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SOME PROPERTIES OF CERTAIN SUBCLASS OF GENERALIZED p-VALENT LOGARITHMIC λ-BAZILEVIC FUNCTIONS

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Abstract. The aim of this paper is to study the properties of a subclass of analytic functions related to the generalized p-valent logarithmic λ-Bazilevic functions by using the concept of differential subordination. We obtain some results concerned with inclusion relations, radius problems, argument properties, and some other interesting properties.

1. Introduction and preliminaries

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions f_1 and f_2 in $\mathcal{H}$, we say that the function f_1 is subordinate to f_2 in $\mathbb{D}$, and write $f_1(z) \prec f_2(z)$ $(z \in \mathbb{D})$, if there exists a Schwarz function ω, which is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f_1(z) = f_2(\omega(z))$ $(z \in \mathbb{D})$. Furthermore, if the function f_2 is univalent in $\mathbb{D}$, then we have the following equivalence (see, for details, [11],[22],[35]):

$$f_1(z) \prec f_2(z)\ (z \in \mathbb{D}) \Longleftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

Let P denote the class of functions $p(z)$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ (z \in \mathbb{D}),$$

which are analytic in $\mathbb{D}$ and satisfy the condition $\Re(p(z)) > 0$.

Let P_ϕ denote the class of analytic functions $\phi(z)$ with positive real part in $\mathbb{D}$ with $\phi(0) = 1$ and $\phi^{'}(0) > 0$, which map the unit disk $\mathbb{D}$ onto a region starlike with respect to 1 and which are symmetric with respect to the real axis.

Let $\mathcal{A}_p$ be the class of analytic functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \ (p \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{1}$$

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defined in the open unit disc $\mathbb{D}$, and let $\mathcal{A}_1 = \mathcal{A}$. We denote by S^* and K the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike and convex in $\mathbb{D}$ (see, e.g., Srivastava and Owa [35]).

Definition 1 A function f in $\mathcal{A}_p$ is said to be in the class $J_p[\lambda, A, B]$ if and only if

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\frac{(zf'(z))'}{f'(z)} \prec p\frac{1+Az}{1+Bz} \ (\lambda \geq 0,\ -1 \leq B < A \leq 1,\ z \in \mathbb{D}).$$

Obviously, for $p = 1,\ \lambda = 0,\ A = 1 - 2\rho\ (0 \leq \rho < 1)$ and $B = -1$ in Definition 1, we have the well-known classes $J(\alpha, \rho)$ (see [14] and [16]; also see [26]). When $p = 1,\ A = 1$ and $B = -1$, if we set $\lambda = 0$ and $\lambda = 1$ in Definition 1, respectively, we have the well-known classes S^* and K. Also, for $\lambda = 0$ and $p = 1$, we obtain the class $J_1[0, A, B]$ of Janowski starlike functions (see [9],[33],[34]). Furthermore, for the function classes $S_p^*(\rho)$ $(0 \leq \rho \leq 1)$ and $K_p(\rho)$ (see [32]), it is easily seen that $J_p[0, 1-2\rho, -1] = S_p^*$ and $J_p[1, 1-2\rho, -1] = K_p(\rho)$.

It is clear that $f \in J_p[\lambda, A, B]$ if and only if

$$\left|(1-\lambda)\frac{zf'(z)}{pf(z)} + \lambda\frac{(zf'(z))'}{pf'(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \ (-1 < B < A \leq 1,\ z \in \mathbb{D})$$

and

$$\Re\left\{(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\frac{(zf'(z))'}{f'(z)}\right\} > \frac{p(1-A)}{2} \ (B = -1,\ z \in \mathbb{D}).$$

In particular, when $A = 1 - 2\rho$ and $B = -1$, we have $f \in J_p(\lambda, \rho) = J_p[\lambda, 1 - 2\rho, -1]$ if and only if

$$\Re\left\{(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\frac{(zf'(z))'}{f'(z)}\right\} > p\rho \ (0 \leq \rho < 1,\ z \in \mathbb{D}). \tag{2}$$

Definition 2 A function f in $\mathcal{A}_p$ is said to be in the class $J_p[\lambda, \alpha, \beta, A, B]$ of p-valent λ-Bazilevic function of type (α, β) if and only if

$$(1-\lambda)\left[\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\right] + \lambda\left[\frac{(zf'(z))'}{pf'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f'(z)}{pz^{p-1}}\right)^{i\beta}\right] \prec \frac{1+Az}{1+Bz}, \tag{3}$$

where $\lambda \geq 0,\ \alpha \geq 0,\ \beta \in \mathbb{R},\ -1 \leq B \leq 1,\ A \neq B$ and $g \in \mathcal{S}_p^* = \mathcal{S}_p^*(0)$.

For $p = 1,\ \lambda = 0,\ A = 1$ and $B = -1$, the class $J_1[0, \alpha, \beta, 1, -1]$ was introduced by Bazilevic (see, for instance, [8],[24],,[23],[25],[2],[3],[4],[5],[6],[29],[7]). For $p = 1$ and $a_n = 0$ $(n = 2, 3, \cdots, k)$, the class $J_1[\lambda, \alpha, \beta, A, B]$ was introduced by Wang et al. [36]. Also, for $p = 1,\ a_n = 0\ (n = 2, 3, \cdots, k),\ A = 1 - 2\rho\ (0 \leq \rho \leq 1)$ and $B = -1$, the class $J_1[\lambda, \alpha, \beta, 1-2\rho, -1]$ was introduced by Li [13].

By making use of the principle of subordination between analytic functions, Ma and Minda [17] introduced the following subclasses $L_{p,\alpha,\beta,\rho}(\lambda, \mu, \phi)$ and $N_{p,\alpha,\beta,\rho}(\lambda, \phi)$ of the class $\mathcal{A}_p$ for $\alpha \geq 0,\ \beta \in \mathbb{R},\ \lambda \geq 0,\ \mu \geq 0$ and $\phi(z) \in P_\phi$.

Definition 3 Let $\alpha \geq 0,\ \beta \in \mathbb{R},\ \lambda \geq 0,\ \mu \geq 0$ and $\phi(z) \in P_\phi$. A function f be in the class $L_{p,\alpha,\beta,\rho}(\lambda, \mu, \phi)$ if it satisfies the condition

$$\frac{1}{p}\left[G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) + \mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right] \prec \phi(z) \ (z \in \mathbb{D}), \tag{4}$$

where

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)=\left[\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\right]^{(1-\lambda)}\left[\frac{(zf'(z))'}{f'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f'(z)}{pz^{p-1}}\right)^{i\beta}\right]^{\lambda}, \tag{5}$$

$$H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)=\left[(1-\lambda)\frac{(zf'(z))'}{f'(z)}+\lambda\frac{(z(zf'(z))')'}{(zf'(z))'}\right]+(\alpha+i\beta-1)\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\frac{(zf'(z))'}{f'(z)}\right]$$
$$-\alpha\left[(1-\lambda)\frac{zg'(z)}{g(z)}+\lambda\frac{(zg'(z))'}{g'(z)}\right]-ip\beta \tag{6}$$

and $g\in J_p(\lambda,\rho)$.

For $\mu=0$, we have the following logarithmic λ-Bazilevic functions of $\mathcal{A}_p$.

Definition 4 Let $\alpha\geq 0,\ \beta\in\mathbb{R},\ \lambda\geq 0$ and $\phi(z)\in P_\phi$. A function f be in the class $N_{p,\alpha,\beta,\rho}(\lambda,\phi)$ if it satisfies the condition

$$\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\prec\phi(z)\ (z\in\mathbb{D}),$$

where $g\in J_p(\lambda,\rho)$ and $G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ is given by (5).

For

$$\phi(z)=\frac{1+[(1-\eta)A+\eta B]z}{1+Bz}\ (0\leq\eta\leq 1,\ -1\leq B<A\leq 1)$$

in Definitions 3 and 4, respectively, we have the following subclasses.

A function f in $\mathcal{A}_p$ is said to be in the class $L_{p,\alpha,\beta,\rho}(\lambda,\mu,\eta,A,B)$ if and only if

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\prec p\frac{1+[(1-\eta)A+\eta B]z}{1+Bz}\ (z\in\mathbb{D}),$$

where $g\in J_p(\lambda,\rho)$, $G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ and $H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ are given by (5) and (6), respectively.

A function f in $\mathcal{A}_p$ is said to be in the class $N_{p,\alpha,\beta,\rho}(\lambda,\eta,A,B)$ if and only if

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\prec p\frac{1+[(1-\eta)A+\eta B]z}{1+Bz}\ (z\in\mathbb{D}),$$

where $g\in J_p(\lambda,\rho)$ and $G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ is given by (5).

Obviously, for $\eta=0,\ A=1-2\xi\ (0\leq\xi<1)$ and $B=-1$ in $L_{p,\alpha,\beta,\rho}(\lambda,\mu,\eta,A,B)$ and $N_{p,\alpha,\beta,\rho}(\lambda,\eta,A,B)$, respectively, we have the following equivalence relationships.

$$f\in L_{p,\alpha,\beta,\rho}(\lambda,\mu,\xi)=L_{p,\alpha,\beta,\rho}(\lambda,0,\mu,1-2\xi,-1)$$
$$\Longleftrightarrow\Re\{G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\}>\xi p$$

and

$$f\in N_{p,\alpha,\beta,\rho}(\lambda,\xi)=N_{p,\alpha,\beta,\rho}(\lambda,0,1-2\xi,-1)\Longleftrightarrow\Re\{G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\}>\xi p.$$

For suitable choices of the parameters $\alpha,\ \beta,\ \lambda,\ \mu,\ \rho,\ \eta,\ A,\ B,\ p$ and the function ϕ involved in Definitions 3 and 4, we also obtain the following subclasses which were studied in many earlier works:

(1) $N_{1,0,0,0}(\lambda,\phi)=L(\lambda,\phi)$ (logarithmic λ-convex functions) (Ali et al. [1]);
(2) $L_{1,\alpha,0,0}(0,\mu,\phi)(g(z)=z)=M_{\alpha,\mu}(\phi)$ (Rosy et al. [31]);
(3) $L_{p,\alpha,0,0}(0,0,\mu,0,A,B)=M_p(\alpha,\mu,A,B)$ (Patel [27]);
(4) $L_{p,\alpha,0,0}(0,0,\mu,1-2\xi,-1)=M_p(\alpha,\mu,\xi)$ $(0\leq\xi<1)$ (Wang et al. [37]);

(5) $L_{p,\alpha,\beta,0}(0,0,\mu,A,B) = M_p(\alpha,\beta,\mu,A,B)$ and $L_{p,\alpha,\beta,0}(0,0,\mu,1-2\xi,-1) = B_p(\alpha,\beta,\mu,\xi)$ $(0 \le \xi < 1)$ (Raza et al. [30]).

In this paper, we focus on discussing the properties of the classes $L_{p,\alpha,\beta,\rho}(\lambda,\mu,\eta,A,B)$ and $N_{p,\alpha,\beta,\rho}(\lambda,\eta,A,B)$. In order to prove our main results, we shall require the following lemmas.

Lemma 1 ([21]) Let $-1 \le B < A \le 1$ and $t > 0$. If a complex number γ satisfies $\Re\{\gamma\} \ge -\frac{t(1-A)}{1-B}$, then the differential equation

$$q(z) + \frac{zq'(z)}{tq(z)+\gamma} = \frac{1+Az}{1+Bz} \ (z \in \mathbb{D})$$

has a univalent solution in $\mathbb{D}$ given by

$$q(z) = \begin{cases} \frac{z^{t+\gamma}(1+Bz)^{t((A-B)/B)}}{t\int_0^z s^{t+\gamma-1}(1+Bs)^{t((A-B)/B)}ds} - \frac{\gamma}{t}, & B \neq 0, \\ \\ \frac{z^{t+\gamma}e^{tAz}}{t\int_0^z s^{t+\gamma-1}e^{tAs}ds} - \frac{\gamma}{t}, & B = 0. \end{cases}$$

If the function $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{D}$ and satisfies

$$h(z) + \frac{zh'(z)}{th(z)+\gamma} \prec \frac{1+Az}{1+Bz} \ (z \in \mathbb{D}),$$

then

$$h(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \ (z \in \mathbb{D})$$

and $q(z)$ is the best dominant.

Lemma 2 ([38]) Let $a_1,\ b_1$ and $c_1 \neq 0,-1,-2,\dots$ be complex numbers. Then, for $\Re c_1 > \Re b_1 \neq 0$,

(i) ${}_2F_1(a_1,b_1,c_1;z) = \frac{\Gamma(c_1)}{\Gamma(c_1-b_1)\Gamma(b_1)}\int_0^1 s^{b_1-1}(1-s)^{c_1-b_1-1}(1-sz)^{-a_1}ds;$

(ii) ${}_2F_1(a_1,b_1,c_1;z) = {}_2F_1(b_1,a_1,c_1;z);$

(iii) ${}_2F_1(a_1,b_1,c_1;z) = (1-z)^{-a_1}{}_2F_1(a_1,c_1-b_1,c_1;\frac{z}{z-1}).$

Lemma 3 ([39]) Let ε be a positive measure on $[0,1]$. Let g be a complex-valued function defined on $\mathbb{D}\times[0,1]$ such that $g(\cdot,t)$ is analytic in $\mathbb{D}$ for each $t\in[0,1]$ and $g(\cdot,t)$ is ε-integrable on $[0,1]$ for all $z\in\mathbb{D}$. In addition, suppose that $\Re g(z,t) > 0$, $g(-r,t)$ is real and $\Re\frac{1}{g(z,t)} \ge \frac{1}{g(-r,t)}$ for $|z| \le r < 1$ and $t \in [0,1]$. If $g(z) = \int_0^1 g(z,s)d\varepsilon(s)$, then $\Re\left\{\frac{1}{g(z)}\right\} \ge \frac{1}{g(-r)}$.

Lemma 4 ([19]) Let $u = u_1 + iu_2,\ v = v_1 + iv_2$, and let $\phi(u,v)$ be a complex-valued function satisfying the conditions:

(i) $\phi(u,v)$ is continuous in a domain $\mathbb{E} \subset \mathbb{C}^2$;

(ii) $(0,1) \in \mathbb{E}$ and $\Re\phi(1,0) > 0$;

(iii) $\Re\{\phi(iu_2,v_1)\} \le 0$ whenever $(iu_2,v_1) \in \mathbb{E}$ and $v_1 \le -\frac{(1+u_2^2)}{2}$.

If the function $h(z) = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{D}$ such that $(h(z), zh'(z)) \in \mathbb{E}$ and $\Re\{\phi(h(z),zh'(z))\} > 0$ for $z \in \mathbb{D}$, then $\Re\{h(z)\} > 0$ in $\mathbb{D}$.

Lemma 5 ([15]) Let $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$. Then

$$\frac{1+A_2 z}{1+B_2 z} \prec \frac{1+A_1 z}{1+B_1 z} \ (z \in \mathbb{D}).$$

Lemma 6 ([15]) Let F is analytic and convex in $\mathbb{D}$. If $f,\ g \in \mathcal{A}_p$ and $f,\ g \prec F$, then

$$\mu f + (1-\mu)g \prec F \quad (0 \le \mu < 1).$$

Lemma 7 ([28]) If $\phi(z)$ is analytic in $\mathbb{D}$ and $|\phi(z)| \leq 1$ for $z \in \mathbb{D}$, then for $|z| = r < 1$,

$$\left|\frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}\right| \leq \frac{1}{1-r}.$$

Lemma 8 ([10] and [18]) If $p(z) = 1 + p_1z + p_2z^2 + \cdots \in P$, then for $|z| = r < 1$,

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r}{1-r^2}, \quad \Re p(z) \geq \frac{1-r}{1+r} \quad \text{and} \quad |p'(z)| \leq \frac{2\Re p(z)}{1-r^2}.$$

These estimates are sharp.

Lemma 9 ([12]) Let h be analytic in $\mathbb{D}$ with $h(0) = 1,\ h(z) \neq 0\ (z \in \mathbb{D})$ and suppose that

$$|\arg(h(z) + mzh'(z))| < \frac{\pi}{2}[l + \frac{2}{\pi}\arctan(ml)]\ (l,\ m > 0)$$

then

$$|\arg h(z)| < \frac{\pi}{2} l\ (z \in \mathbb{D}).$$

Lemma 10 ([20]) If $p(z) = 1 + p_1z + p_2z^2 + \cdots$ is analytic in $\mathbb{D}$, $h(z)$ is a convex function in $\mathbb{D}$ with $h(0) = 1$ and γ is a complex constant such that $\Re\{\gamma\} > 0$, then

$$p(z) + \frac{zp'(z)}{\gamma} = h(z)$$

implies

$$p(z) \prec \gamma z^{-\gamma}\int_0^z t^{\gamma-1}h(t)dt = q(z) \prec h(z).$$

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $\alpha \geq 0,\ \beta \in \mathbb{R},\ \lambda \geq 0,\ 0 \leq \eta < 1,\ \mu \geq 0,\ -1 \leq B < A \leq 1,\ 0 \leq \rho < 1,\ 0 \leq \xi < 1,\ l > 0,\ m > 0,\ \gamma \geq 0,\ p \in \mathbb{N} = \{1, 2, 3, \cdots\}$ and all powers are understood as principle values.

Theorem 1 If $f \in L_{p,\alpha,\beta,\rho}(\lambda, \eta, \mu, A, B)$ $(\mu > 0)$, then

$$\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f, g)(z) \prec q(z)\ (z \in \mathbb{D}), \tag{7}$$

where $G^p_{\alpha,\beta,\lambda,\rho}(f, g)(z)$ and $H^p_{\alpha,\beta,\lambda,\rho}(f, g)(z)$ are given by (5) and (6), respectively,

$$q(z) = \frac{\mu}{pQ(z)} \prec \frac{1 + [(1-\eta)A + \eta B]z}{1 + Bz}$$

and

$$Q(z) = \begin{cases} \int_0^1 s^{\frac{p}{\mu}-1}\left(\frac{1+Bzs}{1+Bz}\right)^{\frac{p}{\mu}(1-\eta)(A-B)/B}ds,\ B \neq 0, \\ \int_0^1 s^{\frac{p}{\mu}-1}e^{\frac{p}{\mu}(s-1)((1-\eta)A+\eta B)z}ds,\ B = 0. \end{cases} \tag{8}$$

In terms of the hypergeometric function,

$$q(z) = \begin{cases} \left[{}_2F_1\left(1, -\frac{p}{\mu}(1-\eta)(A-B)/B; \frac{p}{\mu}+1; \frac{Bz}{Bz+1}\right)\right]^{-1},\ B \neq 0, \\ \left[{}_1F_1(1; \frac{p}{\mu}+1; -\frac{p}{\mu}((1-\eta)A+\eta B)z)\right]^{-1},\ B = 0 \end{cases} \tag{9}$$

and if

$$(1-\eta)A + \eta B < -\frac{\mu B}{p}\ (-1 \leq B < 0),$$

then

$$L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu,A,B)\subset L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu,\xi),$$

where

$$\xi=\left[{}_2F_1\left(1,\frac{p}{\mu}(1-\frac{(1-\eta)A+\eta B}{B});\frac{p}{\mu}+1;\frac{B}{B-1}\right)\right]^{-1}. \tag{10}$$

This result is best possible.

Proof. Let

$$\begin{aligned} h(z) &= \frac{1}{p}G^{p}_{\alpha,\beta,\lambda,\rho}(f,g)(z)\\ &= \left[\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\right]^{1-\lambda}\left[\frac{(zf'(z))'}{pf'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f'(z)}{pz^{p-1}}\right)^{i\beta}\right]^{\lambda}\end{aligned}$$

where $h(z)$ is analytic in $\mathbb{D}$ with $h(0)=1$.

Differentiating logarithmically, we obtain

$$G^{p}_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^{p}_{\alpha,\beta,\lambda,\rho}(f,g)(z)=ph(z)+\frac{\mu zh'(z)}{h(z)}\prec p\frac{1+[(1-\eta)A+\eta B]z}{1+Bz}. \tag{11}$$

Using Lemma 1 with $t=\frac{p}{\mu}$ and $\gamma=0$, we have

$$h(z)\prec q(z)\prec\frac{1+[(1-\eta)A+\eta B]z}{1+Bz}\ (z\in\mathbb{D}),$$

where $q(z)$ is given as (9) and is the best dominant of (11).

Next, in order to prove

$$L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu,A,B)\subset L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu,\xi),$$

we show that

$$\inf_{|z|<1}\{\Re q(z)\}=q(-1).$$

Now, if we set

$$a=-\frac{p}{\mu}\frac{(1-\eta)(A-B)}{B},\ b=\frac{p}{\mu}\text{ and }c=\frac{p}{\mu}+1,$$

then it is clear that $c>b>0$.

Therefore, for $B\neq 0$, by using Lemma 2, it follows from (8) that

$$Q(z)=(1+Bz)^a\int_0^1 s^{b-1}(1+Bsz)^{-a}ds=\frac{\Gamma(b)}{\Gamma(c)}\ {}_2F_1\left(1,a;c;\frac{Bz}{Bz+1}\right). \tag{12}$$

To prove that

$$\inf_{|z|<1}\{\Re q(z)\}=q(-1),$$

we need to show that

$$\Re\left\{\frac{1}{Q(z)}\right\}\geq\frac{1}{Q(-1)}.$$

Since

$$(1-\eta)A+\eta B<-\frac{\mu B}{p}\text{ and }-1\leq B<0$$

imply that $c>a>0$, hence, by using Lemma 3, (12) yields

$$Q(z)=\int_0^1 g(z,s)ds,$$

where

$$g(z,s)=\frac{1+Bz}{1+(1-s)Bz}\ (0\leq s\leq 1)$$

and

$$d\varepsilon(s)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)}s^{a-1}(1-s)^{c-a-1}ds,$$

which is a positive measure on $[0,1]$.

For $-1\leq B<0$, it is clear that $\Re g(z,s)>0$ and $g(-r,s)$ is real for $0\leq |z|\leq r<1$ and $s\in[0,1]$. Also,

$$\Re\left\{\frac{1}{g(z,s)}\right\}=\Re\left\{\frac{1+(1-s)Bz}{1+Bz}\right\}\geq\frac{1-(1-s)Br}{1-Br}=\frac{1}{g(-r,s)}$$

for $|z|\leq r<1$.

Again, using Lemma 3, we have

$$\Re\{\frac{1}{Q(z))}\}\geq\frac{1}{Q(-r)}.$$

Now, letting $r\to 1^{-}$, it follows that

$$\Re\{\frac{1}{Q(z)}\}\geq\frac{1}{Q(-1)}.$$

Thus, we have $L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu,A,B)\subset L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu,\xi)$.

Corollary 1 If $f\in L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu,A,B)$ $(\mu>0)$, then $f\in N_{p,\alpha,\beta,\rho}(\lambda,\eta,A,B)$.

Putting $\lambda=0,\ \rho=0$ and $\eta=0$ in Theorem 1, we have the following result proved in [30].

Corollary 2 If $f\in L_{p,\alpha,\beta,0}(0,0,\mu,A,B)$ $(\mu>0)$, then

$$\frac{1}{p}G^{p}_{\alpha,\beta,0,0}(f,g)(z)=\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^{p}}\right)^{i\beta}\prec q(z)\ (z\in\mathbb{D}),$$

and if

$$A<-\frac{\mu B}{p}\ \text{and}\ -1\leq B<0,$$

then

$$L_{p,\alpha,\beta,0}(0,0,\mu,A,B)\subset L_{p,\alpha,\beta,0}(0,0,\mu,\xi),$$

where $q(z)=\frac{\mu}{pQ(z)}$, $Q(z)$, $q(z)$ and ξ are given by (8), (9) and (10) with $\eta=0$. This result is best possible.

Putting $\lambda=1$ and $\eta=0$ in Theorem 1, we get the following corollary.

Corollary 3 If $f\in L_{p,\alpha,\beta,\rho}(1,0,\mu,A,B)$ $(\mu>0)$, then

$$\frac{1}{p}G^{p}_{\alpha,\beta,1,\rho}(f,g)(z)=\frac{(zf'(z))'}{pf'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f'(z)}{pz^{p-1}}\right)^{i\beta}\prec q(z)\ (z\in\mathbb{D})$$

and if

$$A<-\frac{\mu B}{p}\ \text{and}\ -1\leq B<0,$$

then

$$L_{p,\alpha,\beta,\rho}(1,0,\mu,A,B)\subset L_{p,\alpha,\beta,\rho}(1,0,\mu,\xi),$$

where $q(z)=\frac{\mu}{pQ(z)}$, $Q(z)$, $q(z)$ and ξ are given by (8), (9) and (10) with $\eta=0$. This result is best possible.

Theorem 2 If $f\in\mathcal{A}_p$ satisfies

$$\Re\{G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)(G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z))\}>p^2\gamma\ (0\le\gamma<1,\ z\in\mathbb{D}), \tag{13}$$

then $f\in N_{p,\alpha,\beta,\rho}(\lambda,\xi)$, where

$$\xi=\frac{-\mu+\sqrt{\mu^2+8(2p^2\gamma+\mu p)}}{4p}\in(0,1)$$

and $G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ and $H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ are given by (5) and (6), respectively.

Proof. Setting

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)=p(1-\xi)h(z)+p\xi, \tag{14}$$

then $h(z)$ is analytic in $\mathbb{D}$ and $h(0)=1$.

Differentiating (14) and using the identity (13), we have

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)=\frac{p(1-\xi)\mu zh'(z)+[p(1-\xi)h(z)+p\xi]^2}{p(1-\xi)h(z)+p\xi}. \tag{15}$$

Using (14) and (15), we get

$$\begin{aligned}&G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)[G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)]\\&=p(1-\xi)\mu zh'(z)+[p(1-\xi)h(z)+p\xi]^2.\end{aligned} \tag{16}$$

From (13) and (16), we obtain

$$\Re\frac{p(1-\xi)\mu zh'(z)+p^2(1-\xi)^2h^2(z)+2p^2\xi(1-\xi)h(z)+p^2\xi^2-p^2\gamma}{p^2(1-\gamma)}>0.$$

Next, we construct the function $\phi(u,v)$ by choosing $u=h(z)$ and $v=zh'(z)$, that is,

$$\phi(u,v)=p(1-\xi)\mu v+2p^2\xi(1-\xi)u+p^2(1-\xi)^2u^2+p^2\xi^2-p^2\gamma. \tag{17}$$

Clearly, the conditions (i) and (ii) of Lemma 4 are satisfied. Now, we verify the condition (iii) as follows:

$$\begin{aligned}\Re\{\phi(iu_2,v_1)\}&=p(1-\xi)\mu v_1-p^2(1-\xi)^2u_2^2+p^2\xi^2-p^2\gamma\\&\le-\frac{(1+u_2^2)}{2}\cdot[p(1-\xi)\mu-p^2(1-\xi)^2u_2^2+p^2\xi^2-p^2\gamma]\\&=X+Yu_2^2,\end{aligned}$$

where

$$X=-\frac{1}{2}p\mu(1-\xi)+p^2\xi^2-p^2\gamma\text{ and }Y=-\left(p^2(1-\xi)^2+\frac{p\mu(1-\xi)}{2}\right).$$

We note that $\Re\{\phi(iu_2,v_1)\}<0$ if and only if $X=0$ and $Y<0$. From $X=0$, we have

$$\xi=\frac{-\mu+\sqrt{\mu^2+8(2p^2\gamma+\mu p)}}{4p}\in(0,1).$$

Thus, by applying Lemma 4, we get $f\in N_{p,\alpha,\beta,\rho}(\lambda,\xi)$.

Theorem 3 If $\mu_2\ge\mu_1\ge0$ and $-1\le B_1\le B_2<A_2\le A_1\le1$, then

$$L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu_2,A_2,B_2)\subset L_{p,\alpha,\beta,\rho}(\lambda,\eta,\mu_1,A_1,B_1).$$

Proof. Let $f(z) \in L_{p,\alpha,\beta,\rho}(\lambda, \eta, \mu_2, A_2, B_2)$. Then

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) + \mu_2 H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) \prec p\frac{1+[(1-\eta)A_2+\eta B_2]z}{1+B_2 z}.$$

Since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, so, by Lemma 5, we have

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) + \mu_2 H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) \prec p\frac{1+[(1-\eta)A_1+\eta B_1]z}{1+B_1 z},$$

which implies that $f(z) \in L_{p,\alpha,\beta,\rho}(\lambda, \eta, \mu_2, A_1, B_1)$. Thus, for $\mu_2 = \mu_1 \ge 0$, we have the required result.

When $\mu_2 > \mu_1 \ge 0$, Theorem 1 implies that

$$G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) \prec p\frac{1+[(1-\eta)A_1+\eta B_1]z}{1+B_1 z}.$$

Also, because

$$\begin{aligned}&G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) + \mu_1 H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\\ &= \left(1-\frac{\mu_1}{\mu_2}\right) G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) + \frac{\mu_1}{\mu_2}\left\{G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) + \mu_2 H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right\},\end{aligned}$$

by using Lemma 6, we get the required result.

Putting $\lambda = 0$, $\eta = 0$ and $\rho = 0$ in Theorem 3, we obtain the following result proved in [30].

Corollary 4 If $\mu_2 \ge \mu_1 \ge 0$ and $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, then

$$L_{p,\alpha,\beta,0}(0, 0, \mu_2, A_2, B_2) \subset L_{p,\alpha,\beta,0}(0, 0, \mu_1, A_1, B_1).$$

Setting $\lambda = 1$ and $\eta = 0$ in Theorem 3, we have the following corollary.

Corollary 5 If $\mu_2 \ge \mu_1 \ge 0$ and $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, then

$$L_{p,\alpha,\beta,\rho}(1, 0, \mu_2, A_2, B_2) \subset L_{p,\alpha,\beta,\rho}(1, 0, \mu_1, A_1, B_1).$$

Theorem 4 If $f(z) \in \mathcal{A}_p$ satisfies

$$\Re\left[\left(\frac{f(z)}{z^p}\right)^{(1-\lambda)}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\lambda}\right] > 0$$

and

$$\left|G^p_{\alpha,\beta,\lambda,\rho}(f,g) - p\right| < \sigma p \ (0 < \sigma \le 1)$$

for $g \in J_p(\lambda, \rho)$. Then

$$\Re\left[(1-\lambda)\frac{(zf'(z))'}{f'(z)} + \lambda\frac{(z(zf'(z))')'}{(zf'(z))'}\right] > 0$$

in $|z| < r_1$, where

$$r_1 = \frac{2|1-\alpha-i\beta| + 2\alpha p(1-\rho) + \sigma - \sqrt{(2|1-\alpha-i\beta| + 2\alpha p(1-\rho)+\sigma)^2 - 4pN}}{2N} \tag{18}$$

and

$$N = 2\alpha p - p(1+2\alpha\rho) - \sigma.$$

Proof. Let

$$h(z) = \frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) - 1,$$

where $h(z)$ is analytic in $\mathbb{D}$ with $h(0) = 0$ and $|h(z)| < \sigma$. By using the Schwarz lemma, we get

$$h(z) = \sigma z \phi(z),$$

where $\phi(z)$ is analytic in $\mathbb{D}$ with $|\phi(z)| < 1$. Differentiating logarithmically, we have

$$(1-\lambda)\frac{(zf'(z))'}{f'(z)} + \lambda\frac{(z(zf'(z))')'}{(zf'(z))'} = (1-\alpha-i\beta)\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\frac{(zf'(z))'}{f'(z)}\right]$$
$$+\alpha\left[(1-\lambda)\frac{zg'(z)}{g(z)} + \lambda\frac{(zg'(z))'}{g'(z)}\right] + \frac{\sigma z(z\phi'(z)+\phi(z))}{1+\sigma z\phi(z)} + ip\beta.$$

Since

$$\Re\left[\left(\frac{f(z)}{z^p}\right)^{(1-\lambda)}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\lambda}\right] > 0.$$

Let

$$\psi(z) = \left(\frac{f(z)}{z^p}\right)^{(1-\lambda)}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\lambda},$$

then,

$$(1-\lambda)\frac{(zf'(z))'}{f'(z)} + \lambda\frac{(z(zf'(z))')'}{(zf'(z))'} = p + \frac{z\psi'(z)}{\psi(z)} \text{ and } \Re\psi(z) > 0.$$

This implies that

$$\begin{aligned}
&\Re\left[(1-\lambda)\frac{(zf'(z))'}{f'(z)} + \lambda\frac{(z(zf'(z))')'}{(zf'(z))'}\right] \\
&\geq (1-\alpha)p + \alpha\Re\left[(1-\lambda)\frac{zg'(z)}{g(z)} + \lambda\frac{(zg'(z))'}{g'(z)}\right] \\
&\quad -|1-\alpha-i\beta|\left|\frac{z\psi'(z)}{\psi(z)}\right| - \sigma\left|\frac{z(z\phi'(z)+\phi(z))}{1+\sigma z\phi(z)}\right|.
\end{aligned}$$

Now, using the well-known results for the class $J_p(\lambda,\rho)$, Lemmas 7 and 8, we have

$$\begin{aligned}
&\Re\left[(1-\lambda)\frac{(zf'(z))'}{f'(z)} + \lambda\frac{(z(zf'(z))')'}{(zf'(z))'}\right] \\
&\geq (1-\alpha)p + \alpha p\frac{1-(1-2\rho)r}{1+r} - |1-\alpha-i\beta|\frac{2r}{1-r^2} - \frac{\sigma r}{1-r} \\
&= \frac{(2\alpha p - p(1+2\alpha\rho)-\sigma)r^2 - (2|1-\alpha-i\beta| + 2\alpha p(1-\rho)+\sigma)r + p}{1-r^2}.
\end{aligned}$$

Suppose that

$$p(r) = (2\alpha p - p(1+2\alpha\rho)-\sigma)r^2 - (2|1-\alpha-i\beta| + 2\alpha p(1-\rho)+\sigma)r + p.$$

Since $p \in \mathbb{N}$ and $0 < \sigma \leq 1$, so we have

$$p(0) = p > 0 \text{ and } p(1) = -2(|1-\alpha-i\beta| + \sigma) < 0.$$

It follows that the root lies in $(0,1)$. This implies that

$$\Re\left[(1-\lambda)\frac{(zf'(z))'}{f'(z)} + \lambda\frac{(z(zf'(z))')'}{(zf'(z))'}\right] > 0$$

if $r < r_1$, where r_1 is given by (18).

Putting $\lambda = 0$ and $\rho = 0$ in Theorem 4, we obtain the following result proved in [30].

Corollary 6 If $f(z) \in \mathcal{A}_p$ satisfies

$$\Re\left[\frac{f(z)}{z^p}\right] > 0 \text{ and } \left|G^p_{\alpha,\beta,0,0}(f,g) - p\right| < \sigma p \ (0 < \sigma \leq 1)$$

for $g \in S^*_p$. Then f is p-valent convex in $|z| < r_2$, where

$$r_2 = \frac{(2|1-\alpha-i\beta| + 2\alpha p + \sigma) - \sqrt{(2|1-\alpha-i\beta| + 2\alpha p + \sigma)^2 - 4p(2\alpha p - p - \sigma)}}{2(2\alpha p - p - \sigma)}.$$

Theorem 5 If $f(z) \in \mathcal{A}_p$ satisfies

$$\left|G^p_{\alpha,0,\lambda,\rho}(f,g)(z) - p\right| < \sigma p \ (\alpha > 0,\ 0 < \sigma \leq 1)$$

for $g \in J_p(\lambda, \rho)$. Then

$$\Re\left\{\frac{\lambda(z(zf'(z))')'}{\alpha(zf'(z))'} + \left[\frac{1}{\alpha}(1-\lambda) + \lambda\left(1 - \frac{1}{\alpha}\right)\right]\frac{(zf'(z))'}{f'(z)} + \left(1 - \frac{1}{\alpha}\right)(1-\lambda)\frac{zf'(z)}{f(z)}\right\} > 0$$

in $|z| < r_3$, where

$$r_3 = \frac{(2\alpha p(1-\rho) + \sigma) - \sqrt{(2\alpha p(1-\rho) + \sigma)^2 - 4\alpha p(\alpha p(1-2\rho) - \sigma)}}{2(\alpha p(1-2\rho) - \sigma)}. \tag{19}$$

Proof. Let

$$h(z) = \frac{1}{p}G^p_{\alpha,0,\lambda,\rho}(f,g)(z) - 1 = \left[\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\right]^{1-\lambda}\left[\frac{(zf'(z))'}{pf'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\right]^{\lambda} - 1,$$

where $h(z)$ is analytic in $\mathbb{D}$ with $h(0) = 0$ and $|h(z)| < \sigma$. By using the Schwarz lemma, we get

$$h(z) = \sigma z\phi(z),$$

where $\phi(z)$ is analytic in $\mathbb{D}$ with $|\phi(z)| < 1$. Differentiating logarithmically, we have

$$\begin{aligned}&\frac{\lambda(z(zf'(z))')'}{\alpha(zf'(z))'} + \left[\frac{1}{\alpha}(1-\lambda) + \lambda(1 - \frac{1}{\alpha})\right]\frac{(zf'(z))'}{f'(z)} + (1 - \frac{1}{\alpha})(1-\lambda)\frac{zf'(z)}{f(z)} \\ &= \left[(1-\lambda)\frac{zg'(z)}{g(z)} + \lambda\frac{(zg'(z))'}{g'(z)}\right] + \frac{\sigma z(z\phi'(z) + \phi(z))}{\alpha(1 + \sigma z\phi(z))}.\end{aligned}$$

This implies that

$$\begin{aligned}&\Re\left\{\frac{\lambda(z(zf'(z))')'}{\alpha(zf'(z))'} + \left[\frac{1}{\alpha}(1-\lambda) + \lambda(1 - \frac{1}{\alpha})\right]\frac{(zf'(z))'}{f'(z)} + (1 - \frac{1}{\alpha})(1-\lambda)\frac{zf'(z)}{f(z)}\right\} \\ &\geq \Re\left\{(1-\lambda)\frac{zg'(z)}{g(z)} + \lambda\frac{(zg'(z))'}{g'(z)}\right\} - \frac{\sigma}{\alpha}\left|\frac{z(z\phi'(z) + \phi(z))}{1 + \sigma z\phi(z)}\right|.\end{aligned}$$

Now, using the well-known results for the class

$$\begin{aligned}&\Re\frac{1}{p}\left\{\frac{\lambda(z(zf'(z))')'}{\alpha(zf'(z))'} + \left[\frac{1}{\alpha}(1-\lambda) + \lambda(1 - \frac{1}{\alpha})\right]\frac{(zf'(z))'}{f'(z)} + (1 - \frac{1}{\alpha})(1-\lambda)\frac{zf'(z)}{f(z)}\right\} \\ &\geq \frac{1 - (1-2\rho)r}{1 + r} - \frac{\sigma r}{\alpha p(1-r)} \\ &= \frac{[\alpha p(1-2\rho) - \sigma]r^2 - [2\alpha p(1-\rho) + \sigma]r + \alpha p}{\alpha p(1-r^2)}.\end{aligned}$$

Suppose that

$$m(r) = (\alpha p(1-2\rho) - \sigma)r^2 - (2\alpha p(1-\rho) + \sigma)r + \alpha p.$$

Since $p \in \mathbb{N}$ and $0 < \sigma \leq 1$, so we have

$$m(0) = \alpha p > 0 \text{ and } m(1) = -2\sigma < 0.$$

It follows that the root lies in $(0,1)$. This implies that

$$\Re\left\{\frac{\lambda(z(zf'(z))')'}{\alpha(zf'(z))'} + \left[\frac{1}{\alpha}(1-\lambda) + \lambda(1-\frac{1}{\alpha})\right]\frac{(zf'(z))'}{f'(z)} + (1-\frac{1}{\alpha})(1-\lambda)\frac{zf'(z)}{f(z)}\right\} > 0$$

if $r < r_3$, where r_3 is given by (19).

Putting $\lambda = 0$ and $\rho = 0$ in Theorem 5, we get the following result proved in [30].

Corollary 7 If $f(z) \in \mathcal{A}_p$ satisfies

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha} - p\right| < \sigma p \ (\alpha > 0,\ 0 < \sigma \leq 1)$$

for $g \in S_p^*$. Then f is p-valent α^{-1}-convex in $|z| < r_4$, where

$$r_4 = \frac{(2\alpha p + \sigma) - \sqrt{(2\alpha p + \sigma)^2 - 4\alpha p(\alpha p - \sigma)}}{2(\alpha p - \sigma)}.$$

Setting $\lambda = 1$ and $\rho = 0$ in Theorem 5, we have the following corollary.

Corollary 8 If $f(z) \in \mathcal{A}_p$ satisfies

$$\left|\frac{(zf'(z))'}{f'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha} - p\right| < \sigma p \ (\alpha > 0,\ 0 < \sigma \leq 1)$$

for $g \in S_p^*$. Then

$$\Re\left\{\frac{(z(zf'(z))')'}{\alpha(zf'(z))'} + (1-\frac{1}{\alpha})\frac{(zf'(z))'}{f'(z)}\right\} > 0$$

in $|z| < r_5$, where

$$r_5 = \frac{(2\alpha p + \sigma) - \sqrt{(2\alpha p + \sigma)^2 - 4\alpha p(\alpha p - \sigma)}}{2(\alpha p - \sigma)}.$$

Theorem 6 If $f \in N_{p,\alpha,\beta,\rho}(\lambda,\xi)$, then $f(z) \in L_{p,\alpha,\beta,\rho}(\lambda,\mu,\xi)$ $(\mu > 0)$ in $|z| < r_6$, where r_6 is the only root of the equation

$$p(1-2\xi)r^2 - 2(p(1-\xi) + \mu)r + p = 0. \tag{20}$$

In the interval $(0,1)$, the value of r_6 is the best possible.

Proof. Let the function $h(z)$ be defined by

$$\begin{aligned}
&\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\\
&= \left[\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\right]^{1-\lambda}\left[\frac{(zf'(z))'}{f'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f'(z)}{pz^{p-1}}\right)^{i\beta}\right]^{\lambda}\\
&= (1-\xi)h(z) + \xi \qquad (z \in \mathbb{D})
\end{aligned} \tag{21}$$

Then $h(z) \in \mathcal{A}$, $h(z) = 1 + c_1z + c_2z^2 + \cdots$ and $\Re h(z) > 0$.

A logarithmic differentiation of (21) and application of Lemma 8, yield

$$\begin{aligned}
&\Re\left\{\frac{\frac{1}{p}\left[G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right]-\xi}{1-\xi}\right\}\\
&=\Re\left\{h(z)+\frac{\mu z h'(z)}{p(1-\xi)h(z)+p\xi}\right\}\\
&\geq \Re\left\{h(z)-\frac{\mu |zh'(z)|}{p(1-\xi)|h(z)|+p\xi}\right\}\\
&\geq \Re h(z)\left\{1-\frac{2\mu r}{p(1-\xi)(1-r)^2+p\xi(1-r^2)}\right\}\\
&=\Re h(z)\frac{p(1-2\xi)r^2-2(\mu+p(1-\xi))r+p}{p(1-\xi)(1-r)^2+p\xi(1-r^2)}. \qquad (22)
\end{aligned}$$

If $|z| < r_6$, where r_6 is the only root in the interval $0 < r < 1$ of the equation given by (20), then we find from (22) that $f(z) \in L_{p,\alpha,\beta,\rho}(\lambda,\mu,\xi)$ $(\mu > 0)$ for $|z| < r_6$.

To show that the bound r_6 is sharp, we consider the function $f(z) \in \mathcal{A}_p$ defined by

$$\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z) = (1-\xi)\frac{1-z}{1+z}+\xi \ (z \in \mathbb{D}),$$

or, equivalently

$$\frac{\frac{1}{p}\left[G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)+\mu H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right]-\xi}{1-\xi} = \frac{p(1-2\xi)z^2-2(\mu+p(1-\xi))z+p}{p(1+z)(1-(1-2\xi)z)} = 0$$

for $z = r_6$, which completes the proof of Theorem 6.

For $\rho = 0$, if we set $\lambda = 0$ and $\lambda = 1$ in Theorem 6, respectively, we have the following corollaries.

Corollary 9 If $f \in N_{p,\alpha,\beta,0}(0,\xi)$, then $f(z) \in L_{p,\alpha,\beta,0}(0,\mu,\xi)$ $(\mu > 0)$ in $|z| < r_6$, where r_6 is given by (20).

Corollary 10 If $f \in N_{p,\alpha,\beta,0}(1,\xi)$, then $f(z) \in L_{p,\alpha,\beta,0}(1,\mu,\xi)$ $(\mu > 0)$ in $|z| < r_6$, where r_6 is given by (20).

Theorem 7 Let $\mu > 0,\ l > 0$. If $f(z) \in \mathcal{A}_p$ satisfies

$$\left|\arg\left\{\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\left[1+\frac{\mu}{p}H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right]\right\}\right| < \frac{\pi}{2}\left[l+\frac{2}{\pi}\arctan\left(\frac{\mu l}{p}\right)\right], \qquad (23)$$

then

$$\left|\arg\left(\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right)\right| < \frac{\pi}{2}l, \qquad (24)$$

where $G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ and $H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ are given by (5) and (6), respectively.

Proof. Let

$$\begin{aligned}
h(z) &= \frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\\
&= \left[\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\right]^{1-\lambda}\left[\frac{(zf'(z))'}{pf'(z)}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f'(z)}{pz^{p-1}}\right)^{i\beta}\right]^{\lambda}, \qquad (25)
\end{aligned}$$

then $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{D}$ with $h(0) = 1$ and $h'(0) = 1 \neq 0$.

Differentiating (25) logarithmically with respect to z and multiplying by z, we have

$$\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\left\{1+\frac{\mu}{p}H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right\}=h(z)+\frac{\mu}{p}zh'(z).$$

By using Lemma 9, the proof of Theorem 7 is completed.

Putting $\lambda=0$ in Theorem 7, we have the following corollary.

Corollary 11 Let $\mu>0,\ l>0$. If $f(z)\in\mathcal{A}_p$ satisfies

$$\left|\arg\left\{\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\left[1+\mu\left(\frac{(zf'(z))'}{pf'(z)}+(\alpha+i\beta-1)\frac{zf'(z)}{pf(z)}-\alpha\frac{zg'(z)}{pg(z)}-i\beta\right)\right]\right\}\right|$$
$$<\frac{\pi}{2}\left[l+\frac{2}{\pi}\arctan\left(\frac{\mu l}{p}\right)\right],$$

then

$$\left|\arg\left\{\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\right\}\right|<\frac{\pi}{2}l.$$

Further, for $\beta=0$, if we set $\alpha=1$ and $\alpha=0$ in Corollary 11, respectively, we get the following corollaries.

Corollary 12 Let $\mu>0,\ l>0$. If $f(z)\in\mathcal{A}_p$ satisfies

$$\left|\arg\left\{\frac{zf'(z)}{pg(z)}\left[1+\mu\left(\frac{(zf'(z))'}{pf'(z)}-\frac{zg'(z)}{pg(z)}\right)\right]\right\}\right|<\frac{\pi}{2}\left[l+\frac{2}{\pi}\arctan\left(\frac{\mu l}{p}\right)\right],$$

then

$$\left|\arg\left\{\frac{zf'(z)}{pg(z)}\right\}\right|<\frac{\pi}{2}l.$$

Corollary 13 Let $\mu>0,\ l>0$. If $f(z)\in\mathcal{A}_p$ satisfies

$$\left|\arg\left\{\frac{zf'(z)}{pf(z)}\left[1+\mu\left(\frac{(zf'(z))'}{pf'(z)}-\frac{zf'(z)}{pf(z)}\right)\right]\right\}\right|<\frac{\pi}{2}\left[l+\frac{2}{\pi}\arctan\left(\frac{\mu l}{p}\right)\right],$$

then

$$\left|\arg\left\{\frac{zf'(z)}{pf(z)}\right\}\right|<\frac{\pi}{2}l.$$

Theorem 8 If $f(z)\in\mathcal{A}_p$ satisfies

$$\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\left\{1+\frac{\mu}{p}H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right\}\prec\frac{1+[(1-\eta)A+\eta B]z}{1+Bz}\ (\mu>0), \tag{26}$$

then

$$\frac{1}{p}G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)\prec q(z)\prec\frac{1+((1-\eta)A+\eta B)z}{1+Bz}, \tag{27}$$

where

$$q(z)=(1+Bz)^{-1}\left[{}_2F_1\left(1,1;1+\frac{p}{\mu};\frac{Bz}{1+Bz}\right)+\frac{p[(1-\eta)A+\eta B]z}{p+\mu}{}_2F_1\left(1,1;2+\frac{p}{\mu};\frac{Bz}{1+Bz}\right)\right] \tag{28}$$

and $q(z)$ is the best dominant, $G^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ and $H^p_{\alpha,\beta,\lambda,\rho}(f,g)(z)$ are given by (5) and (6), respectively.

Proof. Let the function $h(z)$ be given by (25), then $h(z)=1+c_1z+c_2z^2+\cdots$ is analytic in $\mathbb{D}$ with $h(0)=1$ and $h'(0)=c_1\neq 0$.

Differentiating (25) logarithmically with respect to z and multiplying by z, we have

$$\frac{1}{p}G^{p}_{\alpha,\beta,\lambda,\rho}(f,g)(z)\left\{1+\frac{\mu}{p}H^{p}_{\alpha,\beta,\lambda,\rho}(f,g)(z)\right\}=h(z)+\frac{\mu}{p}zh'(z).$$

Thus, by using Lemma 10, we obtain

$$\frac{1}{p}G^{p}_{\alpha,\beta,\lambda,\rho}(f,g)(z)\prec\frac{p}{\mu}z^{-\frac{p}{\mu}}\int_0^z\frac{t^{\frac{p}{\mu}-1}(1+((1-\eta)A+\eta B)t)dt}{1+Bt}=q(z).$$

Now, using the conditions (i) and (iii) of Lemma 2, we can rewritten the function $q(z)$ as (28). This completes the proof of Theorem 8.

Putting $\lambda=0$ and $\eta=0$ in Theorem 8, we have the following corollary.

Corollary 14 If $f(z)\in\mathcal{A}_p$ satisfies

$$\left\{\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\left[1+\mu\left(\frac{(zf'(z))'}{pf'(z)}+(\alpha+i\beta-1)\frac{zf'(z)}{pf(z)}-\alpha\frac{zg'(z)}{pg(z)}-i\beta\right)\right]\right\}$$
$$\prec\frac{1+Az}{1+Bz}\quad(\mu>0),$$

then

$$\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z^p}\right)^{i\beta}\prec q(z)\prec\frac{1+Az}{1+Bz},$$

where $q(z)$ is given by (28) with $\eta=0$.

Further, for $\beta=0$, if we set $\alpha=1$ and $\alpha=0$ in Corollary 14, respectively, we get the following corollaries.

Corollary 15 If $f(z)\in\mathcal{A}_p$ satisfies

$$\left\{\frac{zf'(z)}{pg(z)}\left[1+\mu\left(\frac{(zf'(z))'}{pf'(z)}-\frac{zg'(z)}{pg(z)}\right)\right]\right\}\prec\frac{1+Az}{1+Bz}\ (\mu>0),$$

then

$$\frac{zf'(z)}{pg(z)}\prec q(z)\prec\frac{1+Az}{1+Bz},$$

where $q(z)$ is given by (28) with $\eta=0$.

Corollary 16 If $f(z)\in\mathcal{A}_p$ satisfies

$$\left\{\frac{zf'(z)}{pf(z)}\left[1+\mu\left(\frac{(zf'(z))'}{pf'(z)}-\frac{zf'(z)}{pf(z)}\right)\right]\right\}\prec\frac{1+Az}{1+Bz}\ (\mu>0),$$

then

$$\frac{zf'(z)}{pf(z)}\prec q(z)\prec\frac{1+Az}{1+Bz},$$

where $q(z)$ is given by (28) with $\eta=0$.

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