Electronic Journal of Mathematical Analysis and Applications,
Vol. 4(1) Jan. 2016, pp. 11-14
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

SOME RESULTS ON 2-BANACH ALGEBRAS

R. TAPDIGOĞLU, Ü. KÖSEM

Abstract. We consider a 2-Banach algebra and prove some new results, including Gelfand-Mazur type theorems.

1. Introduction

In this article we give some new results for 2-Banach algebras. In particular, we prove an analogy of famous Gelfand-Mazur theorem for 2-Banach algebras.

Recall that the concept of 2-Banach algebra, apparently, was introduced by Mohammed and Siddiqui [1]. Following by Mohammed and Siddiqui [1], note that a 2-Banach algebra B is an algebra with $\dim B > 2$ which is a 2-Banach space (with respect to 2-norm topology) and in addition, the following condition being satisfied:

$$\|a, bc\| \leq M \|a, b\| . \|a, c\| , \ a, b, c \in B$$

$M > 0$ is a constant.

Note that the concept of 2-Banach space was introduced by Gähler [2]. Later, the various aspect of this concept have been studied in [3, 4, 5, 6, 7, 8, 9]. In particular, Mohammed and Siddique [1] proved analog of some known results of the usual Banach algebras in 2-Banach algebras.

Before giving our results, let us give some necessary definitions and notations.

Let X be a vector space of dimension greater than 1 and $\|., .\|$ be a real function on $X \times X$ satisfying the following conditions:

1) $\|a, b\| = 0$ if and only if a and b are linearly dependent;
2) $\|a, b\| = \|b, a\|$;
3) $\|\lambda a, b\| = |\lambda| . \|a, b\|$ for any number λ ;
4) $\|a + b, c\| \leq \|a, c\| + \|b, c\|$ for every $a, b, c \in X$.

$\|., .\|$ is called a 2-norm and X equipped with $\|., .\|$ is a 2-normed space (see [2]). Gähler [2] has proved that $\|., .\|$ is a non-negative function.

A sequence $\{x_n\}$ in 2-Banach space X is called a Cauchy sequence if there exists $y, z \in X$ such that y and z are linearly independent, the $\lim \|x_n - x_m, y\| = 0$ and

---

2010 *Mathematics Subject Classification.* 46J15.
Key words and phrases. 2-Banach space, 2-Banach algebra, 2-norm topology, Gelfand-Mazur theorem.
Submitted April 8, 2015.

the $\lim \|x_n - x_m, z\| = 0$. A sequence $\{x_n\}$ in a 2-normed space X is said to be convergent if there is an $x \in X$ such that the $\lim \|x_n - x, y\| = 0$, for every $y \in X$.

2. **Results**

In this section, we give some new results for 2-Banach algebras.

Let $G(B)$ denote the set of all invertible elements of B. The following theorems shows that $G(B)$ is an open set of B and the map $x \to x^{-1}$is continuous with respect to 2-norm topology.

Theorem 1. *Let B be a unital 2-Banach algebra, $x \in G(B)$, $h \in B$ and $\|h, b\| < \frac{1}{2}\|x^{-1}, b\|^{-1}$ for all $b \in B$. Then $x + h \in G(B)$ and*

$$\left\|(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}, b\right\| \leq 2\left\|x^{-1}, b\right\|^3 \|h, b\|^2$$

for any $b \in B$.

Proof. Let us write the element $x+h$ in the form $x+h = x(e+x^{-1}h)$, where e is an identity element of the 2-Banach algebra B, that is for every $a \in B$, $a.e = e.a = a$ and $\|a, e\| \neq 0$. Since $\|x^{-1}h, b\| < \frac{1}{2}$, it is not difficult to show that $e + x^{-1}h$ is invertible, and hence, $x(e + x^{-1}h)$ is an invertible element in B. Indeed, since $\|(-x^{-1}h)^n, b\| \leq \| - x^{-1}h, b\|^n$ for any $b \in B$, we assert that the sequence

$$S_n := e - x^{-1}h + (x^{-1}h)^2 - (x^{-1}h)^3 + ... + (x^{-1}h)^n \tag{1}$$

is a Cauchy sequence in B. By considering that B is complete with respect to 2-norm topology, $S_n \to s$ $(n \to \infty)$ for some $s \in B$. Using that $(-x^{-1}h)^n \to 0$ $(n \to \infty)$ and

$$S_n.(e - x) = e - x^{n+1} = (e - x).S_n,$$

then its follows from continuity of multiplication with respect to 2-norm in B (see, for example, [1]) that an element $s \in B$ is an inverse of the element $e + x^{-1}h$. Further, it follows from (1) that

$$\begin{aligned} \left\|s - e + x^{-1}h, b\right\| &= \|(x^{-1}h)^2 - (x^{-1}h)^3 + ..., b\| \\ &\leq \sum_{n=2}^{\infty} \|x^{-1}h\|^n = \frac{\|x^{-1}h, b\|^2}{1 - \|x^{-1}h, b\|} \end{aligned} \tag{2}$$

for every $b \in B$.

On the other hand, since

$$(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1} = [(e + x^{-1}h)^{-1} - e + x^{-1}h]x^{-1},$$

by considering (2) we have:

$$\begin{aligned} \|(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}, b\| &= \|[(e + x^{-1}h)^{-1} - e + x^{-1}h]x^{-1}, b\| \\ &\leq 2\|x^{-1}h, b\|^2\|x^{-1}, b\| \end{aligned}$$

for all $b \in B$, which gives the desired result. The theorem is proved. □

Corollary 1. *If B is a 2-Banach algebra, then $G(B)$ is an open set in B, and the map $x \to x^{-1}$ is a 2-homeomorphism of $G(B)$ onto $G(B)$.*

Note that as in the usual Banach algebra, it can be proved (see, for example, Rudin [10]) that the spectrum of element x in 2-Banach algebra is non-empty set, i.e., $\sigma(x) \neq \varnothing$. This allow us to prove Gelfand-Mazur type theorem in 2-Banach algebra B.

Theorem 2. *Let B be a 2-Banach algebra such that every nonzero element x in B is invertible. Then B is isometrically isomorphic to the field of complex numbers $\mathbb{C}$.*

Proof. If $x \in B$ and $\lambda_{1 \neq} \lambda_2$, then only one of these elements can be equal to 0. Therefore, at least one of this is invertible. Since $\sigma(x) \neq \varnothing$, it follows that $\sigma(x) = \{\lambda(x)\}$ for every $x \in B$. By considering that $\lambda(x)e - x$ is noninvertible, we have that $\lambda(x)e - x = 0$, that is $x = \lambda(x)e$, and therefore, the map $x \to \lambda(x)$ is an isomorphism between B and $\mathbb{C}$, and moreover, this map is isometric isomorphism, because

$$|\lambda(x)| = \|\lambda(x)e, b\| = \|x, b\|$$

for all $x \in B$ and $b \in B$,which completes the proof of theorem. □

Theorem 3. *Let B be a 2-Banach algebra, $x_n \in G(B)$, $n = 1, 2, ...,$ and let $x \in \partial G(B)$ (the boundary of the set $G(B)$). If $\| x_n - x, b \| \to 0$ $(n \to \infty)$ for any $b \in B$, then $\left\|x_n^{-1}, b\right\| \to \infty$ $(n \to \infty)$.*

Proof. Suppose in contrary that there exists a finite number $M > 0$ such that $\left\|x_n^{-1}, b\right\| < M$ for any $b \in B$ and infinite numbers n .We can then choose the number n such that $\|x_n - x, b\| < \frac{1}{M}$ for all $b \in B$. Then, for such n we have that

$$\begin{aligned}\left\|e - x_n^{-1}x, b\right\| &= \left\|x_n^{-1}(x_n - x), b\right\| \\ &\leq \left\|x_n^{-1}, b\right\| . \|x_n - x, b\| < M.\frac{1}{M} = 1\end{aligned}$$

for all $b \in B$, and hence, $x_n^{-1}x \in G(B)$. Since $x = x_n(x_n^{-1}x)$ and $G(B)$ is a group, we obtain that $x \in G(B)$. But this is contradiction, because $G(B)$ is an open set in B. The theorem is proved. □

Our next result gives Gelfand-Mazur type theorem. Its proof uses Theorem 3.

Theorem 4. *Let B be a 2-Banach algebra such that*

$$\|x, b\|.\|y, b\| \leq M\|xy, b\|(x \in B, y \in B)$$

for all $b \in B$ and some positive number M. Then B is isometrically isomorphic to $\mathbb{C}$.

Proof. Let y be a boundary point for the $G(B)$. Then, obviously, there exist a sequence $\{y_n\}$ with $y_n \in G(B)$ such that $y = \lim_n y_n$ in 2-norm topology. According to Theorem 3, we obtain that $\lim \left\|y_n^{-1}, b\right\| = \infty$ for any $b \in B$. By condition of theorem $\|y_n, b\| . \left\|y_n^{-1}, b\right\| \leq M \|e, b\|$ $(n = 1, 2, ...)$, that is

$$\|y_n, b\| \leq \frac{M \parallel e, b \parallel}{\parallel y_n^{-1}, b \parallel} \to 0(n \to \infty),$$

which shows that $0 = \lim_n \|y_n, b\| = \|y, b\|$ for any $b \in B$, and thus, $\|y, b\| = 0$, which implies that y and b are linearly dependent. This means that $\lambda b = y$ for any $b \in B$. On the other hand, if $x \in B$ and μ is a boundary point in $\sigma(x)$, that is $\mu \in \partial\sigma(x)$, then $\mu e - x$ is a boundary point in any $G(B)$. Then, $\mu e - x = \eta b$ for any $b \in B$. In particular, $\mu e - x = \tau e$, and hence $x = (\mu - \tau)e$, which means that $B = \{\zeta e : \zeta \in \mathbb{C}\}$, as desired. The theorem is proved. □

In conclusion note that a 2-Banach algebra need not be in general a Banach algebra.

Acknowledgement

The authors would like to thank the referees for giving useful comments and suggestions for the improvement of this paper. We would like to thank to Prof. M. Gürdal for his guidance, assistance and advice through out we work on this project. This work is supported by the Scientific and Technological Research Council of Turkey with 2209.

References

[1] N. Mohammad, and A.H. Siddiqui, On 2-Banach algebras, International Centre for theoretical Physics, Preprint, 1988.
[2] S. Gähler, Über 2-Banach raüme, Math. Nachr., 42, 335-347, 1969.
[3] S. Gähler, A.H. Siddiqui, and S.C. Gupta, Contribution to non-Archimidean functional analysis, Math. Nachr., 69, 163-171, 1975.
[4] M. Gürdal and S. Pehlivan, The Statistical Convergence in 2-Banach spaces, Thai Journal of Mathematics, 2, 1, 107-113, 2004.
[5] S.L. Singh, B.M.L. Tiwari, V.K. Gupta, Commen fixed points of commuting mapping in 2-metric space and applicatious, Math. Nachr., 95, 1, 293-297, 1980.
[6] Y. Je Cho, Paul C. S. Lin, S.S. Kim, A. Miksiak, Theory of 2-Inner Product Spaces, Nova Science Publishers Inc., 2001.
[7] S.N. Lal, S. Bhattacharya, C. Sreedhar, Complex 2-normed linear space and extension of 2-functionals, Zeitschrift für analysis and ihre Anwendungen, 20, 35-53, 2001.
[8] A. White, 2-Banach space, Math. Nachr., 42, 43-60, 1969.
[9] N. Srivastava, S. Bhattacharya and S.N. Lal, On Hahn-Banach Extension of linear n-functionals in n-normed spaces, Math. Maced., 4, 25-32, 2006.
[10] W. Rudin, Function Analysis, McGRAW-Hill Book Company, 1973.

Ramiz Tapdigoğlu
Süleyman Demirel University, Department of Mathematics, 32260, Isparta, Turkey
E-mail address: racer_rmz@hotmail.com

Ümit Kösem
Süleyman Demirel University, Department of Mathematics, 32260, Isparta, Turkey