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EXISTENCE OF SOLUTIONS FOR NONLOCAL PROBLEMS IN ORLICZ-SOBOLEV SPACES VIA MONOTONE METHOD

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Abstract. In this paper, using theory of monotone operators, we study the existence of weak solutions for a class of nonlocal nonvariational problems in the Orlicz-Sobolev spaces.

1. Introduction

In this article, we are concerned with a class of nonlocal problems in Orlicz-Sobolev spaces of the form

$$\begin{cases} -\mathcal{M}(u)div(a(|\nabla u|)\nabla u - a(|u|)u) &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad\;\; \text{on } \partial\Omega, \end{cases} \tag{1}$$

and

$$\begin{cases} -\mathcal{M}(u)div(a(|\nabla u|)\nabla u - g(x,u,\nabla u)) &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad\;\; \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, f is a Carathéodory function and $\mathcal{M}$ is a continuous and bounded functional. The function $\varphi(t) := a(t)t$ is an increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. For the case $a(t) = |t|^{p-2}$, equations (1) and (2) turn into the well-known $p-$Laplacian equation $(p > 1)$ which has been extensively studied during the last decades by many authors. We refer to ([4, 9, 12, 16, 18, 22]) and the references therein for detailed background.

The study of variational problems in the classical Sobolev and Orlicz-Sobolev spaces is an interesting topic of research due to its significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasiconformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory, and probability theory (see [2, 3, 6, 7, 8, 13, 19, 20]). Moreover, problems (1) and (2) possess more complicated nonlinearities, for example, it is inhomogeneous,

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so in the discussions, some special techniques will be needed. However, the inhomogeneous nonlinearities have important physical background. Therefore, equations (1) and (2) may represent a variety of mathematical models corresponding to certain phenomenons (see [13]), e.g.,

(1) Nonlinear elasticity: $\varphi(t) = \left(1+t^2\right)^{\alpha} - 1$, $\alpha > \frac{1}{2}$,
(2) Plasticity: $\varphi(t) = t^{\alpha}\left(\log\left(1+t\right)\right)^{\beta}$, $\alpha \geq 1, \beta > 0$,
(3) Generalized Newtonian fluids: $\varphi(t) = \int_0^t s^{1-\alpha}\left(\sinh^{-1} s\right)^{\beta} ds$, $0 \leq \alpha \leq 1, \beta > 0$.

In the present paper, we study the nonlocal problems (1) and (2) in the Orlicz-Sobolev spaces. However, unlike the above mentioned papers, problems (1) and (2) cannot be settled in the variational framework because of the functional $\mathcal{M}$. Indeed, the existence of $\mathcal{M}$ makes these problems very complicated and make us force to apply different tools, such as the monotone operator theory, for this class of problems. So, in that context, we use two well-known theorems named as Browder theorem and Leray-Lions theorem. We want to remark that, to our best knowledge, there is no paper which considers nonvariational nonlocal problems via monotone operator methods in the Orlicz-Sobolev spaces, and hence, the results of the present paper are new and original.

2. Preliminaries

To study problems (1) and (2), let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz and Orlicz-Sobolev spaces, useful for what follows, for more details we refer the readers to the monographs ([1, 17, 21, 23]) and the papers ([5, 7, 8, 10, 15]).

The function $a : (0,\infty) \to \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) := \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0, & \text{for} t = 0, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

For the function φ, let us define

$$\Phi(t) = \int_0^t \varphi(s) ds \quad \forall t \in \mathbb{R}.$$

The function Φ introduced above is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \text{ and } \lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty,$$

the function Φ is then called an *N-function.*

The function Φ^* defined by the formula

$$\Phi^*(t) = \int_0^t \varphi^{-1}(s) ds \quad \forall t \in \mathbb{R}$$

is called the *complementary* function of Φ and it satisfies the condition

$$\Phi^*(t) = \sup\{st - \Phi(s) : \ s \geq 0\} \quad \forall t \geq 0.$$

Notice that the function Φ^* is also an N-function in the sense of the following Young inequality

$$st \leq \Phi(s) + \Phi^*(t) \quad \forall s, t \geq 0.$$

We define the numbers

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}. \tag{3}$$

Throughout this paper, we assume that

$$1 < \varphi_0 \leq \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 < \infty \quad \forall t \geq 0, \tag{4}$$

which implies that Φ satisfies the Δ_2-*condition*, i.e.,

$$\Phi(2t) \leq K\Phi(t) \quad \forall t \geq 0, \tag{5}$$

where K is a positive constant (see,e.g., [21]).
In this paper, we also need the following condition

$$\text{the function } t \mapsto \Phi(\sqrt{t}) \text{ is convex for all } t \in [0, \infty). \tag{6}$$

Under the condition (4), for a N-function Φ, the *Orlicz space* $L_\Phi(\Omega)$ coincides with the set (equivalence classes) of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi\left(|u\left(x\right)|\right) \, dx < +\infty$$

and is equipped with the (Luxemburg) norm, i.e.

$$\|u\|_{L_\Phi(\Omega)} = \|u\|_\Phi := \inf\left\{k > 0 : \int_\Omega \Phi\left(\frac{|u\left(x\right)|}{k}\right) \, dx \leq 1\right\},$$

which makes the pair $(L_\Phi(\Omega), \|\cdot\|_\Phi)$ a Banach space.
For the Orlicz spaces, the Hölder inequality holds (see [23]):

$$\int_\Omega uv \, dx \leq 2\|u\|_{L_\Phi(\Omega)}\|u\|_{L^*_\Phi(\Omega)} \quad u \in L_\Phi(\Omega), \, v \in L_{\Phi^*}(\Omega).$$

The *Orlicz-Sobolev space* $W^1L_\Phi(\Omega)$ is the space defined by

$$W^1L_\Phi(\Omega) := \left\{u \in L_\Phi(\Omega) : \; \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \; i = 1, 2, ..., N\right\},$$

and $W^1L_\Phi(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W^1L_\Phi(\Omega)} = \|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

Now, we introduce the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$. It turns out that the space $W_0^1L_\Phi(\Omega)$ can be renormed by using as an equivalent norm

$$\|u\| := \||\nabla u|\|_\Phi.$$

Lemma 2.1 [see [3, 19, 20]] Let $u \in W_0^1L_\Phi(\Omega)$. Then we have

(i) $\|u\|^{\varphi^0} \leq \int_\Omega \Phi(|\nabla u(x)|) \, dx \leq \|u\|^{\varphi_0}$ if $\|u\| < 1$.
(ii) $\|u\|^{\varphi_0} \leq \int_\Omega \Phi(|\nabla u(x)|) \, dx \leq \|u\|^{\varphi^0}$ if $\|u\| > 1$.

We notice that the Orlicz-Sobolev spaces, unlike the Sobolev spaces they generalize, are in general neither separable nor reflexive. A key tool to guarantee these properties is represented by the Δ_2-condition (5). Actually, condition (5) assures that both $L_\Phi(\Omega)$ and $W_0^1 L_\Phi(\Omega)$ are separable (see [1]). Conditions (5) and (6) assure that $L_\Phi(\Omega)$ is a uniformly convex space and thus, a reflexive Banach space (see [20]); consequently, the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ is also a reflexive Banach space.

We also note that with the help of condition (4), Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_0}(\Omega)$, as a result, $W_0^1 L_\Phi(\Omega)$ is continuously and compactly embedded in the classical Lebesgue space $L^q(\Omega)$ for all $1 \leq q < \varphi_0^* := \frac{N\varphi_0}{N-\varphi_0}$.
We can give certain examples of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are odd, increasing homeomorphisms from $\mathbb{R}$ onto itself and satisfy conditions (4) and (6):

(1) Let $\varphi(t) = p\,|t|^{p-2}\,t \;\forall t \in \mathbb{R}$, with $p > 1$: For this function it can be proved that $\varphi_0 = \varphi^0 = p$. Besides, in this particular case the corresponding Orlicz space$L_\Phi(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ while the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ is the classical Sobolev space $W_0^{1,p}(\Omega)$.
(2) Let $\varphi(t) = \log(1 + |t|^\alpha)\,|t|^{p-2}\,t \;\; \forall t \in \mathbb{R}$, with $p, \alpha > 1$: For this function it can be shown that $\varphi_0 = p$, $\varphi^0 = p + \alpha$.
(3) Let $\varphi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$ if $t \neq 0$, $\varphi(0) = 0$, with $p > 2$: For this function we have $\varphi_0 = p - 1$, $\varphi^0 = p$.

The main results of the present paper are based on the following well-known lemmas (see, e.g., [11]).

Lemma 2.2 [Browder] Let X be a reflexive real Banach space. Moreover, let $T : X \to X^*$ be an operator satisfying the conditions:

(i) T is bounded;
(ii) T is demicontinuous;
(iii) T is coercive;
(iv) T is monotone on the space X, i.e., for all $u, v \in X$ we have

$$(T(u) - T(v), u - v) \geq 0. \tag{7}$$

Then the equation

$$T(u) = h^* \tag{8}$$

has at least one solution $u \in X$ for every $h^* \in X^*$. If, moreover, the inequality (7) is strict for all $u, v \in X$, $u \neq v$, then the equation (8) has precisely one solution $u \in X$ for every $h^* \in X^*$.

Lemma 2.3 [Leray-Lions] Let X be a reflexive real Banach space. Moreover, let $T : X \to X^*$ be an operator satisfying the conditions:

(i) T is bounded;
(ii) T is demicontinuous;
(iii) T is coercive.

Moreover, let there exists a bounded mapping $\Psi : X \times X \to X^*$ such that

(iv) $\Psi(u, u) = T(u)$ for every $u \in X$;

(v) For all $u, w, h \in X$ and any sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers such that $t_n \to 0$, we have
$$\Psi(u + t_n h, w) \rightharpoonup \Psi(u, w);$$
(vi) For all $u, w \in X$ we have (the so-called condition of monotonicity in the principal part)
$$(\Psi(u, u) - \Psi(w, u), u - w) \geq 0,$$
(vii) If $u_n \rightharpoonup u$ and
$$\lim_{n\to\infty} (\Psi(u_n, u_n) - \Psi(u, u_n), u_n - u) = 0,$$
then we have
$$\Psi(w, u_n) \rightharpoonup \Psi(w, u) \text{ for arbitrary } w \in X;$$
$(viii)$ If $w \in X$, $u_n \rightharpoonup u$ and $\Psi(w, u_n) \rightharpoonup z$, then
$$\lim_{n\to\infty} (\Psi(w, u_n), u_n) = (z, u).$$
Then the equation (8) has at least one solution $u \in X$ for every $h^* \in X^*$.

In the sequel, for function a introduced above, we assume that:
Define $a : [0, +\infty) \to (0, +\infty)$ such that there exist a constant $a_0 > 0$ such that
$$a(t) \geq a_0 > 0, \quad \forall t \in [0, +\infty). \tag{9}$$
We point out that $a(t) = |t|^{p-2} + a_0$ satisfies the conditions mentioned for function a.

In the rest of this section, we prove the following auxiliary result.

Lemma 2.4 Assume that the condition (9) holds. Then for any $k, l > 0$, there exists a positive constant $C(\delta)$, $\delta = \min\{1, a_0, k, l\}$, such that
$$(ka(|\xi|)\xi - la(|\eta|)\eta, \xi - \eta) \geq C(\delta)\Phi(|\xi - \eta|) \quad \forall \xi, \eta \in \mathbb{R}^N. \tag{10}$$
Proof. If we consider the homogeneity of norm, we can assume that $|\xi| = 1$ and $|\eta| \leq 1$. Moreover, according to a convenient basis in $\mathbb{R}^N$, we can assume
$$\xi = (1, 0, ..., 0) \text{ and } \eta = (\eta_1, \eta_2, 0, ..., 0) \text{ with } |\eta| \leq 1.$$
Let also denote $a(|\xi|) = a_\xi$ and $a(|\eta|) = a_\eta$. Then we can proceed as follows: To verify (10) holds, we will show
$$\frac{(ka_\xi - la_\eta\eta_1)(1 - \eta_1) + la_\eta\eta_2^2}{(1 - \eta_1)^2 + \eta_2^2} \geq C,$$
which is equivalent to (10). Then we must show that the function
$$\beta(t, s) = \frac{ka_\xi - (la_\eta t + ka_\xi t)s + la_\eta t^2}{(1 - 2st + t^2)} \tag{11}$$
is bounded from below by a positive constant. Then for fixed t, we get
$$\frac{\partial \beta}{\partial s} = \frac{-(la_\eta t + ka_\xi t)\left(1 - 2st + t^2\right) - \left(ka_\xi - (la_\eta t + ka_\xi t)s + la_\eta t^2\right)(-2t)}{(1 - 2st + t^2)^2}.$$
Then, we obtain
$$ka_\xi - (la_\eta t + ka_\xi t)s + la_\eta t^2 = \frac{(la_\eta + ka_\xi)\left(1 - 2st + t^2\right)}{2}, \tag{12}$$

when $\frac{\partial\beta}{\partial s} = 0$. Therefore, replacing (12) in (11), we get

$$\beta(t,s) = \frac{(la_\eta + ka_\xi)\left(1 - 2st + t^2\right)}{2\left(1 - 2st + t^2\right)} = \frac{(la_\eta + ka_\xi)}{2} \geq \frac{C(\delta)}{2} > 0,$$

where $\delta = \min\{1, a_0, k, l\} > 0$. The proof is completed.

3. Main results

In this section, we shall state and prove the main results of the paper by using the monotone operator method. For brevity, we denote by X and X^* the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ and its dual space $\left(W_0^1 L_\Phi(\Omega)\right)^*$, respectively.

Let us define the operators $J, F : X \to X^*$ by

$$(J(u), v) = \mathcal{M}(u) \int_\Omega (a(|\nabla u|)\nabla u \nabla v + a(|u|)uv)\, dx, \quad \forall u, v \in X,$$

$$(F(u), v) = \int_\Omega f(x,u)v\, dx, \quad \forall u, v \in X,$$

and set

$$T := J - F.$$

Then, from the monotone operator theory, the solution function $u \in X$ of (1) satisfying the operator equation

$$T(u) = J(u) - F(u) = 0. \tag{13}$$

is also the solution of the integral equation

$$\mathcal{M}(u) \int_\Omega (a(|\nabla u|)\nabla u \nabla v + a(|u|)uv)\, dx - \int_\Omega f(x,u)v dx = 0 \ \forall v \in X. \tag{14}$$

Namely, the existence of weak solution of problem (1) is equivalent to the existence of solution of the operator equation (13) (see, e.g., [11, 14]).

Our first result is given by the following theorem.

Theorem 3.1 Assume that the following assertions hold:

(f_0) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and assume that there exist $C_2 > 0$ such that

$$|f(x,t)| \leq C_2\left(1 + |t|^{q-1}\right),$$

for all $t \in \mathbb{R}$ and a.e. $x \in \overline{\Omega}$, where $\varphi^0 < q < \varphi_0^*$;

(f_1) $f(x,0) = 0$ and $(f(x,t) - f(x,s))(t-s) \leq 0$ for all $s, t \in \mathbb{R}$ and a.e. $x \in \overline{\Omega}$;

(M_0) $\mathcal{M} : X \to (0, +\infty)$ is continuous and bounded on any bounded subset of X such that there are constants $m_0, m_1 > 0$ such that $m_0 \leq \mathcal{M}(u) \leq m_1$ for all $u \in X$.

Then problem (1) has precisely one weak solution.

Proof. It is obvious from (f_0) that T is well defined. Indeed, we first see that the functional

$$\int_\Omega (a(|\nabla u|)\nabla u \nabla v + a(|u|)uv)\, dx$$

is well defined. Moreover, taking into account that $\mathcal{M}$ is bounded and continuous then boundedness and continuity (and hence demicontinuity) of T is derived.
From (f_1) and (M_0), for sufficiently large $\|u\|$, using Lemma 2.1

$$\begin{aligned}(T(u),u) &= \mathcal{M}(u)\int_\Omega \left(a(|\nabla u|)\,|\nabla u|^2 + a(|u|)\,|u|^2\right)dx - \int_\Omega f(x,u)u dx\\ &\geq m_0\,\|u\|^{\varphi_0}\,,\end{aligned}$$

which shows that T is coercive. Let us show the monotonicity of T. If $u = v$, $T(u) = T(v)$. For the case $u \neq v$, from (f_1) and (M_0) and Lemma 2.4 we have

$$\begin{aligned}&(T(u)-T(v),u-v)\\ &\geq \mathcal{M}(u)\int_\Omega \left(a(|\nabla u|)\nabla u(\nabla u-\nabla v) + a(|u|)u(u-v)\right)\,dx\\ &- \mathcal{M}(v)\int_\Omega \left(a(|\nabla v|)\nabla v(\nabla u-\nabla v) + a(|v|)v(u-v)\right)\,dx\\ &= \int_\Omega \left(\mathcal{M}(u)a(|\nabla u|)\nabla u - \mathcal{M}(v)a(|\nabla v|)\nabla v\right)(\nabla u-\nabla v)\,dx\\ &+ \int_\Omega \left(\mathcal{M}(u)a(|u|)u - \mathcal{M}(v)a(|v|)v)\right)(u-v)\,dx\\ &\geq C(\delta)\int_\Omega \left(\Phi(|\nabla u-\nabla v|) + \Phi(|u-v|)\right)\,dx,\end{aligned}$$

where $\delta = \min\{1, a_0, \mathcal{M}(u), \mathcal{M}(v)\}$. This implies the monotonicity of T. As a consequence of Lemma 2.2, the equation

$$T(u) = J(u) - F(u) = h^*$$

has at least one solution $u \in X$ for every $h^* \in X^*$. Moreover, since the last inequality above is strict, it follows then from Lemma 2.2 that there is a unique solution of (13), which in turn is a unique weak solution of (1). The proof is completed.

We want to remark that the operator

$$T_{h^*}(u) := T(u) - h^*$$

also satisfies all the conditions of Lemma 2.2. Therefore, it suffices to prove that the operator equation (13) has at least one solution, as we showed.

Now, we consider problem (2). Similarly, we define the operators $I, K : X \to X^*$ by

$$\begin{aligned}(I(u),v) &= \mathcal{M}(u)\int_\Omega a(|\nabla u|)\nabla u\nabla v\,dx \quad \forall u,v\in X,\\ (K(u),v) &= \mathcal{M}(u)\int_\Omega g(x,u,\nabla u)v\,dx \quad \forall u,v\in X,\end{aligned}$$

and set

$$G := I + K - F,$$

where operator F is defined as previous. Then, the solution $u \in X$ of (2) satisfying the operator equation

$$G(u) = I(u) + K(u) - F(u) = 0 \tag{15}$$

is also the solution of the integral equation

$$\mathcal{M}(u)\int_{\Omega}(a(|\nabla u|)\nabla u\nabla v+g(x,u,\nabla u)v)\,dx-\int_{\Omega}f(x,u)v\,dx=0\quad\forall v\in X.\tag{16}$$

This fact shows that the solutions of (15) correspond to the weak solutions of problem (2).

The second result of the present paper is:

Theorem 3.2 Assume that the conditions $(f_0),(f_1)$ and (M_0) hold. In addition, suppose that the function $g:\overline{\Omega}\times\mathbb{R}^{N+1}\to\mathbb{R}$ is a Carathéodory function, and for all $(t,\xi_1,\xi_2,...\xi_N)\in\mathbb{R}^{N+1}$ and a.e. $x\in\overline{\Omega}$, it satisfies the followings:

(g_0) there exist a constant $C>0$ and a function $h\in L^{\varphi_0'}(\Omega)$ such that

$$|g(x,t,\xi_1,\xi_2,...,\xi_N)|\le h(x)+C\sum_{i=1}^{N}|\xi_i|^{r-\epsilon},$$

where $r=\varphi_0-1+\epsilon$ with $\epsilon\in(0,1)$;

(g_1) the inequality

$$g(x,t,\xi_1,\xi_2,...,\xi_N)t\ge 0,$$

holds.

Then problem (2) has at least one solution.

Proof. From Theorem 3.1, we know that I is bounded and continuous. Moreover, by compact embedding $X\hookrightarrow L^{q}(\Omega)$ and assumption (f_0) the operator F is bounded and (sequentially weakly-strongly) continuous. Let us proceed for K. First of all, it is immediately seen that $K(u)$ is linear in v. To establish its continuity it suffices to check that

$$\sup_{\|v\|\le 1}|(K(u),v)|<\infty.\tag{17}$$

Therefore, using (M_0), (g_0), Hölder inequality and continuous embedding, we have

$$\begin{aligned}\|K(u)\| &= \sup_{\|v\|\le 1}|(K(u),v)|=\sup_{\|v\|\le 1}\left|\mathcal{M}(u)\int_{\Omega}g(x,u,\nabla u)v\,dx\right|\\ &\le m_1\sup_{\|v\|\le 1}\int_{\Omega}\left|h(x)+C\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{r-\epsilon}\right||v|\,dx\\ &\le m_1\sup_{\|v\|\le 1}\left(\int_{\Omega}|h|_{\varphi_0'}|v|_{\varphi_0}+C\left||\nabla u|^{r-\epsilon}\right|_{\varphi_0'}|v|_{\varphi_0}\right)\\ &\le Cm_1\|u\|^{r-\epsilon}\sup_{\|v\|\le 1}\|v\|\\ &\le Cm_1\|u\|^{r-\epsilon},\quad \varphi_0'=\frac{\varphi_0}{\varphi_0-1}.\end{aligned}$$

Therefore, K is bounded, and hence it is continuous. As a consequence, G is bounded and continuous. From $(f_1),(M_0)$ and (g_1), for sufficiently large $\|u\|$, we have

$$\begin{aligned}(G(u),u) &= \mathcal{M}(u)\int_{\Omega}\left(a(|\nabla u|)|\nabla u|^2+g(x,u,\nabla u)u\right)\,dx-\int_{\Omega}f(x,u)\,udx\\ &\ge m_0\|u\|^{\varphi_0},\end{aligned}$$

which implies that G is coercive. The assumptions (i)-(iii) are verified.
Let us define an operator $\Psi : X \times X \to X^*$ by

$$(\Psi(u,w),v) := (I(u),v) + (L(w),v), \quad \forall u,w,v \in X,$$

where $L := K - F$, and hence $L : X \to X^*$ is also bounded and continuous. Then

$$(\Psi(u,u),v) = \mathcal{M}(u)\int_\Omega (a(|\nabla u|)\nabla u \nabla v + g(x,u,\nabla u)v)\ dx - \int_\Omega f(x,u)\, v = (G(u),v),$$

i.e., $\Psi(u,u) = G(u)$ for all $u \in X$, so assumption (iv) is verified. In order to verify the assumption (v), let $u,w,h \in X$ and any sequence of real numbers $\{t_n\}_{n=1}^\infty$ such that $t_n \to 0$ as $n \to \infty$. Then

$$\Psi(u + t_n h, w) = I(u + t_n h) + L(w).$$

Since I is continuous, we have

$$I(u + t_n h, w) + L(w) \to I(u) + L(w),$$

or

$$\Psi(u + t_n h, w) \to \Psi(u,w).$$

Let us continue for the assumption (vi). Using Lemma 2.4 and Lemma 2.1, respectively, we have

$$\begin{aligned}(\Psi(u,u) - \Psi(w,u), u-w) &= (I(u),u-w) + (L(u),u-w) - (I(w), u-w) - (L(u), u-w)\\ &= \int_\Omega (\mathcal{M}(u)a(|\nabla u|)\nabla u - \mathcal{M}(w)a(|\nabla w|)\nabla w, \nabla u - \nabla w)\ dx\\ &\geq C(\delta)\int_\Omega \Phi(|\nabla u - \nabla w|)\, dx\\ &\geq C(\delta)\min\left\{\|u-w\|^{\varphi_0}, \|u-w\|^{\varphi^0}\right\} \geq 0,\end{aligned}$$

where $\delta = \min\{1, a_0, \mathcal{M}(u), \mathcal{M}(w)\}$. For the assumption (vii), let us assume that $u_n \rightharpoonup u$ in X and

$$\lim_{n\to\infty}(\Psi(u_n,u_n) - \Psi(u,u_n), u_n - u) = 0,$$

which is equivalent to

$$\lim_{n\to\infty}(I(u_n) - I(u), u_n - u) = 0. \tag{18}$$

Since $u_n \rightharpoonup u$ in X, the sequence $\{u_n\}$ is bounded in X. Therefore, from conditions (9) and (M_0), the functions a and $\mathcal{M}$ are bounded for any $n \in \mathbb{N}$. Then from Lemma 2.4 and Lemma 2.1, we have

$$\begin{aligned}(I(u_n) - I(u), u_n - u) &= \int_\Omega (\mathcal{M}(u_n)a(|\nabla u_n|)\nabla u_n - \mathcal{M}(u)a(|\nabla u|)\nabla u, \nabla u_n - \nabla u)\ dx\\ &\geq C(\delta_n)\int_\Omega \Phi(|\nabla u_n - \nabla u|)\, dx\\ &\geq C(\delta_n)\min\left\{\|u_n-u\|^{\varphi_0}, \|u_n-u\|^{\varphi^0}\right\} \geq 0,\end{aligned}$$

where $\delta_n = \min\{1, a_0, \mathcal{M}(u_n), \mathcal{M}(u)\}$. From the proof of Lemma 2.4 and the condition (M_0), it is easy to see that $C(\delta_n) > 0$ for all n. Therefore, considering (18), we must have that $\|u_n - u\| \to 0$ as $n \to \infty$, i.e., $u_n \to u$ in X.
In addition, by continuity of L, we have

$$\Psi(w,u_n) = I(w) + L(u_n) \to I(w) + L(u) = \Psi(w,u)$$

for arbitrary $w \in X$.
Finally, let $w \in X$ and $u_n \rightharpoonup u$ in X. Then, by continuity of L, we get

$$(\Psi(w, u_n), u_n) = (I(w), u_n) + (L(u_n), u_n) \to (I(w), u) + (L(u), u) = (\Psi(w, u), u).$$

Since $u_n \rightharpoonup u$ in X, we also have $\Psi(w, u_n) \to I(w) + L(u) = \Psi(w, u)$, which verifies the assumption $(viii)$. Consequently, from Lemma 2.3, operator equation (15) has at least one solution, which is a weak solution of problem (2). The proof is completed.

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