Electronic Journal of Mathematical Analysis and Applications,
Vol. 4(1) Jan. 2016, pp. 74 - 85.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

GRADIENT RECOVERY TECHNIQUES IN ONE-DIMENSIONAL GOAL-ORIENTED PROBLEMS

M. EL-AGAMY, A. ELSAID, H. M. NOUR

Abstract. In this paper, we propose a new technique to evaluate a posteriori error estimate for goal-oriented problems using recovery techniques. In our technique, we replace the gradient in the goal-oriented error estimate by the recovered gradient obtained by the polynomial preserving recovery technique. Also, we present a new local refinement algorithm suitable to the proposed technique. Finally, the validity of the proposed technique is illustrated by numerical examples.

1. Introduction

Adaptive control based on a posteriori error estimates has become standard in finite element methods. Having established a finite element space, we iterate the procedure (Solve − Estimate − Refine) until a stopping criterion is satisfied [1]. Generally, error estimators can be classified under two categories. The first one is the residual type estimators, as in [2], and the second one is the recovery type estimators, as in [3].

Goal-oriented methods have been conceived in the 1990s as a generalization to classical a posteriori error estimation methods. The idea of goal-oriented methods indicates that the numerical simulations are generally performed to study specific features of solutions to initial and boundary-value problems. These are the quantities of interest and represent the goals of the predictions. Obviously, it is important to be able to estimate the error in such quantities of interest and use these estimates to adapt the mesh in order to control their accuracy. Despite considerable progress, many of the popular techniques are based on the residual type estimators, as in [4]-[11].

Finite element recovery techniques are post-processing methods that reconstruct numerical approximations from finite element solutions to obtain the improved solutions. The practical usage of the recovery technique is not only to improve the quality of the approximation, but also to construct recovery type a posteriori error estimators in adaptive computation [12].

1991 *Mathematics Subject Classification.* 65L10, 65L60, 65L70.
Key words and phrases. A posteriori error estimate; Gradient recovery; Goal-oriented; Polynomial preserving recovery.
Submitted May 17, 2015.

In the literature of finite element recovery methods, gradient recovery has attracted considerable attentions from scientists and engineers. Different kinds of gradient recovery techniques are developed based on weighted averaging [13], [14], local or global projections [15]-[17], smoothing techniques [18], [19] and least-square type methods [20]-[22]. Gradient recovery technique has been widely used in engineering practice for its robustness as an a posteriori error estimator, its superconvergence of the recovered derivatives, and its efficiency in implementation; see, e.g., [23]-[30] and references therein.

In this work, a new approach for evaluating a posteriori error estimate for goal-oriented problems is presented. This approach is based on replacing the gradient in the goal-oriented error estimate by the recovered gradient obtained by the polynomial preserving recovery (PPR) introduced in [21], [26]. Also, a new local refinement algorithm that properly implements the proposed technique is suggested. We conclude this work by presenting the results of some numerical examples that illustrate the efficiency of the new technique.

2. Goal-oriented error estimation

Suppose that we are given the elliptic boundary-value problem

$$-\Delta u = f \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1}$$

and a linear functional G such that $G(u)$ is a quantity of physical, engineering or scientific interest. In order to approximate $G(u)$, one may compute $G(u_h)$, where u_h is the linear finite element approximation to u over a conforming mesh T of Ω.

We are interested in estimating the goal-oriented error e_h

$$e_h = |G(u-u_h)| = |\langle u-u_h\ ,\ G\rangle|\,, \tag{2}$$

where $\langle .\ ,\ .\rangle$ denotes the standard scalar product. Numerous contributions have been made to have an upper bound to e_h in both the mathematical and engineering literature and the reader is referred to the approaches [5]-[10] for further details and references. In [11], the authors discussed the disadvantages of these approaches and also develop and compare a number of alternative approaches to obtain guaranteed and fully computable bounds to e_h of arbitrary order finite element approximations in the context of a linear second-order elliptic problem.

In fact, combining the solution of the dual problem

$$-\Delta z = f \quad \text{in } \Omega, \quad z = 0 \text{ on } \partial\Omega, \tag{3}$$

with Galerkin orthogonality yields the representation formula

$$\begin{aligned} G(u-u_h) &= \langle -\Delta z\ ,\ u-u_h\rangle \\ &= \langle \nabla z\ ,\ \nabla(u-u_h)\rangle \\ &= \langle \nabla(u-u_h)\ ,\ \nabla(z-z_h)\rangle\,, \end{aligned} \tag{4}$$

where z_h is the linear finite element approximation of z [9]. We develop an alternative approach to obtain bound to the goal-oriented error. The exact gradients in (4) are replaced by the recovered gradients obtained by the PPR technique. So, the estimated error η_h is calculated by the scalar product

$$\eta_h = \langle R_h u_h - \nabla u_h \ , \ R_h z_h - \nabla z_h \rangle \,, \tag{5}$$

where $R_h u_h$ and $R_h z_h$ are the PPR recovered gradients of u_h and z_h, respectively.

3. PPR technique

Gradient recovery techniques are classified into three types: averaging techniques [31], projection techniques [32], and filtering techniques [13]. Any of the aforementioned techniques has a drawback: complexity, long computational time, meshes must have special structure, or finite elements must be low in order. In an attempt to overcome these problems, Zienkiewicz and Zhu [29] developed the well-known superconvergent patch recovery (SPR) which enjoys several advantages.

In [21] and [26], the authors aimed to design a new gradient recovery technique that systematically works in finite element methods of all orders in 2D and 3D and at the same time inherits the good properties of the SPR. To achieve this goal, the authors introduced the PPR and they analyzed and showed that the PPR is as good as or better than the SPR.

To recover the gradient using the PPR technique at a mesh node p, a patch κ_p of elements is selected. Then, a polynomial that best fits the finite element solution, in least-squares sense, at the mesh nodes in κ_p is constructed. The recovered gradient is defined to be the gradient of the fitting polynomial. Nodes on $\partial\Omega$ are handled in the same way, although they need extra care in constructing their patches [26].

It was found that the convergence rate at some exceptional points of the domain exceeds the global known optimal rates. The term "superconvergence" was suggested as a name for this phenomenon. Currently, this term is used in a broader sense where, at least, three types of superconvergence are identified:

1. Pointwise superconvergence in which the known global convergence rates are exceeded at some a priori known points.

2. Interpolantwise superconvergence. In this type u_h (∇u_h) is a higher order perturbation of a special projection of u (∇u) onto the finite element space.

3. Superconvergence by postprocessing. This type is observed in approximations of u or ∇u obtained with various means of postprocessing where the resulting approximations may have accelerated convergence rates in $\Omega' \subseteq \Omega$.

The PPR-recovered gradient has a superconvergence property. As it is known, if the recovered gradient is superconvergent to the exact gradient, then the a posteriori error estimator based on this recovered gradient is exact in asymptotic sense.

4. Local refinement algorithm

After having computed the local error estimate, we now face the problem of marking the elements that have to be refined. Different approaches for marking strategies can be found in [1]. Let $\tau_{h,0}$ be the initial mesh. Now we start the procedure to compute a sequence of meshes and approximate solutions. Compute the local error estimate $\eta_{h,T}$ for every element T of the mesh $\tau_{h,k}$ for some integer $k \geq 0$ and set an appropriate tolerance for the error estimate and choose $\mu, \nu \in (0, 1)$.

The approaches in [1] provide a measure for the total error by

$$\eta_h^2 = \sum_{T \in \tau_h} \eta_{h,T}^2 \ . \tag{6}$$

We propose a new algorithm in the following procedure, compatible with estimate (5) we proposed for the goal-oriented problems.

```
while (|η_h| > tolerance) do
    sum = 0 ;
    t = 1 ;
    while (|sum| < μ . |η_h|) do
        t = t − ν ;
        if (t ≠ 0)
          for all T ∈ τ_{h,k}
              if (T is not marked)
                 if (|η_{h,T}| > t . |η_{h,T}|_max) mark T ;
                   sum = sum + η_{h,T} ;
```

By choosing ν we can control how fine the procedure should work. One may choose μ depending on the complexity of f. Note that this algorithm is not expensive in its computational cost, because all local errors have already been computed. Having marked the elements, we refine the mesh appropriately and then solve again.

5. Numerical examples

We apply the proposed approach to obtain an estimate for the goal-oriented error to one-dimensional problems. Numerical examples are presented to show the resulting estimator provide tight bound with the effectivity index λ_h defined by

$$\lambda_h = |\eta_h| / e_h \ , \tag{7}$$

tending to unity.

We suppose that the main goal is to obtain an 'accurate' value of the solution u or du/dx at a given point x_0 in $[0, 1]$. We appeal here to the use of the mollification introduced in [4], which allows us to introduce the following quantity of interest

$$L_\epsilon(u; x_0) = \int_0^1 u(x).G_\epsilon(x - x_0)\ dx, \tag{8}$$

or

$$L_\epsilon(u; x_0) = \int_0^1 \frac{du}{dx}(x).G_\epsilon(x - x_0)\ dx. \tag{9}$$

The mollifying process can be viewed as an averaging of the quantity u or du/dx over a small neighborhood of the point x_0. This approach is also suited to estimate the pointwise error in the first derivatives of the solution, since pointwise derivatives are generally not defined at the element interfaces for the finite element solution. It is customary to choose the mollifiers G_ϵ of the form

$$G_\epsilon(x - x_0) = C\ .\ exp\left(\frac{\epsilon^2}{(x - x_0)^2 - \epsilon^2}\right), \tag{10}$$

if $|x - x_0| < \epsilon$ and $G_\epsilon(x - x_0) = 0$ elsewhere. The constant C, depends on ϵ and x_0, is selected to satisfy

$$\int_{x_0-\epsilon}^{x_0+\epsilon} G_\epsilon(x - x_0) dx = 1, \tag{11}$$

a numerical integration of the last integral provides that $C \approx 2.2523\ \epsilon^{-1}$ [4]. In the following examples, we fix the following parameters

$$\nu = 0.1,\ \ \mu = 0.25,\ \ x_0 = 0.5 \text{ and } tolerance\ 10^{-6}$$

and explore the influence of the parameter ϵ with respect to the mesh size h on the effectivity index λ_h defined by (7).

Example 1.

Consider the problem

$$-u'' = f \quad \text{in } \Omega, u = 0 \text{ on } \partial\Omega, \tag{12}$$

and the dual problem

$$-z'' = G \quad \text{in } \Omega, z = 0 \text{ on } \partial\Omega, \tag{13}$$

where Ω is $[0, 1]$.
We choose f so that the exact solution is

$$u(x) = x(x-1). \tag{14}$$

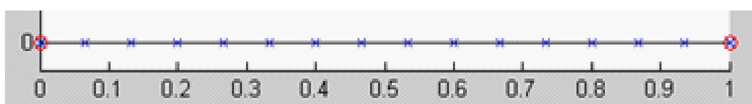

Figure 1. The initial mesh (16 nodes)

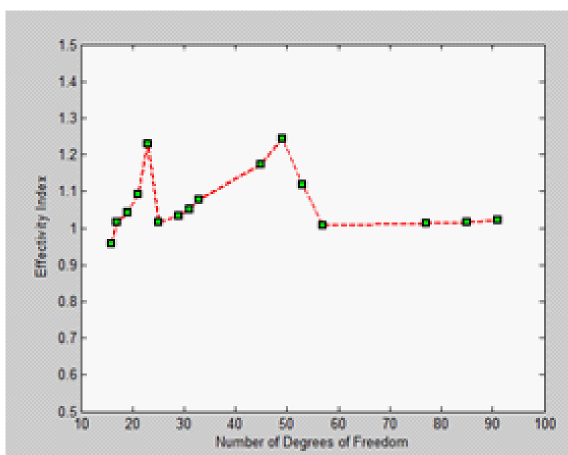

Figure 2. Effectivity index for $\epsilon = 0.1$

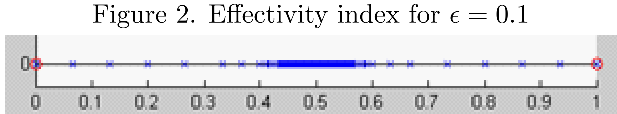

Figure 3. Final mesh for $\epsilon = 0.1$

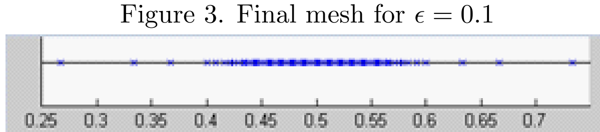

Figure 4. Zoom in for final mesh for $\epsilon = 0.1$

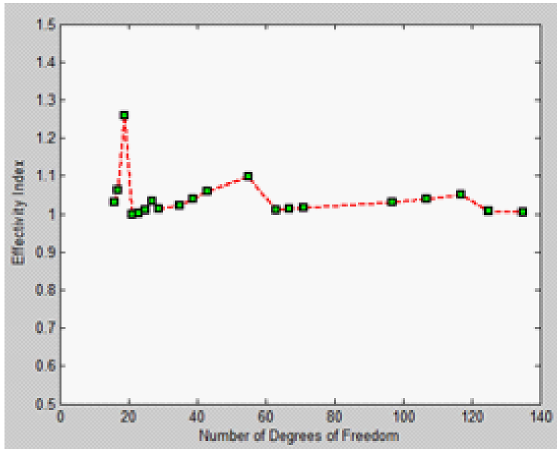

Figure 5. Effectivity index for $\epsilon = 0.15$

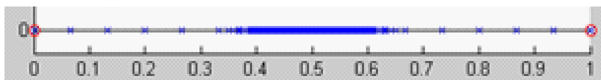

Figure 6. Final mesh for $\epsilon = 0.15$

Figure 7. Zoom in for final mesh for $\epsilon = 0.15$

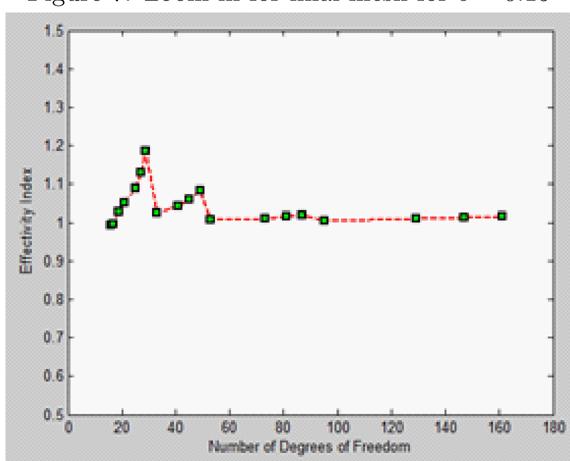

Figure 8. Effectivity index for $\epsilon = 0.2$

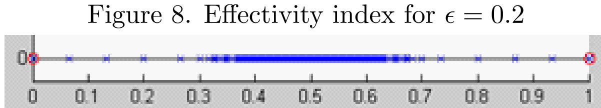

Figure 9. Final mesh for $\epsilon = 0.2$

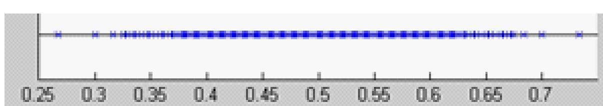

Figure 10. Zoom in for final mesh for $\epsilon = 0.2$

The graphs show that the effectivity index tends to unity in a fast rate and for different values of ϵ. We note that the solution to this problem has an extrema at $x_0 = 0.5$.

Example 2.

For the equations in (12) and (13), we choose another f so that

$$u(x) = x^4\ (x-1), \tag{15}$$

and start with the same initial mesh as in Example 1 (Figure 1).

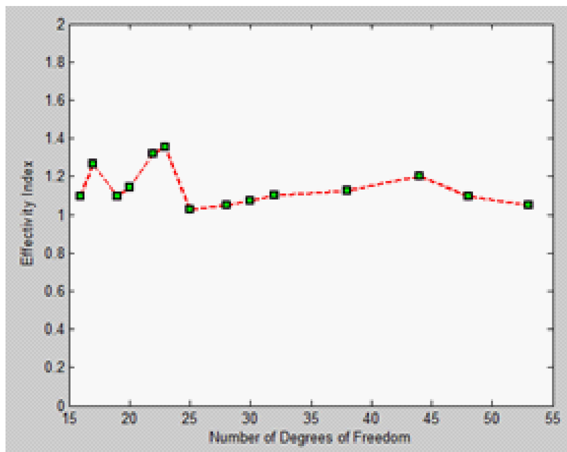

Figure 11. Effectivity index for $\epsilon = 0.1$

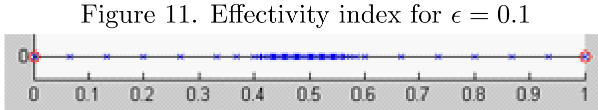

Figure 12. Final mesh for $\epsilon = 0.1$

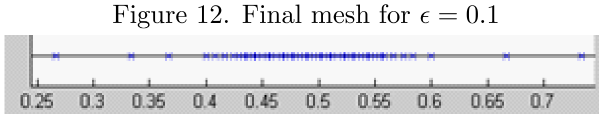

Figure 13. Zoom in for final mesh for $\epsilon = 0.1$

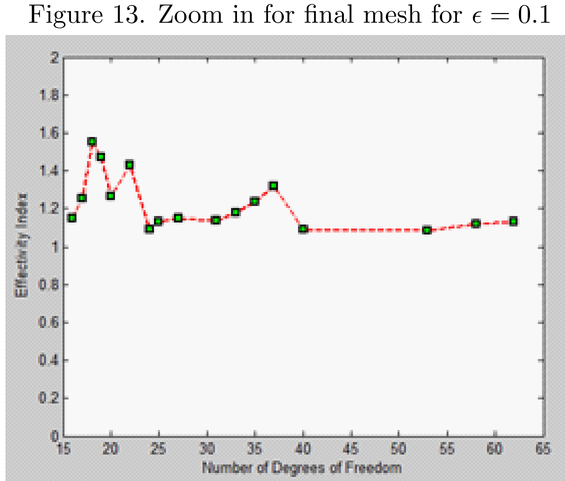

Figure 14. Effectivity index for $\epsilon = 0.15$

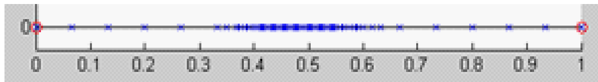

Figure 15. Final mesh for $\epsilon = 0.15$

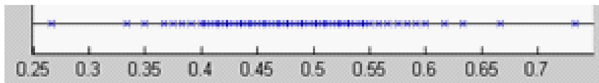

Figure 16. Zoom in for final mesh for $\epsilon = 0.15$

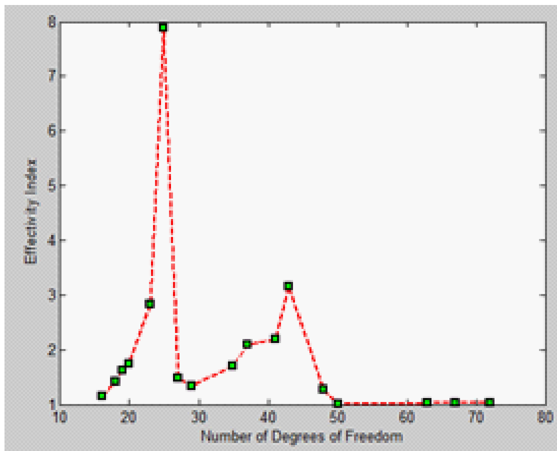

Figure 17. Effectivity index for $\epsilon = 0.2$

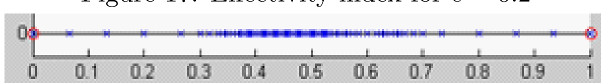

Figure 18. Final mesh for $\epsilon = 0.2$

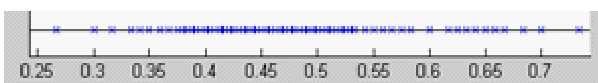

Figure 19. Zoom in for final mesh for $\epsilon = 0.2$

The solution of this example has an extrema at $x = 0.8$. The figures show that when ϵ is large enough such that the neighborhood of this extrema intersects with the compact support of the function G, the effectivity index exhibits high oscillation before it reaches unity.

Example 3.

We extend the methodology to another problem

$$-\frac{d}{dx}[(x+1)\ u'] + u = f \text{ in } \Omega,\ \ u = 0 \text{ on } \partial\Omega, \tag{16}$$

where Ω is $[0, 1]$. Set f so that the solution of this problem given by (14). Also, we start with the initial mesh in Figure 1.

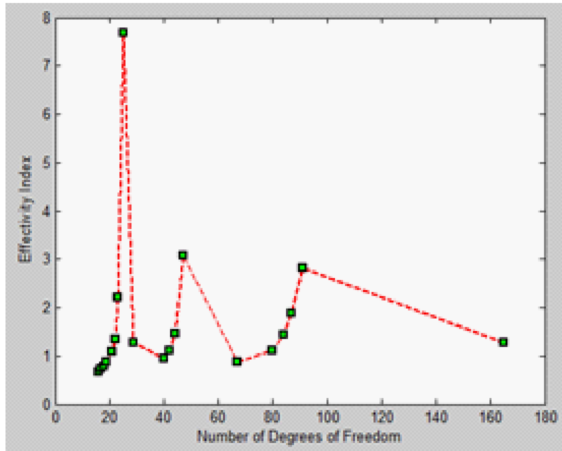

Figure 20. Effectivity index for $\epsilon = 0.1$

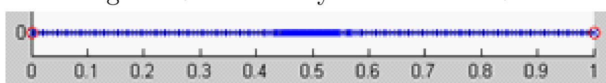

Figure 21. Final mesh for $\epsilon = 0.1$

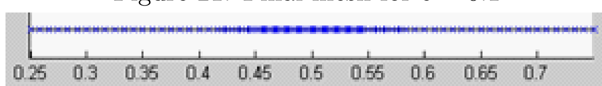

Figure 22. Zoom in for final mesh for $\epsilon = 0.1$

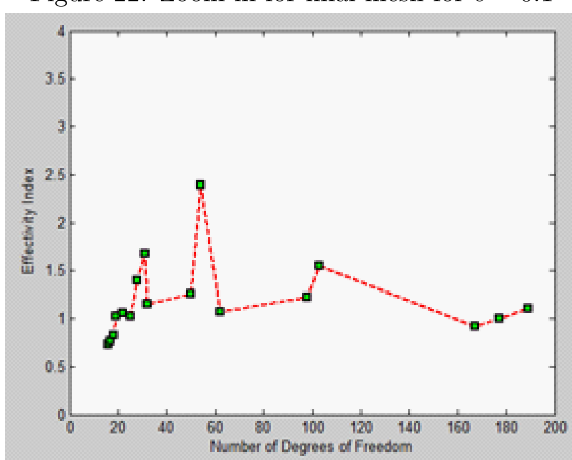

Figure 23. Effectivity index for $\epsilon = 0.15$

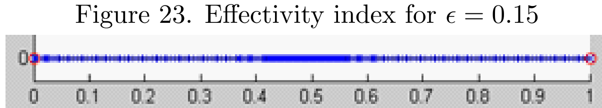

Figure 24. Final mesh for $\epsilon = 0.15$

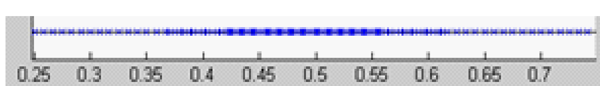

Figure 25. Zoom in for final mesh for $\epsilon = 0.15$

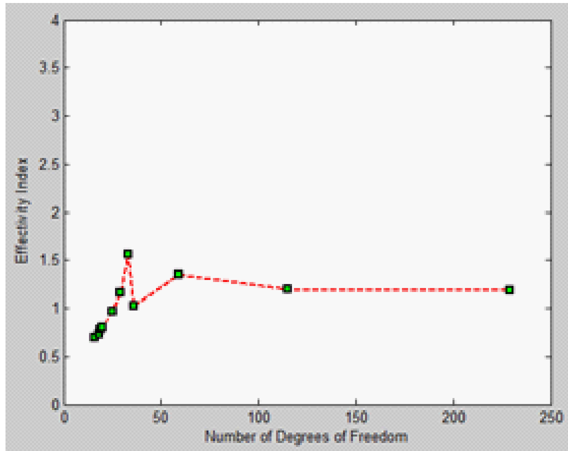

Figure 26. Effectivity index for $\epsilon = 0.2$

Figure 27. Final mesh for $\epsilon = 0.2$

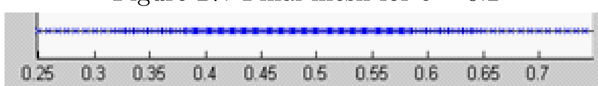

Figure 28. Zoom in for final mesh for $\epsilon = 0.2$

The graphs of the mesh refinement of this problem show that some refinement is made outside the domain of the function G near the left boundary of the problem domain. Also, though the effectivity index tends to unity, the oscillations in the graphs indicate that the proposed algorithm requires some modifications to tackle problems with variable coefficients in a faster way.

6. Conclusion

In this work, we proposed a new technique for evaluating a posteriori error estimate for one-dimensional goal-oriented problems. In the error estimate, the gradients for the exact solution to the original and dual problems are replaced by the corresponding recovered gradients obtained by the PPR technique.

The results obtained in the numerical examples illustrate that, due to the superconvergence of the recovered gradients, the error estimate provides a tight bound with effectivity index tending to unity. In comparison with the other techniques, the results are accomplished with smaller number of degrees of freedom. We note that there are several parameters that contribute to the results of the proposed technique. This includes the initial mesh and the parameters of the local refinement algorithm such as ν, μ and the error tolerance.

In view of these promising results, the authors plan to investigate the performance of the method on two-dimensional problems, to analyze the behavior for various linear functionals of general interest, and to extend the methodology to other classes of problems.

7. Acknowledgement

The authors wish to thank Prof. Zhimin Zhang for sharing his expertise about the main idea of this work.

References

[1] W. Dörfler, A convergent adaptive algorithm for Poisson's equation, SIAM J. Numer. Anal. 33 (3) (1996) 1106-1124.
[2] R.E. Bank, A. Weiser, Some a posteriori error estimators for elliptic partial differential equations, Math. Comp. 44 (1985) 283-301.
[3] O.C. Zienkiewicz, J.Z. Zhu, A simple error estimator and adaptive procedure for practical engineering analysis, Internat. J. Numer. Methods Eng. 24 (1987) 337-357.
[4] S. Prudhomme, J.T. Oden, On goal-oriented error estimation for elliptic problems: application to the control of pointwise errors, Comput. Methods Appl. Mech. Engrg. 176 (1999) 313-331.
[5] W. Bangerth, R. Rannacher, Adaptive Finite Element Methods for Differential Equations. Lectures in Mathematics ETH Zürich, Birkhäuser Verlag: Basel, (2003).
[6] R. Becker, R. Rannacher, An optimal control approach to a posteriori error estimation in finite element methods. Acta Numerica 10 (2001) 1–102.
[7] M.B. Giles, N.A. Pierce, Adjoint error correction for integral outputs. In Error Estimation and Adaptive Discretization Methods in Computational Fluid Dynamics, Lecture Notes in Computational Science and Engineering, Springer: Berlin 25 (2003) 47–95.
[8] M.B. Giles, E. Süli, Adjoint methods for PDEs: a posteriori error analysis and postprocessing by duality. Acta Numerica 11 (2002) 145–236.
[9] R. Nochetto, A. Veeser, M. Verani, A safeguard dual weighted residual method, IMA Journal of Numerical Analysis 29 (1) (2009) 126-140.
[10] F.T. Suttmeier, Reliable approximation of weight factors entering residual-based error bounds for FE-discretisations. Computing 73(3) (2004) 199–205.
[11] M. Ainsworth, R. Rankin, Guaranteed computable bounds on quantities of interest in finite element computations, Internat. J. Numer. Methods Eng. 89 (2012) 1605-1634.
[12] Y. Huang, W. Yang, N. Yim, Superconvergence analysis for the explicit polynomial recovery method, Journal of Computational and Applied Mathematics 265 (2014) 187–198.
[13] J. Bramble, A. Schatz, Higher order local accuracy by averaging in the finite element method, Math. Comp. 31 (137) (1977) 94–111.
[14] Y. Huang, K. Jiang, N. Yi, Some weighted averaging methods for gradient recovery, Adv. Appl. Math. Mech. 4 (2012) 131–155.
[15] R. Bank, J. Xu, Asymptotically exact a posteriori error estimators, part I: grids with superconvergence, SIAM J. Numer. Anal. (2004) 2294–2312.
[16] B. Heimsund, X. Tai, J. Wang, Superconvergence for the gradient of finite element approximations by L2 projections, SIAM J. Numer. Anal. (2003) 1263–1280.
[17] H. Wei, L. Chen, Y. Huang, Superconvergence and gradient recovery of linear finite elements for the Laplace–Beltrami operator on general surfaces, SIAM J. Numer. Anal. 45 (2007) 1064–1080.
[18] E. Hinton, J. Campbell, Local and global smoothing of discontinuous finite element functions using a least squares method, Internat. J. Numer. Methods Engrg. 8 (3) (1974) 461–480.
[19] M. Tabbara, T. Blacker, T. Belytschko, Finite element derivative recovery by moving least square interpolants, Comput. Methods Appl. Mech. Eng. 117 (1–2) (1994) 211–223.
[20] Y. Huang, N. Yi, The superconvergent cluster recovery method, J. Sci. Comput. 44 (3) (2010) 301–322.
[21] Z. Zhang, A. Naga, A new finite element gradient recovery method: superconvergence property, SIAM J. Sci. Comput. 26 (4) (2005) 1192–1213.
[22] J. Zhu, O. Zienkiewicz, Superconvergence recovery technique and a posteriori error estimators, Internat. J. Numer. Methods Engrg. 30 (7) (1990) 1321–1339.
[23] M. Ainsworth, J. Oden, A Posteriori Error Estimation in Finite Element Analysis, Wiley Interscience 37 (2000).
[24] I. Babuska, T. Strouboulis, The Finite Element Method and its Reliability, Oxford University Press, Oxford (2001).

[25] C. Carstensen, S. Bartels, Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids, part I: low order conforming, nonconforming, and mixed FEM, Math. Comp. 71 (239) (2002) 945–970.

[26] A. Naga, Z. Zhang, A posteriori error estimates based on the polynomial preserving recovery, SIAM J. Numer. Anal. (2005) 1780–1800.

[27] S. Repin, A Posteriori Estimates for Partial Differential Equations, Walter de Gruyter, Berlin (2008).

[28] J. Xu, Z. Zhang, Analysis of recovery type a posteriori error estimators for mildly structured grids, Math. Comp. 73 (247) (2004) 1139–1152.

[29] O. Zienkiewicz, J. Zhu, The superconvergent patch recovery and a posteriori error estimates, part I: the recovery technique, Internat. J. Numer. Methods Engrg. 33 (7) (1992) 1331–1364.

[30] O. Zienkiewicz, J. Zhu, The superconvergent patch recovery and a posteriori error estimates, part II: error estimates and adaptivity, Internat. J. Numer. Methods Engrg. 33 (7) (1992) 1365–1382.

[31] N. Levine, Superconvergent recovery of the gradient from piecewise linear finite element approximations, IMA J. Numer. Anal. 5(1985) 407–427.

[32] J. Oden, H. Brauchli, On the calculation of consistent stress distributions in finite element applications, Internat. J. Numer. Methods Engrg. 3 (1971) 317–325.

[33] S. Brenner, L. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag, New York (2008).

[34] D. Braess, Finite elements: Theory, fast solvers and applications in solid mechanics, Cambridge University Press, Cambridge (2001).

M. El-Agamy, Mathematics & Engineering Physics Department, Faculty of Engineering, Mansoura University, Mansoura, 35516, Egypt

E-mail address: mustafa_agamy@mans.edu.eg

A. Elsaid, Mathematics & Engineering Physics Department, Faculty of Engineering, Mansoura University, Mansoura, 35516, Egypt

E-mail address: a.elsaid@ymail.com

H. M. Nour, Mathematics & Engineering Physics Department, Faculty of Engineering, Mansoura University, Mansoura, 35516, Egypt

E-mail address: hanour@mans.edu.eg