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SOME FIXED POINT THEOREMS IN GENERALIZED TYPES OF METRIC SPACES

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Abstract. In this paper, the concept of generalized contraction mapping has been used in proving fixed point theorems. We have establish a fixed point theorem in complete dislocated quasi-metric space using some new types of contraction conditions. We have also prove a unique fixed point theorem in right dislocated metric (*rd*-metric) as well as in left dislocated metric (*ld*-metric) spaces. Our establish results extend and generalize some well-known results of the literature.

1. Introduction

The notion of metric space, introduced by Frechet in 1906, is one of the useful topic not only in mathematics but also in several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many ways. An incomplete list of such attempts are following: symmetric space, b-metric space, partial metric space, quasi metric space, fuzzy metric space, dislocated metric space, dislocated quasi-metric space, right and left dislocated metric spaces etc.

The increasing applications of fixed point theorems of metric spaces and their generalizations in various branches of engineering and other sciences attracted several researchers to work on them in recent past. For instance, the generalization of well-known Banach contraction principle of metric space to the dislocated metric space proved by Hitzler and Seda [4] play a key role in topology, logic programming semantics and electronic engineering etc. Zeyada et al. [11] initiated the concept of dislocated quasi-metric space and generalized the result of Hitzler and Seda [4] in dislocated quasi-metric space. Results on fixed point in dislocated quasi-metric space are followed by Aage and Salunke [1], Isufati [5], Muraliraj and Hussain [7] and recently by Mujeeb and Sarwar [8, 10].

The concept of dislocated metric space was further generalized by Ahmad et al. [2] followed by the result established in [9] by Rao. The purpose of this article is to prove some fixed point results in generalized types of dislocated metric space.

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We have derived a fixed point result in dislocated quasi-metric space for a single continuous self-mapping using some new type of rational contraction conditions. We have also proved a fixed point theorem in right as well as in left dislocated metric spaces for a single self-mapping. Our established results generalize some well-known existing results in the literature.

2. Preliminaries

Throughout this paper $\mathbb{R}^+$ will represent the set of non negative real numbers.
Definition 2.1. [4]. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a distance function satisfying the conditions

d_1) $d(x,x) = 0$;
d_2) $d(x,y) = d(y,x) = 0$ implies that $x = y$;
d_3) $d(x,y) = d(y,x)$;
d_4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

If d satisfy the conditions from d_1 to d_4 then it is called metric on X, if d satisfy conditions d_2 to d_4 then it is called dislocated metric (d-metric) on X and if d satisfy conditions d_2 and d_4 only then it is called dislocated quasi-metric(dq-metric) on X.

Clearly every metric space is a dislocated metric space and dislocated quasi-metric space. Also every dislocated metric space is dislocated quasi-metric space but the converse is not necessarily true as clear form the following example.
Example 2.2 Let X=$\mathbb{R}^+$. Define the distance function $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = |x| \quad \text{for all } x, y \in X.$$

Clearly X is dislocated quasi-metric space but not a metric nor dislocated metric space.
In our main work we will use the definition of dq-convergent sequence, Cauchy sequence, contraction and completeness in dislocated quasi-metric space which can be found in [11].

Now we state the following simple but important lemma and theorem without proof in the context of dislocated quasi-metric space.
Lemma 2.3. [11]. Limit of a convergent sequence in dislocated quasi-metric space is unique.
Theorem 2.4. [11]. Let (X, d) be a complete dislocated quasi-metric space and $T : X \to X$ be contraction. Then T has a unique fixed point.

Since dislocated quasi-metric space is the proper generalization of dislocated metric space, this concept (dislocated metric space) was further generalized by Ahmad et al. in [2] to right and left dislocated metric spaces.
Definition 2.5. A distance function d is called right dislocated metric (rd-metric) iff it satisfies (d_2) and the condition

(Rd) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

Definition 2.6. A distance function d is called left dislocated metric (ld-metric) iff it satisfies (d_2) and the condition

(Ld) $d(x,y) \leq d(z,x) + d(z,y)$ for all $x, y, z \in X$

and the pair (X, d) is called left (right) dislocated metric space.

Obviously every d-metric, ld-metric and rd-metric are different generalizations of a metric function.
The following examples show that the conditions (d_4), (Rd) and (Ld) of a distance on X satisfying (d_1) are independent of each other.

Example 2.7. Let $X = \{0, 1\}$ define

$$d(0,0) = d(1,1) = 0, d(0,1) = 1 \text{ and } d(1,0) = 0.$$

Then d is dislocated quasi-metric but not rd-metric nor ld-metric on X. Because

$$d(0,1) > d(0,0) + d(1,0),\ d(0,1) > d(1,0) + d(1,1).$$

Example 2.8. Let $X = \{0, 1, 2\}$ define

$$d(0,0) = d(1,1) = d(1,2) = 0, d(2,2) = 2,$$
$$d(1,0) = d(0,1) = d(0,2) = d(2,0) = d(2,1) = 1.$$

Then d is rd-metric but not dq-metric nor ld-metric on X, to see that

$$d(2,2) > d(2,1) + d(1,2),\ d(2,2) > d(1,2) + d(1,2).$$

If we define d^* on $X \times X$ by $d^*(x,y) = d(y,x)$ then d^* is a ld-metric but not rd-metric nor dq-metric on X. Because

$$d^*(2,2) > d^*(2,1) + d^*(2,1),\ d^*(2,2) > d^*(2,1) + d^*(1,2).$$

Now we recall some basic definitions in the context of right and left dislocated metric spaces which can be seen in [2].

Definition 2.9. A sequence $\{x_n\}$ in distance space (X, d) is rd-converges (resp. ld-converges) to x in X if

$$\lim_{n\to\infty} d(x, x_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, x) = 0.$$

In this case x is called rd-limit (resp. ld-limit) of $\{x_n\}$ in X.

Definition 2.10. A sequence $\{x_n\}$ in rd-metric (ld-metric) space (X, d) is called Cauchy sequence iff for $\epsilon > 0$ and there exist $n_0 \in N$ (N= the set of natural numbers) such that

$$d(x_n, x_m) < \epsilon \ \ \forall \ \ m, n \geq n_0.$$

Definition 2.11. A rd-metric (ld-metric) space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Lemma 2.12.[2]. rd-limit (ld-limit) of a convergent sequence in rd-metric (ld-metric) space is unique.

Theorem 2.13.[2]. Let (X, d) be a complete rd-metric (ld-metric) space if $T : X \to X$ is contraction. Then T has a unique fixed point in X.

Remark 2.14. In a distance space (X, d) we will denote the set $\{x \in X : d(x, x) = 0\}$ by X_0.

Lemma 2.15.[9]. Let (X, d) be a rd-metric (ld-metric) space and $\{x_n\}$ be a sequence in X having rd-limit (ld-limit) $x \in X$. Then $x \in X_0$.

Aage and Salunke [1] proved the following theorem in the context of dislocated quasi-metric space.

Theorem 2.16. Let (X, d) be a complete dq-metric space and suppose there exist non-negative constants a_1, a_2, a_3 with $a_1 + a_2 + a_3 < 1$. Let $T : X \to X$ be a continuous mapping satisfying

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Recently, Rao in [9] generalized the Kannan type mapping [6] in complete rd-metric as well as in ld-metric spaces.

3. Main Results

The following theorem generalize Theorem 2.4.

Theorem 3.1. Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous mapping satisfying the condition

$$d(Tx,Ty) \le \alpha \cdot d(x,y) + \beta \cdot \frac{d^2(x,Tx)[1+d(x,y)]}{d(x,y)[1+d(x,Tx)]} + \gamma \cdot \frac{d(y,Ty)[1+d^2(x,Tx)]}{1+d^2(x,y)} + \mu \cdot \frac{d(x,Ty)d^2(x,Tx)}{d(x,y)[d(x,y)+d(y,Ty)]} + \delta \cdot [d(x,Ty)+d(x,y)] \quad (1)$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \mu, \delta \ge 0$ with $\alpha+\beta+\gamma+\mu+3\delta < 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, ..., x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Using (1) we have

$$d(x_n,x_{n+1}) \le \alpha \cdot d(x_{n-1},x_n) + \beta \cdot \frac{d^2(x_{n-1},Tx_{x-1})[1+d(x_{n-1},x_n)]}{d(x_{n-1},x_n)[1+d(x_{n-1},Tx_{n-1})]} + \gamma \cdot \frac{d(x_n,Tx_n)[1+d^2(x_{n-1},Tx_{n-1})]}{1+d^2(x_{n-1},x_n)} + \mu \cdot \frac{d(x_{n-1},Tx_n)d^2(x_{n-1},Tx_{n-1})}{d(x_{n-1},x_n)[d(x_{n-1},x_n)+d(x_n,Tx_n)]} + \delta \cdot [d(x_{n-1},Tx_n)+d(x_{n-1},x_n)]$$

$$\le \alpha \cdot d(x_{n-1},x_n) + \beta \cdot \frac{d^2(x_{n-1},x_n)[1+d(x_{n-1},x_n)]}{d(x_{n-1},x_n)[1+d(x_{n-1},x_n)]} + \gamma \cdot \frac{d(x_n,x_{n+1})[1+d^2(x_{n-1},x_n)]}{1+d^2(x_{n-1},x_n)} + \mu \cdot \frac{d(x_{n-1},x_{n+1})d^2(x_{n-1},x_n)}{d(x_{n-1},x_n)[d(x_{n-1},x_n)+d(x_n,x_{n+1})]} + \delta \cdot [d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)]$$

$$\le \alpha \cdot d(x_{n-1},x_n) + \beta \cdot d(x_{n-1},x_n) + \gamma \cdot d(x_n,x_{n+1}+ \mu \cdot d(x_{n-1},x_n) + 2\delta \cdot d(x_{n-1},x_n) + \delta \cdot d(x_n,x_{n+1}).$$

Finally we have

$$d(x_n,x_{n+1}) \le \frac{\alpha+\beta+\mu+2\delta}{1-(\gamma+\delta)} \cdot d(x_{n-1},x_n).$$

Let

$$h = \frac{\alpha+\beta+\mu+2\delta}{1-(\gamma+\delta)} < 1.$$

So

$$d(x_n,x_{n+1}) \le h \cdot d(x_{n-1},x_n).$$

Also

$$d(x_{n-1},x_n) \le h \cdot d(x_{n-2},x_{n-1}).$$

Therefore

$$d(x_n,x_{n+1}) \le h^2 \cdot d(x_{n-2},x_{n-1}).$$

Similarly proceeding we get

$$d(x_n, x_{n+1}) \leq h^n \cdot d(x_0, x_1).$$

Taking limit $n \to \infty$, as $h < 1$ so $h^n \to 0$ implies

$$d(x_n, x_{n+1}) \to 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete dq-metric space. So there must exist $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now to show that u is a fixed point of T. Since T is continuous therefore

$$\lim_{n\to\infty} Tx_n = Tu \quad \Rightarrow \quad \lim_{n\to\infty} x_{n+1} = Tu \quad \Rightarrow \quad Tu = u.$$

Hence u is the fixed point of T.

Uniqueness: Let $u \neq v$ are two distinct fixed points of T then by (1)

$$d(u,v) = d(Tu,Tv) \leq \alpha\cdot d(u,v) + \beta\cdot\frac{d^2(u,Tu)[1+d(u,v)]}{d(u,v)[1+d(u,Tu)]} + \gamma\cdot\frac{d(v,Tv)[1+d^2(u,Tu)]}{1+d^2(u,v)}$$

$$+\mu\cdot\frac{d(u,Tv)d^2(u,Tu)}{d(u,v)[d(u,v)+d(v,Tv)]} + \delta\cdot[d(u,Tv)+d(u,v)].$$

Since u and v are fixed points of T so the above inequality becomes

$$d(u,v) \leq \alpha\cdot d(u,v) + \beta\cdot\frac{d^2(u,u)[1+d(u,v)]}{d(u,v)[1+d(u,u)]} + \gamma\cdot\frac{d(v,v)[1+d^2(u,u)]}{1+d^2(u,v)}$$

$$+\mu\cdot\frac{d(u,v)d^2(u,u)}{d(u,v)[d(u,v)+d(v,v)]} + \delta\cdot[d(u,v)+d(u,v)].$$

Using (1) and the fact that u, v are the fixed points of T one can get

$$d(u,u) = d(v,v) = 0.$$

Thus the above inequality take the form

$$d(u,v) \leq (\alpha + 2\delta)\cdot d(u,v).$$

Which is possible only if $d(u,v) = 0$. Similarly we can show that $d(v,u) = 0$. Hence $u = v$. Therefore fixed point of T is unique.

Now we prove right and left dislocated version of Theorem 2.

Theorem 3.2. Let (X,d) be a complete rd-metric space and $T : X \to X$ is a mapping satisfying

$$d(Tx,Ty) \leq \alpha\cdot d(x,y) + \beta\cdot d(x,Tx) + \gamma\cdot d(y,Ty) \tag{2}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma < 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots\ldots\ldots\ldots, x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Using (2) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha\cdot d(x_{n-1}+x_n) + \beta\cdot d(x_{n-1}, Tx_{n-1}) + \gamma\cdot d(x_n, Tx_n).$$

By the definition of the sequence we get

$$d(x_n, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_n) + \beta \cdot d(x_{n-1}, x_n) + \gamma \cdot d(x_n, x_{n+1}).$$

Simplification yields

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \beta}{1 - \gamma} \cdot d(x_{n-1}, x_n).$$

Let

$$h = \frac{\alpha + \beta}{1 - \gamma} < 1.$$

So the above inequality become

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n).$$

Also

$$d(x_{n-1}, x_n) \leq h \cdot d(x_{n-2}, x_{n-1}).$$

Thus

$$d(x_n, x_{n+1}) \leq h^2 \cdot d(x_{n-2}, x_{n-1}).$$

Similarly proceeding we get

$$d(x_n, x_{n+1}) \leq h^n \cdot d(x_0, x_1).$$

Since $h < 1$. Taking limit $n \to \infty$, so $h^n \to 0$ and

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete rd-metric space so there must exist $u \in X$ such that

$$\lim_{n \to \infty} (u, x_n) = 0.$$

Now to show that u is the fixed point of T consider

$$\begin{aligned} d(u, Tu) &\leq d(u, x_n) + d(Tu, x_n) = d(u, x_n) + d(Tu, Tx_{n-1}) \\ &\leq d(u, x_n) + \alpha \cdot d(u, x_{n-1}) + \beta \cdot d(u, Tu) + \gamma \cdot d(x_{n-1}, Tx_{n-1}) \\ &\leq d(u, x_n) + \alpha \cdot d(u, x_{n-1}) + \beta \cdot d(u, Tu) + \gamma \cdot d(x_{n-1}, x_n) \\ &\leq d(u, x_n) + \alpha \cdot d(u, x_{n-1}) + \beta \cdot d(u, Tu) + h^{n-1}\gamma \cdot d(x_0, x_1). \end{aligned}$$

Taking limit $n \to \infty$ we have

$$d(u, Tu) \leq \beta \cdot d(u, Tu).$$

Which is possible only if

$$d(u, Tu) = 0. \tag{3}$$

Also

$$d(Tu, u) \leq d(Tu, Tu) + d(u, Tu)$$
$$d(Tu, u) \leq \alpha \cdot d(u, u) + \beta \cdot d(u, Tu) + \gamma \cdot d(u, Tu) + d(u, Tu).$$

So by (3) and Lemma 2 we have

$$d(Tu, u) = 0. \tag{4}$$

So by (3) and (4) $Tu = u$. Hence u is the fixed point of T.

Uniqueness: Uniqueness follows as in Theorem 2 in [1].

Theorem 3.3. Let (X,d) be a complete ld-metric space and $T : X \to X$ is a mapping satisfying

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot d(x, Tx) + \gamma \cdot d(y, Ty) \tag{5}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point in X.

Proof. Proceeding like Theorem 3 one can get that $\{x_n\}$ is a Cauchy sequence in complete ld-metric space X. So there must exist $u \in X$ such that

$$\lim_{n\to\infty}(x_n, u) = 0.$$

Now to show that u is the fixed point of T consider

$$d(u, Tu) \leq d(x_n, u) + d(x_n, Tu) = d(x_n, u) + d(Tx_{n-1}, Tu)$$

$$\begin{aligned} d(u, Tu) &\leq d(x_n, u) + \alpha \cdot \cdot d(x_{n-1}, u) + \beta \cdot d(x_{n-1}, Tx_{n-1}) + \gamma \cdot d(u, Tu) \\ &\leq d(x_n, u) + \alpha \cdot d(x_{n-1}, u) + \beta \cdot \cdot d(x_{n-1}, x_n) + \gamma \cdot \cdot d(u, Tu) \\ &\leq d(x_n, u) + \alpha \cdot d(x_{n-1}, u) + h^{n-1}\beta \cdot d(x_0, x_1) + \gamma \cdot d(u, Tu). \end{aligned}$$

Taking limit $n \to \infty$ we have,

$$d(u, Tu) \leq \gamma \cdot d(u, Tu).$$

Which is possible only if

$$d(u, Tu) = 0. \tag{6}$$

Also

$$d(Tu, u) \leq d(u, Tu) + d(u, u).$$

Using (6) and Lemma 2 we have

$$d(Tu, u) = 0. \tag{7}$$

From (6) and (7) $Tu = u$. Hence u is the fixed point of T.

Uniqueness: Uniqueness follows as in Theorem 2 in [1].

Remark 3.4. We have the following remarks from the above theorems.

- Our established Theorem 3.2 and 3.3, generalize the result of [1] in to right and left dislocated metric spaces.
- From Examples 2.7 and 2.8, it is concluded that Theorem 3.2 and 3.3, are independent of each other and from the result established by Aage and Salunke [1].

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