main theorem.

Theorem 2 Let $S=(s_n)_n$ be a sequence in a normed space X and f an unbounded modulus. Then $s_n\to L$ $\left(\mathcal{I}^f\text{-}stat\right)$ if and only if there is a subset A of $\mathbb{N}$ such that $s_{n_k}\to L$ (in the usual sense) as $k\to\infty$, where $\mathbb{N}\backslash A=\{n_k:k\in\mathbb{N}\}$ and A has $\mathcal{I}^f$-density zero.

Proof. Assume $s_n \to L$ $\left(\mathcal{I}^f\text{-}stat\right)$. For each $\varepsilon_n = \frac{1}{n}$, $n = 1,2,...$, there exists a positive integer r_n with the sequence $\{r_n\}_{n=1}^{\infty}$ strictly increasing such that $r_n \in B_n := \left\{k \in \mathbb{N} : \|s_k - L\| > \frac{1}{n}\right\}$ and $\dfrac{f\left(|B_n(k)|\right)}{f(k)} \leq \frac{1}{n}$ whenever $k \geq r_n$. Set

$$A = \cup_{n\in\mathbb{N}} \left\{k : r_n \leq k < r_{n+1} \text{ and } \|s_k - L\| > \frac{1}{n}\right\}.$$

The subsequence of S obtained by removing the terms with in A clearly converges to L in the ordinary sense.

Now we will show that A has $\mathcal{I}^f$-density zero. For every $k \geq r_1$ there exists $n \in \mathbb{N}$ such that $r_n \leq k < r_{n+1}$ and if $j \in A(k)$ then $j < r_{n+1}$, which implies $j \in B_n$. Therefore $A(k) \subseteq B_n(k)$ and thus

$$\frac{f\left(|A(k)|\right)}{f(k)} \leq \frac{f\left(|B_n(k)|\right)}{f(k)} \leq \frac{1}{n}$$

so A has $\mathcal{I}^f$-density zero.

Now we look at converse. Assume that $n_k = \mathbb{N}\backslash A$, $s_{n_k} \to L$ (in the ordinary sense) and A has $\mathcal{I}^f$-density zero. We must show that $s_n \to L$ $\left(\mathcal{I}^f\text{-}stat\right)$. Given $\varepsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that $\|s_{n_k} - L\| < \varepsilon$ for each $k > k_0$. Then for every $\delta > 0$

$$\left\{n \in \mathbb{N} : \frac{f\left(|\{k \in \mathbb{N} : \|s_n - L\| > \varepsilon\}|\right)}{f(n)} \geq \delta\right\} \subseteq A \cup \{n_1 < n_2 < ... < n_{k_0}\}.$$

The set on the right-hand side belongs to $\mathcal{I}$ and so it follows that $s_n \to L$ $\left(\mathcal{I}^f\text{-}stat\right)$.

Definition 2 A sequence $S = (s_k)_k$ is $\mathcal{I}^f$-statistically Cauchy if for every $\varepsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\left\{n \in \mathbb{N} : \frac{f\left(|\{k \in \mathbb{N} : \|s_k - s_N\| > \varepsilon\}|\right)}{f(n)} \geq \delta\right\} \in \mathcal{I}.$$

It is easy to show that if a sequence is $\mathcal{I}^f$-statistically convergent then it is $\mathcal{I}^f$-statistically Cauchy. The converse is true if the complete.

Theorem 3 Let X be a Banach space, f an unbounded modulus. Then every $\mathcal{I}^f$-statistically Cauchy sequence is $\mathcal{I}^f$-statistically convergent.

Proof. For every $k \in \mathbb{N}$ let s_{N_k} be such that

$$\left\{n \in \mathbb{N} : \frac{f\left(|\{i \in \mathbb{N} : \|s_i - s_{N_k}\| > \varepsilon\}|\right)}{f(n)} > \delta\right\} \in \mathcal{I}.$$

Consider the sets $I_k = \underset{j\leq k}{\cap} B\left(s_{N_j}, \frac{1}{j}\right)$ and $J_k = \{i \in \mathbb{N} : s_i \notin I_k\}$. Then we have $diam\left(I_k\right) \leq \frac{2}{k}$ and $J_k = \cup_{j\leq k} \left\{i \in \mathbb{N} : \left\|s_i - s_{N_j}\right\| > \dfrac{1}{j}\right\}$ for every each $k \in \mathbb{N}$, which implies that J_k has $\mathcal{I}^f$-density zero. Consequently each $I_k \neq \varnothing$. Note also that $I_1 \supseteq I_2 \supseteq ...$ and $J_1 \subseteq J_2 \subseteq ....$

As in the proof of the previous theorem we can find a positive integer r_k with the sequence $\{r_k\}_{k=1}^{\infty}$ strictly increasing such that $r_k \in J_k$ and if $i \geq r_k$ then $\frac{f(|J_k(i)|)}{f(i)} \leq \frac{1}{k}$. Let $A = \cup_{k\in\mathbb{N}} \{n : r_k \leq i < r_{k+1} \text{ and } J_k\}$. Therefore $A(i) \subseteq J_k(i)$ and thus

$$\frac{f(|A(i)|)}{f(i)} \leq \frac{f(|J_k(i)|)}{f(i)} \leq \frac{1}{k}$$

so A has $\mathcal{I}^f$-density zero. By the Cantor theorem for complete spaces, $\bigcap_{k\in\mathbb{N}} I_k = s$. Let us prove that $\lim_{i\in\mathbb{N}\setminus A} s_i = s$.

Given $\varepsilon > 0$, choose $j \in \mathbb{N}$ with $\frac{2}{j} < \varepsilon$. If $i \geq r_j$ and $i \in \mathbb{N}\setminus A$ then there exists $k \geq j$ such that $r_k \leq i < r_{k+1}$ and thus $i \notin J_k$, which implies $s_i \in I_k$ and thus $\|s_i - s\| \leq \frac{2}{k} \leq \frac{2}{j} < \varepsilon$, as desired. The theorem is proved.

The following lemma can be proved by the same method in [2], and therefore the proof is omitted.

Lemma 4 If $A \subset \mathbb{N}$ is infinite then there exists an unbounded module f such that $\delta_{\mathcal{I}^f}(A) = 1$.

Theorem 5 Let $(s_n)_n$ be sequence in X. If for every unbounded modulus f there exists $\mathcal{I}^f$-$st\lim s_n$ then all these limits are the same $L \in X$ and $\{s_n\}_{n\in\mathbb{N}}$ also converges to L in the norm topology.

Proof. That is limit is common for all the moduli is a direct consequence of Corollary 1. If it is false that $\lim s_n = L$ there exists $\varepsilon > 0$ such that $A = \{n \in \mathbb{N} : \|s_n - L\| > \varepsilon\}$ is infinite. Choose an unbounded modulus f satisfying $\delta_{\mathcal{I}^f}(A) = 1$, this clearly contradicts the assumption that $\mathcal{I}^f$-$st\lim s_n = L$.

4. Conclusion

In this paper, by using modulus functions, we introduce a new type of summability notion, namely, $\mathcal{I}^f$-statistical convergence and $\mathcal{I}^f$-statistical Cauchy for real sequences in normed spaces. Further, note that, if we take the notions of $\mathcal{I}^{2,f}$-statistical convergence and $\mathcal{I}^{2,f}$-statistical Cauchy of double sequences for unbounded moduli f in normed spaces analogously Aizpuru et al. [1], then we get the generalized results of Theorems 3.1, 3.2, 3.6. of [1] for double sequences.

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M. Gürdal
Department of Mathematics, Süleyman Demirel University, Isparta, Turkey
E-mail address: gurdalmehmet@sdu.edu.tr

M. O. Özgür
Department of Mathematics, Süleyman Demirel University, Isparta, Turkey
E-mail address: on_es0755@hotmail.com