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NUMERICAL APPROACH FOR SOLVING A CLASS OF NONLINEAR MIXED VOLTERRA FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT. This paper is interested in solving a class of nonlinear mixed Volterra Fredholm integral equations (MVFIE) numerically. In most traditional numerical methods, the original problem is transformed to its corresponding system of nonlinear algebraic equations in which the problem of uniqueness and convergence appear. In this work, based on Adomian decomposition method (ADM), we present a direct numerical technique called a discrete Adomian decomposition method (DADM). The main advantage of DADM is its direct applicability to the problem. Another advantage is that the coefficient matrices are not changed during the computation of all components. Based on a new formula of Adomian polynomials, convergence of the technique is discussed and the error is estimated.

1. Introduction

Many problems of mathematical physics are reduced to the solution of mixed integral equations in the nonlinear case. These type of integral equations have rarely been studied to solve numerically and primary works in this area have been done in the last two decades (see [1] -[7]). Most numerical methods reduce the solution of the nonlinear integral equation to the solution of a nonlinear system of algebraic equations. The iteration methods, for example Newton's method, for solving such cumbersome nonlinear system is usually sensitive to the selection of initial guess. DADM can overcome this obstacle and solve nonlinear integral equation, see section 2.

The topic of ADM, introduced by Adomian [8], [9] has been rapidly growing in recent years. ADM possesses a great potential in solving different kinds of functional equations. Application of ADM to different types of integral equations was discussed by many authors for example (see [10] -[14]). In this paper we consider the nonlinear mixed Volterra-Fredholm integral equation

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega k(x,t,\zeta,\tau) F(u(\zeta,\tau)) \, d\zeta d\tau, \tag{1}$$

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where $u(x,t)$ is an unknown function, the functions $f(x,t)$ and $k(x,t,\zeta,\tau)$ are analytical on $D=[0,T]\times\Omega$ and Ω is a closed subset of $\mathbb{R}^n, n=1,2,3$. This type of equation appears in theory of parabolic boundary value problems, the mathematical modelling of the spatio-temporal development of an epidemic and various physical and biological problems [15]. Detailed descriptions and analyses of these models may be found in ([16] -[21]). Several numerical methods for obtaining the approximate solution of equation (1) with continuous kernel are known [1] - [7]. The interested reader should consult the fine expositions by Linz [22], Goldberg [23], Atkinson [24, 25], Delves and Mohammed [26] for numerical methods and consult the book by Tricomi [27] for information concerning analytical solution methods.

In this work we assume $f(x,t)$ is bounded $\forall(x,t)\in D$, the kernel function is bounded such that $|k(x,t,\zeta,\tau)|\leq M, \forall\, 0\leq\tau\leq t\leq T<1$ and $\forall\,(x,\zeta)\in\Omega\times\Omega$. The nonlinear term $F(u(\zeta,\tau))$ is Lipschitz continuous with $|F(u)-F(z)|\leq L\,|u-z|$, L is Lipschitz constant and has Adomian polynomials representation

$$F(u)=\sum_{n=0}^{\infty}A_n(u_0,u_1,u_2,\cdots,u_n) \tag{2}$$

El-Kalla in [28, 29] deduced a new formula to the Adomian's polynomials A_n which can be written in the form

$$A_n=F(S_n)-\sum_{i=0}^{n-1}A_i, \tag{3}$$

where the partial sum $S_n=\sum_{i=0}^{n}u_i$, $A_0=F(u_0)$. Formula (3) is called an accelerated Adomian polynomials and it was used successfully in [13] for solving a class of nonlinear fractional differential equations and in [30] for solving a class of nonlinear partial differential equations. Formula (3) has the advantage of absence of any derivative terms in the recursion, thereby allowing for ease of computation. Applying ADM to (1) yields

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t), \tag{4}$$

where the components $u_i(x,t), i\geq 0$ are computed using the following recursive relations

$$u_0(x,t)=f(x,t), \tag{5}$$

$$u_{m+1}(x,t)=\lambda\int_0^t\int_\Omega k(x,t,\zeta,\tau)\,A_m(\zeta,\tau)\,d\zeta d\tau,\ m\geq 0. \tag{6}$$

The computation of each component $u_i(x,t), i\geq 1$ requires the computation of integrals in equation (6). If the evaluation of these integrals analytically are possible, ADM can be applied in a simple manner. In case where the evaluation of any integral in (6) is analytically impossible, DADM can be directly applied, please see the details in section 2. In section 3, convergence analysis will be introduced including the sufficient condition that guarantees a unique solution to problem (1) (see Theorem 1), convergence of ADM will be discussed (see Theorem 2), the maximum absolute truncation error of the Adomian's series solution (4) will be estimated (see Theorem 3) and equivalence between DADM and ADM will be introduced (see Theorem 4). Finally, to verify the theoretical results, some numerical examples are presented in section 4.

2. Discrete Adomian Decomposition Method

DADM is a numerical version of ADM. In paper [31], DADM is used to solve linear and nonlinear Fredholm integral equation. Also in paper [32], DADM is used to solve a class of nonlinear Fredholm Volterra integral equations in Two Dimensions. DADM arises when the quadrature rules are used to approximate the integrals which can not be computed analytically. Consider any numerical integration scheme to approximate definite integral by the following formula [33] - [34]

$$\int_a^b g(s)ds \approx \sum_{j=0}^{n} w_{n,j}\ g(s_{n,j}), \tag{7}$$

where $g(s)$ is continuous function on $\Omega = [a, b]$, $s_{n,j} = a + jh$ are the nodes of the quadrature rule, $h = (b - a)/n$ and $w_{n,j}$, $j = 0, 1, 2, ..., n$ are the weight functions. The idea is to discretize the independent variables x and t just before applying the quadrature rule. This gives an opportunity to evaluate the integrals in equation (6) numerically but, of course, at the discretization points of the independent variables. Thus, the discrete version of equations (5) and (6) respectively, take the form

$$\widetilde{u}_0(s_{n,i}, s_{n,j}) = f(s_{n,i}, s_{n,j}), \quad and \tag{8}$$

$$\widetilde{u}_{m+1}(s_{n,i}, s_{n,j}) = \lambda \sum_{r=0}^{n} \sum_{q=0}^{j} w_{n,r} w_{n,q}\ k(s_{n,i}, s_{n,j}, s_{n,r}, s_{n,q})\ A_m(s_{n,r}, s_{n,q}) \tag{9}$$

where $m \geq 0$, $s_{n,i} = a + ih$, $i = 0, 1, ..., n$, $s_{n,j} = j\frac{T}{n}$, $j = 0, 1, ..., n$ and $w_{n,r}$ are the weight functions of any numerical integration scheme. The approximate solution of equation (1) using DADM can be computed as

$$\widetilde{u}(s_{n,i}, s_{n,j}) = \sum_{m=0}^{\infty} \widetilde{u}_m(s_{n,i}, s_{n,j}). \tag{10}$$

Rewriting equations (8), (9) and (10) at specific $x = s_{n,i}$ in matrix form

$$\mathbf{U}_0 = \mathbf{F}, \tag{11}$$

$$\mathbf{U}_{m+1} = \mathbf{C}\ \mathbf{A}_m \mathbf{D}\ , \quad m \geq 0, \quad \text{and} \tag{12}$$

$$\mathbf{U} = \sum_{m=0}^{\infty} \mathbf{U}_m \tag{13}$$

where $\mathbf{U}_0$, $\mathbf{U}_m$, $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{F}$ are vectors with dimension $(n + 1) \times 1$, the matrices $\mathbf{C}$ and $\mathbf{A}_m$ have dimension $(n + 1) \times (n + 1)$ such that

$$\begin{aligned}
\mathbf{U}_m &= \left[\widetilde{u}_m(s_{n,i}, s_{n,j})\right],\ j = 0, 1, ..., n,\ m \geq 0, \\
\mathbf{F} &= \left[f(s_{n,i}, s_{n,j})\right],\ j = 0, 1, ..., n, \\
\mathbf{U} &= \left[\widetilde{u}(s_{n,i}, s_{n,j})\right],\ j = 0, 1, ..., n, \\
\mathbf{A}_m &= \left[A_m(s_{n,r}, s_{n,q})\right],\ r = 0, 1, ..., n,\ q = 0, 1, ..., n, \\
\mathbf{C} &= \left[c_{r,q}\right],\ r = 0, 1, ..., n,\ q = 0, 1, ..., n, \\
c_{r,q} &= \begin{cases} \lambda w_{n,q}\ k(s_{n,i}, s_{n,j}, s_{n,r}, s_{n,q}), & j \geq 1 \text{ and } r \leq j \\ 0 \qquad , & j = 0 \text{ or } r > j \end{cases}, \\
\mathbf{D} &= \left[w_{n,r}\right],\ r = 0, 1, ..., n.
\end{aligned}$$

The main advantage of DADM is that the computation of the solution need not to solve nonlinear algebraic system of equations like Nystrom method or projection method. Another advantage of DADM is that the matrices C and D are unchanged during the computation of components U_m, $m \geq 1$ in equation (12).

3. Convergence Analysis

Convergence of the Adomian series solution was studied for different problems and by many authors. In ([35] -[36]), convergence was investigated when the method applied to a general functional equations and to specific type of equations in ([37] -[38]). In convergence analysis, Adomian polynomials play a very important role however, these polynomials cannot utilize all the information concerning the obtained successive terms of the series solution, which could affect directly the accuracy as well as the convergence region and the convergence rate.

3.1. **Uniqueness Theorem.**

Theorem 1. *Problem (1) has a unique solution whenever* $0 < \alpha < 1$, *where,* $\alpha = |\lambda|\, LMT(b-a)$

Proof. Let $E = (C\,[D]\,, \|\cdot\|)$ the Banach space of all continuous functions on $D = [0,T] \times \Omega$, $\Omega = [a,b]$ with the norm $\|u\,(x,t)\| = \max\limits_{\forall (x,t)\in D} |u\,(x,t)|$. Define a mapping $G:\, E \to E$ where,

$$Gu\,(x,t) = f\,(x,t) + \lambda \int_0^t \int_\Omega k\,(x,t,\zeta,\tau)\, F\,(u\,(\zeta,\tau))\, d\zeta d\tau,$$

and let u and $\overset{*}{u}$ be two different solutions to (1) then

$$\begin{aligned}
\left\| u - \overset{*}{u} \right\| &= \max_{\forall (x,t)\in D} \left| \lambda \int_0^t \int_\Omega k\,(x,t,\zeta,\tau) \left[F\,(u) - F\left(\overset{*}{u}\right)\right] d\zeta d\tau \right| \\
&\leq \max_{\forall (x,t)\in D} |\lambda| \int_0^t \int_\Omega |k\,(x,t,\zeta,\tau)| \left| F\,(u) - F\left(\overset{*}{u}\right)\right| d\zeta d\tau \\
&\leq |\lambda|\, LM \left| u - \overset{*}{u} \right| \int_0^t \int_\Omega d\zeta d\tau \\
&\leq |\lambda|\, LMT(b-a) \left| u - \overset{*}{u} \right| \\
&\leq \alpha \left| u - \overset{*}{u} \right|
\end{aligned}$$

Under the condition $0 < \alpha < 1$ the mapping G is contraction therefore, by the Banach fixed-point theorem for contraction , there exist a unique solution to problem (1) and this completes the proof. □

3.2. **Convergence Theorem.**

Theorem 2. *The series solution (4) converges if:* $0 < \alpha < 1$ *and* $|u_1| < \infty$

Proof. Let S_n and S_m be arbitrary partial sums with $n \geq m$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in Banach space E,

$$\begin{aligned}
\|S_n - S_m\| &= \max_{\forall (x,t)\in D} |S_n - S_m| \\
&= \max_{\forall (x,t)\in D} \left| \sum_{i=m+1}^{n} u_i(x,t) \right| \\
&= \max_{\forall (x,t)\in D} \left| \sum_{i=m+1}^{n} \lambda \int_0^t \int_\Omega k(x,t,\zeta,\tau) A_{i-1}(\zeta,\tau)\, d\zeta d\tau \right| \\
&= \max_{\forall (x,t)\in D} \left| \lambda \int_0^t \int_\Omega k(x,t,\zeta,\tau) \sum_{i=m}^{n-1} A_i\ (\zeta,\tau)\, d\zeta d\tau \right|.
\end{aligned}$$

From formula (3) we have $\sum_{i=m}^{n-1} A_i = F(S_{n-1}) - F(S_{m-1})$ so,

$$\begin{aligned}
\|S_n - S_m\| &= \max_{\forall (x,t)\in D} \left| \lambda \int_0^t \int_\Omega k(x,t,\zeta,\tau) \left[F(S_{n-1}) - F(S_{m-1})\right] \right| \\
&\leq \max_{\forall (x,t)\in D} |\lambda| \int_0^t \int_\Omega |k(x,t,\zeta,\tau)| \left|F(S_{n-1}) - F(S_{m-1})\right| d\zeta d\tau \\
&\leq \alpha \|S_{n-1} - S_{m-1}\|.
\end{aligned}$$

Let, $n = m+1$ then

$$\|S_{m+1} - S_m\| \leq \alpha \|S_m - S_{m-1}\| \leq \alpha^2 \|S_{m-1} - S_{m-2}\| \leq ... \leq \alpha^m \|S_1 - S_0\|.$$

From the triangle inequality we have,

$$\begin{aligned}
\|S_n - S_m\| &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\
&\leq \left[\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}\right] \|S_1 - S_0\| \\
&\leq \alpha^m \left[1 + \alpha + \alpha^2 + ... + \alpha^{n-m-1}\right] \|S_1 - S_0\| \\
&\leq \alpha^m \left(\frac{1-\alpha^{n-m}}{1-\alpha}\right) \|u_1(x,t)\|.
\end{aligned}$$

Since $0 < \alpha < 1$ so, $(1-\alpha^{n-m}) < 1$ then,

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1-\alpha} \max_{\forall (x,t)\in D} |u_1(x,t)|. \tag{14}$$

But $|u_1| < \infty$ so, as $m \to \infty$ then $\|S_n - S_m\| \to 0$. We conclude that $\{S_n\}$ is a Cauchy sequence in $C[D]$ so, the series converges and the proof is complete. □

3.3. Error Estimate.

Theorem 3. *The maximum absolute truncation error of the series solution (4) to problem (1) is estimated to be:* $\max_{\forall (x,t)\in D} |u(x,t) - \sum_{i=0}^{m} u_i(x,t)| \leq \frac{K\alpha^{m+1}}{L(1-\alpha)}$ *where* $K = \max_{\forall (x,t)\in D} |F(f(x,t))|$.

Proof. From Theorem 2 inequality (14) we have

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1-\alpha} \max_{\forall (x,t)\in D} |u_1(x,t)|.$$

As $n \to \infty$ then $S_n \to u(x,t)$ and

$$\begin{aligned}\max_{\forall(x,t)\in D}|u_1(x,t)| &= \max_{\forall(x,t)\in D}\left|\lambda\int_0^t\int_\Omega k(x,t,\zeta,\tau)\,A_0(\zeta,\tau)\,d\zeta d\tau\right| \\ &= \max_{\forall(x,t)\in D}\left|\lambda\int_0^t\int_\Omega k(x,t,\zeta,\tau)\,F(u_0)\;d\zeta d\tau\right| \\ &\leq |\lambda|\,MT(b-a)\max_{\forall(x,t)\in D}|F(u_0)|\end{aligned}$$

so,

$$\|x(t)-S_m\| \leq \frac{\alpha^{m+1}}{L(1-\alpha)}\max_{\forall(x,t)\in D}|F(f(x,t))|\,.$$

Finally, the maximum absolute truncation error in the interval D is:

$$\max_{\forall(x,t)\in D}\left|x(t)-\sum_{i=0}^{m}x_i(t)\right| \leq \frac{K\alpha^{m+1}}{L(1-\alpha)}. \tag{15}$$

This completes the proof. □

3.4. **Equivalence between DADM and ADM.** Let $D=[0,T]\times\Omega$ is closed bounded set in $\mathbf{R}^2$. Define the operator κ such that

$$\kappa F(u) = \iint_D k(x,t,\zeta,\tau)\,F(u(\zeta,\tau))\,d\zeta d\tau, \quad t\in[0,T],x\in\Omega,\; u\in C(D), \tag{16}$$

where κ is compact bounded operator on $C(D)$ to $C(D)$ since its norm is defined as

$$\|\kappa F(u)\| \leq \|\kappa\|\cdot\|F(u)\| \quad \text{and}\,\|\kappa\| = \max_{(x,t)\in D}\iint_D |k(x,t,\zeta,\tau)|\,d\zeta d\tau.$$

Now, equation (1) can be written in the form

$$u = f + \lambda\kappa F(u)\,,$$

where u be the solution obtained by using ADM, such that $u=\sum_{m=0}^{\infty}u_m$ and $u_0=f$.

Define the numerical integral operator κ_n as

$$\kappa_n F\left(\widetilde{u}(x,t)\right) = \sum_{r=0}^{n}\sum_{q=0}^{j} w_{n,r}w_{n,q}\;k(x,t,s_{n,r},s_{n,q})\;F\left(\widetilde{u}(s_{n,r},s_{n,q})\right), \tag{17}$$

where κ_n is linear finite rank bounded operator on $C(D)$ to $C(D)$ since its norm is defined as

$$\|\kappa_n\| = \max_{(x,t)\in D}\sum_{r=0}^{n}\sum_{q=0}^{j}|w_{n,r}w_{n,q}\;k(x,t,s_{n,r},s_{n,q})|$$

Using the operators κ_n , equation (1) can be written in the form

$$\widetilde{u} = f + \lambda\kappa_n F(u)\,,$$

where $\widetilde{u}$ here is the solution obtained by using *DADM,* and $\widetilde{u}=\sum_{m=0}^{\infty}\widetilde{u_m}$ and $\widetilde{u}_0=f$.

Theorem 4. *If* $\|\kappa_n\ g - \kappa g\| \to 0$ *as* $n \to \infty$ *where* $g \in C(D)$ [25], *then the solution of equation (1), using DADM converges to the solution using ADM, i.e.*

$$\widetilde{u} \to u \text{ as } n \to \infty,$$

Proof. Since $u = \sum_{m=0}^{\infty} u_m$, $u_0 = f$, $\widetilde{u} = \sum_{m=0}^{\infty} \widetilde{u}_m$, and $\widetilde{u}_0 = f$. Then

$$\left\|\widetilde{u} - u\right\| = \left\|\sum_{m=0}^{\infty}\left(\widetilde{u}_m - u_m\right)\right\| \le \sum_{m=0}^{\infty}\left\|\widetilde{u}_m - u_m\right\|. \tag{18}$$

Since,

$$\left\|\widetilde{u}_0 - u_0\right\| = \|f - f\| = 0, \text{ and} \tag{19}$$

$$\begin{aligned}\left\|\widetilde{u}_m - u_m\right\| &= \|\lambda\kappa_n A_{m-1} - \lambda\kappa A_{m-1}\| \\ &\le \lambda\,\|\kappa_n A_{m-1} - \kappa A_{m-1}\|,\end{aligned} \tag{20}$$

thus, $\left\|\widetilde{u}_m - u_m\right\| \to 0$ as $n \to \infty$. Substituting from equation (19) and equation (20) into inequality (18), this completes the proof. □

4. Numerical Experiment

To verify our analysis, we introduce the following numerical examples.

Example (1) Consider the following linear mixed volterra Fredholm integral equation ([15] -[39])

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega k(x,t,\zeta,\tau)\, u(\zeta,\tau)\, d\zeta d\tau, \quad (x,t) \in [0,2] \times \Omega$$

where $f(x,t) = \exp(-t)\left[\cos(x) + t\cos(x) + \frac{1}{2}t\cos(x-2)\sin(2)\right]$, $k(x,t,\zeta,\tau) = -\cos(x-\zeta)\exp(\tau - t)$, exact solution $u(x,t) = \cos(x)\exp(-t)$ and $\Omega = [0,2]$. In this case (linear case), the A_i reduce to u_i. Here, ADM can be used as well as *DADM.* DADM is applied to obtain solution using equations (8) and (9). Table (1) shows the absolute error $|e_m^n(x,t)| = \left|u(x,t) - \widetilde{u}(x,t)\right|$ at nodes of the quadrature rule, $n+1$ is number of nodes and m is number of components computed form equation (9).

Table (1) the absolute error of example (1)

t	x	$\left\|e_5^{16}(x,t)\right\|$	t	x	$\left\|e_7^{16}(x,t)\right\|$
0.125	0.125	$4.9365e-9$	0.125	0.125	$7.1093e-10$
0.250	0.250	$5.8401e-8$	0.250	0.250	$7.7328e-9$
0.375	0.375	$6.7042e-7$	0.375	0.375	$8.6309e-8$
0.500	0.500	$7.3695e-6$	0.500	0.500	$8.6684e-7$
0.625	0.625	$8.2081e-5$	0.625	0.625	$9.0227e-6$
0.750	0.750	$9.0636e-4$	0.750	0.750	$9.6426e-5$
0.875	0.875	$1.4413e-4$	0.875	0.875	$1.7325e-5$
1.000	1.000	$2.3721e-3$	1.000	1.000	$3.5520e-4$

Example (2) Consider the following nonlinear mixed volterra Fredholm integral equation ([15] -[39])

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega k(x,t,\zeta,\tau) \exp(-u(\zeta,\tau))\, d\zeta d\tau,$$

where $f(x,t) = -\ln\left(1 + \frac{xt}{1+t^2}\right) - \frac{2}{3}\frac{x\tan^{-1}(t)}{1+t} - \frac{1}{8}\frac{xt^2}{(1+t)(1+t^2)}$, $k(x,t,\zeta,\tau) = \frac{x(1-\zeta^2)}{(1+t)(1-\tau^2)}$, exact solution $u(x,t) = -\ln\left(1 + \frac{xt}{1+t^2}\right)$ and $\Omega = [0,1]$. Here, ADM can be used as well as *DADM.* DADM is applied to obtain solution using equations (8) and (9) DADM is applied to obtain solution using equations (8) and (9). Table (2) shows the absolute error at nodes of the quadrature rule.

Table (2) the absolute error of example (2)

t	x	$\left\|e_5^8(x,t)\right\|$	t	x	$\left\|e_7^8(x,t)\right\|$
0.125	0.125	$7.1427e-9$	0.125	0.125	$5.0596e-10$
0.250	0.250	$8.8344e-8$	0.250	0.250	$6.2142e-9$
0.375	0.375	$9.7567e-7$	0.375	0.375	$7.1314e-8$
0.500	0.500	$1.6698e-7$	0.500	0.500	$8.6586e-7$
0.625	0.625	$3.5920e-6$	0.625	0.625	$9.8798e-6$
0.750	0.750	$4.3172e-5$	0.750	0.750	$1.3930e-6$
0.875	0.875	$6.9334e-4$	0.875	0.875	$3.6152e-5$
1.000	1.000	$7.6758e-3$	1.000	1.000	$6.3629e-4$

Example (3) Consider the following nonlinear mixed volterra Fredholm integral equation

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega k(x,t,\zeta,\tau) \exp(-u(\zeta,\tau))\, d\zeta d\tau,$$

where $f(x,t) = xt + \frac{1-\exp(1)}{12}t^4$, $k(x,t,\zeta,\tau) = \tau\exp(\zeta^3)$, exact solution $u(x,t) = xt$ and $\Omega = [0,1]$. In this example, ADM can not be applied in a direct manner because the evaluation of $u_m(x,t), m \geq 1$ requires the evaluation of the integral $\int_0^1 \exp(\zeta^3)d\zeta$. Here, DADM is the suitable method to obtain solution. DADM is applied to obtain solution using equations (8) and (9). Table (3) shows the absolute error at nodes of the quadrature rule.

Table (3) the absolute error of example (3)

t	x	$\left\|e_4^{16}(x,t)\right\|$	t	x	$\left\|e_6^{16}(x,t)\right\|$
0.125	0.125	$9.7214e-9$	0.125	0.125	$7.1605e-10$
0.250	0.250	$9.6183e-8$	0.250	0.250	$8.2721e-9$
0.375	0.375	$9.9275e-7$	0.375	0.375	$8.3813e-8$
0.500	0.500	$9.0368e-6$	0.500	0.500	$9.4965e-7$
0.625	0.625	$1.7350e-6$	0.625	0.625	$1.5087e-7$
0.750	0.750	$2.7532e-5$	0.750	0.750	$2.8339e-6$
0.875	0.875	$4.8114e-4$	0.875	0.875	$3.9461e-5$
1.000	1.000	$5.2647e-3$	1.000	1.000	$4.0536e-4$

5. Conclusion

When using DADM for solving nonlinear MVFIE, there is no need to solve a system of nonlinear algebraic equations like the traditional numerical methods. Another advantage of DADM is that the coefficient matrices **C** and **D** are not changed during the computation of all components U_m, $m \geq 1$ in equation (12). Moreover, only a small number of iterations are needed to obtain a satisfactory result and the given numerical examples support this claim.Using the new formula of Adomian polynomials, no differentiation is needed like the traditional formula and consequently it is easier in computations, moreover; this formula is used directly in convergence analysis and error estimation.

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