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AN APPROXIMATE SOLUTION OF SYSTEMS OF HIGH-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS BY MEANS OF A RATIONAL CHEBYSHEV COLLOCATION METHOD

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Abstract. The purpose of this paper is to investigate the use of rational Chebyshev collocation method for solving systems of high-order linear ordinary differential equations with variable coefficients. Using the rational Chebyshev collocation points, this method transforms the system of high-order linear ordinary differential equations and the given conditions to matrix equations with unknown rational Chebyshev coefficients. By means of the obtained matrix equations, a new system of equations which corresponds to the system of linear algebraic equations is gained. Numerical examples are given to illustrative the validity and applicability of the method. The proposed method is numerically compared with others existing methods where it maintains better accuracy.

1. Introduction

Recently many authors studied the application of Chebyshev polynomials and rational Chebyshev functions for solving different problems of differential equations and some other physical problems with variable coefficients. Akyüz and Sezer [2] developed the use of Chebyshev collocation method for the solution of systems of high-order linear differential equations with variable coefficients. Gökmen and Sezer [5] developed the use of Taylor collocation method for the solution of systems of high-order linear differential difference equations with variable coefficients.Yüksel et al. [11] considered a Chebyshev polynomial approach for higher order linear Fredholm Volterra integro-differential equations. Saeid Abbasbandy et al. [1] investigated the use of rational Chebyshev collocation method to get an approximate solution of Magnetohydrodynamic (MHD) flow of an incompressible viscous fluid over a stretching sheet. Yalçinbaş et al. [10] and Ramadan et al. [8] investigated the use of rational Chebyshev functions to obtain an approximate solution of higher order linear differential equations. In their approach they developed the Chebyshev tau and the Taylor collocation methods in mth order linear nonhomogenous differential equation with mixed conditions where the solution is expressed in terms of

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the rational Chebyshev functions. In [7] Parand and Razzaghi introduced rational Chebyshev functions as a new computational method for solving Volterra model for population growth of a species within a closed system where the Volterra population model is first converted to an equivalent nonlinear ODE, the solution of which is then approximated by a rational Chebyshev functions with unknown coefficients. The operational matrices of derivative and product of rational Chebyshev functions are given.

The organization of this paper is as follows. In Section 2, Preliminaries introduced while in Section 3 Properties of the rational Chebyshev (RC) functions are presented. In Section 4, we formulated the fundamental matrix relation and in Section 5 matrix relations based on collocation Points Section of the problem are derived. In Section 6, method of solution is presented. Section 7 contains numerical illustrations and results are compared with the exact solution and other exist methods to demonstrate the accuracy of the present method.

2. Preliminaries

A system of k linear differential equations with variable coefficients is a set of k equations of the mth order in the form [2]

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{ij}^{n}(t)\, y_j^{(n)}(t) = f_i(t)\,, i = 1, 2, \ldots, k \tag{1}$$

This system can be written in compact notation as

$$\sum_{i=0}^{m} \boldsymbol{P}_i(\mathrm{t})\, \boldsymbol{y}^{(i)}(\mathrm{t}) = \boldsymbol{f}(\mathrm{t}) \tag{2}$$

where

$$\boldsymbol{P}_i(t) = \begin{bmatrix} p_{11}^i & p_{12}^i & \cdots & p_{1k}^i \\ p_{21}^i & p_{22}^i & \cdots & p_{2k}^i \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_{k1}^i & p_{k2}^i & \cdots & p_{kk}^i \end{bmatrix}, \qquad \boldsymbol{y}^{(i)}(t) = \begin{bmatrix} y_1^{(i)}(t) \\ y_2^{(i)}(t) \\ \cdot \\ \cdot \\ \cdot \\ y_k^{(i)}(t) \end{bmatrix}, \qquad \boldsymbol{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \cdot \\ \cdot \\ \cdot \\ f_k(t) \end{bmatrix}$$

We consider the above system under the mixed condition defined as

$$\sum_{i=0}^{m-1} \boldsymbol{a}_i y^{(i)}(a) + \boldsymbol{b}_i y^{(i)}(b) + \boldsymbol{c}_i y^{(i)}(c) = \lambda_i, a \le c \le b \tag{3}$$

where $\boldsymbol{a}_i$, $\boldsymbol{b}_i$, $\boldsymbol{c}_i$ and λ_i are real valued matrices and we assume that the solution of this system can be expressed in terms of a truncated rational Chebyshev series as follows:

$$y_i(t) = \sum_{n=0}^{N} a_{in} R_n(t) \quad , \quad i = 1, 2, ..., k\ ,\ 0 \le \mathrm{t} < \infty \tag{4}$$

where N is chosen any positive integer such that $N \ge m$ and a_{in} are unknown rational Chebyshev coefficients.

3. Properties Of The Rational Chebyshev (Rc) Functions [10]

In cases when errors near the ends of an interval $[a, b]$ are particular importance, a weighting function which is the for$1/\sqrt{(t-a)(b-t)}$is often useful. It is supposed again that a linear change in variables has transformed the given interval into the interval [-1, 1], so that the weighting function becomes $w(t) = 1/\sqrt{1-t^2}$. In other words a great variety of other types of least – square polynomial approximation can be formulated in terms of other weighting functions. In particular, for the weighting function $w(t) = (1-t)^{\alpha}(1+t)^{\beta}$, $(\alpha > -1, \beta > -1)$ over [-1, 1], which reduces to Legendre case when $\alpha = \beta = 0$ and to the Chebyshev case when $\alpha = \beta = -1/2$.The well-known Chebyshev polynomials are orthogonal in the interval [-1, 1] with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$and can be determined with the aid of the recurrence formulae

$$T_0(t) = 1, \qquad T_1(t) = t, \qquad T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t) \qquad \mathrm{n} \geq 1$$

The RC functions are defined by

$$R_n(t) = T_n\left(\frac{t-1}{t+1}\right)$$

or clearly

$$R_0(t) = 1, \quad R_1(t) = \frac{t-1}{t+1}, \quad R_{n+1} = 2\left(\frac{t-1}{t+1}\right)R_n - R_{n-1}, \qquad \mathrm{n} \geq 1 \tag{5}$$

These functions are orthogonal with respect to the weight function$w(t) = 1/((t+1)\sqrt{t})$ in the interval [0, 8).

If we use the expression $v{=}\frac{t-1}{t+1}$ in the RC function (4), we have

$$R(t) = V(t)C^T \tag{6}$$

where *R (t)* and *V (t)* are matrices of the form:

$$R(t) = \left[\begin{array}{cccc} R_0(t) & R_1(t) & ... & R_N(t) \end{array}\right]$$

$$V(t) = \left[\begin{array}{cccc} v^0(t) & v^1(t) & ... & v^N(t) \end{array}\right]$$

and $C^T$is a matrix with its inverse given by

$$C^{-1} = \left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & \cdots & 0 & 0 \\
0 & 3/4 & 0 & 1/4 & & 0 & 0 \\
. & . & . & . & & . & . \\
. & . & . & . & & . & . \\
. & . & . & . & & . & . \\
\frac{1}{2^N}\left(\begin{array}{c} N \\ \frac{N-0}{2} \end{array}\right) & 0 & \frac{1}{2^{N-1}}\left(\begin{array}{c} N \\ \frac{N-2}{2} \end{array}\right) & 0 & \cdots & \frac{1}{2^{N-1}}\left(\begin{array}{c} N \\ 0 \end{array}\right) & 0 \\
0 & \frac{1}{2^{N-1}}\left(\begin{array}{c} N \\ \frac{N-1}{2} \end{array}\right) & 0 & \frac{1}{2^{N-1}}\left(\begin{array}{c} N \\ \frac{N-3}{2} \end{array}\right) & \cdots & 0 & \frac{1}{2^{N-1}}\left(\begin{array}{c} N \\ 0 \end{array}\right)
\end{array}\right]$$

In this case, we are going to use the last row for odd values of N, and otherwise we use the second row from below of matrix C^{-1}.

For example, in the cases $N = 3$ and $N = 4$, the matrix C becomes

$$C = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -3 & 0 & 4 \end{array}\right], \quad C = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 \\ 1 & 0 & -8 & 0 & 8 \end{array}\right]$$

Consequently, thej^{th} derivative of the matrix $R(t)$, can be obtained as

$$R^{(j)}(t) = V^{(j)}(t)C^T \tag{7}$$

where

$$R^{(j)}(t) = \begin{bmatrix} R_0^{(j)}(t) & R_1^{(j)}(t) & ... & R_N^{(j)}(t) \end{bmatrix}$$

$$\mathrm{V}^{(j)}(t) = \begin{bmatrix} (v^0(t))^{(j)} & (v^1(t))^{(j)} & ... & (v^N(t))^{(j)} \end{bmatrix}$$

and

$$v^0(x) = 1\ , \quad v^1(x) = \frac{x-1}{x+1}\ , \quad v^2(x) = \left(\frac{x-1}{x+1}\right)^2, \quad \ldots\ , \quad v^N(x) = \left(\frac{x-1}{x+1}\right)^{\mathrm{N}}$$

4. Fundamental Matrix Relation

Let us first assume that the solution $y_i(t)$of Eq. (1) can be expressed in the form (4), which is a truncated Chebyshev series in terms of RC functions. Then $y_i(t)$ and its derivative $y_i^{(j)}(t)$ can be put in the matrix forms

$$y_i(t) = R(t)A_i \tag{8}$$

and

$$y_i^{(j)}(t) = R^{(j)}(t)A_i i = 1, 2, \ldots, k, j = 0, 1, \ldots, m \leq N \tag{9}$$

where

$$A_i = \begin{bmatrix} a_{i0} & a_{i1} & \cdots & a_{iN} \end{bmatrix}^T,$$

Substituting relation (7) into Eq. (9), we get

$$y_i^{(j)}(t) = V^{(j)}(t)C^T A_i \qquad\qquad j = 0, 1, \ldots, m \leq N \tag{10}$$

Hence, the matrix $\boldsymbol{y}^{(i)}($ t)defined as a column matrix that is formed of ith derivatives of unknown functions, can be expressed by

$$\boldsymbol{y}^{(i)}(\mathrm{t}) = \boldsymbol{V}^{(i)}(\mathrm{t})\boldsymbol{C}^T\boldsymbol{A} \tag{11}$$

where

$$\boldsymbol{V}^{(i)}(t) = \begin{bmatrix} V^{(i)}(t) & 0 & \cdots & 0 \\ 0 & V^{(i)}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V^{(i)}(t) \end{bmatrix}_{k\times k}, \boldsymbol{C}^T = \begin{bmatrix} C^T & 0 & \cdots & 0 \\ 0 & C^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^T \end{bmatrix}_{k\times k}, \boldsymbol{A}= \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_k \end{bmatrix}_{k\times 1}$$

5. Matrix Relation Based On Collocation Points

Now, let us define the collocation points as

$$t_s = \frac{c}{N}s, \qquad\qquad \mathrm{s} = 0, 1, \ldots, N \tag{12}$$

so that $0 \leq t_s \leq c < \infty; c \in IR^+$.
Then we substitute the collocation points (12) into Eq. (2) to obtain the system

$$\sum_{i=0}^{m} \tilde{\boldsymbol{P}}_i \boldsymbol{Y}^{(i)} = \mathbf{F} \tag{13}$$

where

$$\tilde{\boldsymbol{P}}_i = \begin{bmatrix} \boldsymbol{P}_i(\mathrm{t}_0) & 0 & \cdots & 0 \\ 0 & \boldsymbol{P}_i(\mathrm{t}_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{P}_i(\mathrm{t}_N) \end{bmatrix}, \boldsymbol{Y}^{(i)} = \begin{bmatrix} \boldsymbol{y}^{(i)}(\mathrm{t}_0) \\ \boldsymbol{y}^{(i)}(\mathrm{t}_1) \\ \vdots \\ \boldsymbol{y}^{(i)}(\mathrm{t}_N) \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} \boldsymbol{f}(\mathrm{t}_0) \\ \boldsymbol{f}(\mathrm{t}_1) \\ \vdots \\ \boldsymbol{f}(\mathrm{t}_N) \end{bmatrix},$$

By putting the collocation points x_s,$s = 0, 1, \ldots, N$in relation (11) we have the matrix system

$$\boldsymbol{y}^{(i)}(\mathrm{t}_s) = \boldsymbol{V}^{(i)}(\mathrm{t}_s)\boldsymbol{C}^T\mathbf{A} \tag{14}$$

This system can be written as

$$\boldsymbol{Y}^{(i)} = \tilde{\boldsymbol{V}}^{(i)}\boldsymbol{C}^T\boldsymbol{A}$$

where

$$\tilde{\boldsymbol{V}}^{(i)} = \begin{bmatrix} \boldsymbol{V}^{(i)}(\mathrm{t}_0) \\ \boldsymbol{V}^{(i)}(\mathrm{t}_1) \\ \vdots \\ \boldsymbol{V}^{(i)}(\mathrm{t}_N) \end{bmatrix}$$

with the aid of this equation, expression (13) becomes

$$\sum_{i=0}^{m} \tilde{\boldsymbol{P}}_i\ \tilde{\boldsymbol{V}}^{(i)}\boldsymbol{C}^T\boldsymbol{A} = \boldsymbol{F} \tag{15}$$

Similarly, we form the matrix representations of the mixed conditions.

Substituting the matrix $y^{(i)}(a)$ and $y^{(i)}(b)$ which depends on the rational Chebyshev coefficients matrix $\boldsymbol{A}$ into the Eq. (3) and simplifying the result we obtain

$$\sum_{i=0}^{m-1} \{\boldsymbol{a}_i V^{(i)}(a) + \boldsymbol{b}_i V^{(i)}(b) + \boldsymbol{c}_i V^{(i)}(c)\} C^T A = \lambda_i \tag{16}$$

6. Method Of Solution

The fundamental matrix equation (15) for Eq. (1) corresponds to a system of k $(N+1)$ algebraic equations for the k $(N+1)$ unknown coefficients$a_{i0}, a_{i1}, \ldots, a_{iN}; i = 1, 2, \ldots, k$.

Briefly we can write Eq. (15) as

$$\boldsymbol{W}\ \boldsymbol{A}\ = \boldsymbol{F}\ \ or[\boldsymbol{W};\ \boldsymbol{F}] \tag{17}$$

where

$$\mathbf{W} = [w_{pq}] = \sum_{i=0}^{m} \tilde{\boldsymbol{P}}_i\ \tilde{\boldsymbol{V}}^{(i)}\boldsymbol{C}^T \qquad , p, q = 1, 2, \ldots, k(N+1)$$

We can obtain the matrix form for the mixed conditions (3), by means of Eq. (16) briefly as

$$\mathbf{U}\boldsymbol{A} = \lambda_i \tag{18}$$

where

$$\boldsymbol{U} = \sum_{i=0}^{m-1} \{\boldsymbol{a}_i V^{(i)}(a) + \boldsymbol{b}_i V^{(i)}(b) + \boldsymbol{c}_i V^{(i)}(c)\} C^T$$

Consequently, replacing the rows of the matrix $\boldsymbol{U}$ and λ_i by the last rows of the matrix $\boldsymbol{W}$ and $\boldsymbol{F}$ respectively, we have

$$\tilde{\boldsymbol{W}}\boldsymbol{A} = \tilde{\boldsymbol{F}} \qquad \text{or} \qquad \left[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{F}}\right] \tag{19}$$

In case, the number of conditions is *mk,* the augmented matrix of the above system is following

$$\left[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{F}}\right] = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,k(N+1)} & ; & f_1(t_0) \\ w_{2,1} & w_{2.2} & \cdots & w_{2,k(N+1)} & ; & f_2(t_0) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ w_{k,1} & w_{k,2} & \cdots & w_{k,k(N+1)} & ; & f_k(t_0) \\ w_{k+1,1} & w_{k+1,2} & \cdots & w_{k+1,k(N+1)} & ; & f_1(t_1) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ w_{k(N-m+1),1} & w_{k(N-m+1),2} & \cdots & w_{k(N-m+1),k(N+1)} & ; & f_k(t_{N-m}) \\ u_{1,1} & u_{1,2} & \cdots & u_{1,k(N+1)} & ; & \lambda_1 \\ u_{2,1} & u_{2,2} & \cdots & u_{2,k(N+1)} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ u_{mk,1} & u_{mk,2} & \cdots & u_{mk,k(N+1)} & ; & \lambda_{mk} \end{bmatrix}$$

Hence rational Chebyshev coefficients can be simply computed and the solution of system (2) under the mixed conditions (3) can then be obtained.

7. Numerical Examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness of the proposed method. All examples are performed on the computer using a program written in MATHEMATICA 7.0. The absolute errors in the Tables are the values of $|y(t) - y_N(t)|$ at selected points.

Example 1 Consider the linear system

$$\begin{aligned} &x'(t) + y'(t) = \tfrac{17-7t}{2(1+t)^3} \\ &x'(t) - y'(t) + \tfrac{3}{1+t}x(t) - \tfrac{2}{1+t}y(t) = \tfrac{3+11t}{2(1+t)^3} \end{aligned} \quad x \in [0,1]$$

with$x(0) = 0, \quad y(0) = 0$
Then, for N =2, the collocation points are $t_0 = 0, t_1 = 1/2, t_2 = 1$
The fundamental matrix equation (15) of problem is

$$\{\tilde{\boldsymbol{P}}_0\tilde{\boldsymbol{V}}^{(0)} + \tilde{\boldsymbol{P}}_1\tilde{\boldsymbol{V}}^{(1)}\}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{F}$$

where$\tilde{\boldsymbol{P}}_0, \tilde{\boldsymbol{P}}_1, \tilde{\boldsymbol{V}}^{(0)}, \tilde{\boldsymbol{V}}^{(1)}, \boldsymbol{C}^T$are matrices of order (6×6) given, for this example,

$$\tilde{\boldsymbol{P}}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & -1 \end{bmatrix}, \tilde{\boldsymbol{P}}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\tilde{\boldsymbol{V}}^{(0)} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & -\frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{9} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \tilde{\boldsymbol{V}}^{(1)} = \begin{bmatrix} 0 & 2 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & \frac{8}{9} & -\frac{16}{27} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{9} & -\frac{16}{27} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

$$\boldsymbol{C}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, [\tilde{\boldsymbol{W}}; \tilde{\boldsymbol{F}}] = \begin{bmatrix} 0 & 2 & -8 & 0 & 2 & -8 & ; & \frac{17}{2} \\ 3 & -1 & -5 & -2 & 0 & 6 & ; & \frac{3}{2} \\ 0 & \frac{8}{9} & -\frac{32}{27} & 0 & \frac{8}{9} & -\frac{32}{27} & ; & 2 \\ 2 & \frac{2}{9} & -\frac{74}{27} & -\frac{4}{3} & -\frac{4}{9} & \frac{20}{9} & ; & \frac{34}{27} \\ 1 & -1 & 1 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & ; & 0 \end{bmatrix}.$$

Then we obtain the solution

$$\boldsymbol{A} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{3}{4} & -\frac{1}{4} \end{bmatrix}^T$$

Therefore, we find the solution

$$y_i(t) = \sum_{n=0}^{2} a_{in} R_n(t)$$

Then

$$x(t) = R_0 + \frac{1}{2} R_1 - \frac{1}{2} R_2 \quad and \qquad y(t) = R_0 + \frac{3}{4} R_1 - \frac{1}{4} R_2$$

or in the form

$$x(t) = \frac{5t + t^2}{(t+1)^2}, y(t) = \frac{7t + 3t^2}{2(t+1)^2}$$

which is exact solution of this problem.

Example 2 Consider the two solvable linear time varying differential algebraic equations in Campbell paper [3]

$$\begin{bmatrix} -t & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with the exact solutions$x = 1, y = t$. For this example we have,

$$k = 2, m = 1, f_1(t) = 1, \;\; f_2(t) = 0, \;\; p_{11}^0(t) = 0, \;\; p_{12}^0(t) = 0, p_{21}^0(t) = -t,$$
$$p_{22}^0(t) = 1, p_{11}^1(t) = -t, \;\; p_{12}^1(t) = 1, \;\; p_{21}^1(t) = 0, \;\; p_{22}^1(t) = 0$$

Then, for N =4, the collocation points are $t_0 = 0, t_1 = 1/4, t_2 = 1/2, t_3 = 3/4, t_4 = 1$ and the fundamental matrix equation of problem is

$$\{\tilde{\boldsymbol{P}}_0 \tilde{\boldsymbol{V}}^{(0)} \boldsymbol{C}^T + \tilde{\boldsymbol{P}}_1 \tilde{\boldsymbol{V}}^{(1)} \boldsymbol{C}^T\} \boldsymbol{A} = \boldsymbol{F}$$

Following the procedure in Section 7, we find the matrix in (19) for $N = 4$ as:

$$
[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{F}}]=\left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 2 & -8 & 18 & -32 & ; & 1\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & ; & 0\\
0 & -\frac{8}{25} & \frac{96}{125} & -\frac{264}{625} & -\frac{2688}{3125} & 0 & \frac{32}{25} & -\frac{384}{125} & \frac{1056}{625} & \frac{10752}{3125} & ; & 1\\
-\frac{1}{4} & \frac{2}{5} & -\frac{43}{100} & \frac{2}{125} & \frac{527}{2500} & 1 & -\frac{3}{5} & -\frac{7}{25} & \frac{117}{125} & -\frac{527}{625} & ; & 0\\
0 & -\frac{4}{9} & \frac{16}{27} & \frac{20}{27} & -\frac{448}{243} & 0 & \frac{8}{9} & -\frac{32}{27} & -\frac{40}{27} & \frac{896}{243} & ; & 1\\
-\frac{1}{2} & \frac{2}{3} & -\frac{11}{18} & -\frac{11}{18} & -\frac{17}{162} & 1 & -\frac{1}{3} & -\frac{7}{9} & \frac{23}{27} & \frac{17}{81} & ; & 0\\
0 & -\frac{24}{49} & \frac{96}{343} & \frac{3240}{2401} & -\frac{18048}{16807} & 0 & \frac{32}{49} & -\frac{128}{343} & -\frac{4320}{2401} & \frac{24064}{1601} & ; & 1\\
-\frac{3}{4} & \frac{6}{7} & -\frac{153}{196} & \frac{150}{343} & -\frac{6051}{9604} & 1 & -\frac{1}{7} & -\frac{47}{49} & \frac{143}{343} & \frac{2017}{2401} & ; & 0\\
0 & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 0 & ; & 1\\
-1 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & ; & 0
\end{array}\right]
$$

We find the rational Chebyshev coefficients matrix for N =4, as:

$$
\boldsymbol{A}=\left[\begin{array}{cccccccccc} \frac{11899}{16384} & -\frac{241}{512} & -\frac{1301}{4096} & -\frac{73}{512} & -\frac{743}{16384} & \frac{31711}{16384} & \frac{2757}{1024} & \frac{3971}{4096} & \frac{251}{1024} & \frac{533}{16384} \end{array}\right]
$$

We obtain the approximate solutions by rational Chebyshev functions for $t \in [0,1]$ of example2 for $N = 4$, 6 and 8 as shown in tables 1 and 2. In tables 3 and 4 given numerical results of approximate solution and absolute error functions by rational Chebyshev functions for $t \in [0,2]$ of example2 for $N = 8$.

Table 1 Approximate solution and exact values of *x (t)* for Example 2

t_i	**Exact solution**	Present Method		
	$x(t) = 1$	$N = 4$	$N = 6$	$N = 8$
0	1	0.976525	0.997427983539	0.999699592590
0.2	1	1.003405912422	1.000067552918	0.999997454604
0.4	1	0.999560729904	0.999996121724	1.000000248814
0.6	1	1.000672507934	0.999998259446	1.000000085493
0.8	1	1.000672772824	1.000005930571	0.999999900682
1	1	0.99853515625	0.999959812242	0.999998826533

Table 2 Approximate solution and exact values of $y(t)$ for Example 2

t_i	**Exact solution**	Present Method		
	$y(t) = t$	$N = 4$	$N = 6$	$N = 8$
0	0	0	0	0
0.2	0.2	0.200840326003	0.200009991261	0.199999346434
0.4	0.4	0.399742294877	0.399999403611	0.400000091682
0.6	0.6	0.599893569946	0.599998527103	0.600000053991
0.8	0.8	0.800506782502	0.800005053424	0.799999926184
1	1	0.99853515625	0.999959812243	0.999998826534

Table 3 Approximate solution, exact values and absolute error of x (t), $t \in [0, 2]$**of Example2 for $N = 8$**

t_i	Exact solution	$N = 8, x(t_i)$	$e_{x,8}(t_i)$
0.2	1	1.00355470	3.55470e-003
0.4	1	0.99981014	1.89854e-004
0.6	1	0.99999611	3.88115e-006
0.8	1	1.00000853	8.53007e-006
1	1	0.99999571	4.29153e-006
1.2	1	1.00000171	1.71298e-006
1.4	1	0.99999985	1.51436e-007
1.6	1	0.99999866	1.34171e-006
1.8	1	1.00000405	4.04565e-006
2	1	0.99998828	1.17214e-005

Table 4 Approximate solution, exact values and absolute error of y (t), $t \in [0, 2]$**of Example2for $N = 8$**

t_i	Exact solution	$N = 8, x(t_i)$	$e_{x,8}(t_i)$
0	0	0	0
0.2	0.2	0.20085708	8.57081e-004
0.4	0.4	0.39990250	9.74955e-005
0.6	0.6	0.60000112	1.12515e-006
0.8	0.8	0.80000629	6.28638e-006
1	1	0.99999571	4.29153e-006
1.2	1.2	1.20000216	2.16355e-006
1.4	1.4	1.39999965	3.46774e-007
1.6	1.6	1.59999801	1.99441e-006
1.8	1.8	1.80000712	7.11585e-006
2	2	1.99997656	2.34429e-005

Example 3 Let us consider the initial value problem [4]

$$x'(t) + y'(t) + y(t) = t - e^{-t}$$
$$x'(t) + 4y'(t) + x(t) = 1 + 2e^{-t}$$

with the initial conditions $x(0) = 1$, $y(0) = 0$, and the exact solutions

$$x(t) = e^{-t} + 3e^{-t/3} - 3, \quad y(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-t/3} - 1 + t$$

The numerical solutions obtained from the rational Chebyshev collocation method for $N = 5$ are compared with the results, using the Chebyshev collocation method [2] with $N = 5$ and Stehfest's method with $M = 8$, given by Davies and Crann [4] in Tables 5 and 6.

Table 5 Comparison between absolute error functions obtained by present methods and other existed methods for $x(t)$ of Example 3 for *N=5*

t_i	**Stehfestmethod**[4]	**Chebyshev collocation** [2]	**Presentmethod**
0.1	6.7614e-005	4.510522e-005	6.2951e-005
0.2	8.4949e-005	7.985043e-005	1.5714e-004
0.5	3.1897e-003	9.719089e-005	4.3862e-004
0.8	5.2028e-003	8.006002e-005	8.0076e-004
1	1.1937e-002	1.067677e-004	1.0598e-003

Table 6 Comparison between absolute error functions obtained by present method and other existed methods for $y(t)$ of Example 3 for N=5

t_i	**Stehfestmethod**[4]	**Chebyshev collocation** [2]	**Presentmethod**
0.1	8.4086e-006	2.247723e-005	4.5250e-004
0.2	1.9575e-005	3. 984701e-005	6.5580e-004
0.5	2.242e-004	4.890662e-005	6.7891e-004
0.8	4.464e-004	4.064222e-005	7.8896e-004
1	4.710e-004	5.390356e-005	1.2735e-003

Example 4

Consider the following linear differential system [9]

$$\begin{array}{ll} x'(t)+y'(t)+x(t)+y(t)=1 & \\ y'(t)-2x(t)-y(t)=0 & \qquad t\in[0,1] \end{array}$$

with the initial condition $y_1(0)=0, y_2=(0)=1$,and the exact solution $y_1(t)=e^{-t}-1, y_2(t)=2-e^{-t}$.

For the example,

$$\begin{array}{l} k=2, m=1, \\ f_1(t)=1,\ \ f_2(2)=0, \quad p_{11}^0(t)=1, \quad p_{12}^0(t)=1, \quad p_{21}^0(t)=-2, \\ p_{22}^0(t)=-1, \quad p_{11}^1(t)=1, \quad p_{12}^1(t)=1, \quad p_{21}^1(t)=0,\ \ p_{22}^1(t)=1. \end{array}$$

For (15) the fundamental matrix equation of the problem is

$$\{\tilde{\boldsymbol{P}}_0\,\tilde{\boldsymbol{V}}^{(0)}+\tilde{\boldsymbol{P}}_1\,\tilde{\boldsymbol{V}}^{(1)}\}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}=\boldsymbol{F}$$

We obtain the approximate solutions by RC functions of the problem for $N=6$, 8 in tables 7, 8. The numerical results obtained by the present method for $N=6$, are compared with the differential transform method [9], of this system. It is seen form the tables that the present method is better than the differential transform method in [9].

Table 7 Comparison between Exact solution and approximate solutions obtained by present method and other existed methods for $x(t)$of Example 4

t_i	**Exact solution**	**Transform method**[9]	**Present method**	
	$x(t) = e^{-t} - 1$	$N = 6, x(t_i)$	$N = 6, x(t_i)$	$N = 8, x(t_i)$
0	0	0	0	0
0.2	-0.1812692469220	-0.165693777778	-0.1812493115195	-0.18126925925619
0.4	-0.3296799539643	-0.278471111111	-0.3296650977943	-0.32967996462728
0.6	-0.4511883639059	-0.349692000000	-0.4511761317037	-0.45118837253673
0.8	-0.5506710358827	-0.384369777778	-0.5506606459551	-0.55067104269892
1	-0.6321205588285	-0.398611111111	-0.6321177527222	-0.63212049548008

Table 8 Comparison between exact solution and approximate solutions obtained by present and other existed methods for$y(t)$of Example 4

t_i	**Exact solution**	**Transform method**[9]	**Present method**	
	$y(t_i) = 2 - e^{-t}$	$N = 6, y(t_i)$	$N = 6, y(t_i)$	$N = 8, y(t_i)$
0	1	1	1	1
0.2	1.18126924692202	1.18359546667	1.1812493115195	1.1812692592562
0.4	1.32967995396436	1.34654720000	1.3296650977943	1.3296799646273
0.6	1.45118836390597	1.50568720000	1.4511761317037	1.4511883725367
0.8	1.55067103588278	1.68261546667	1.5506606459551	1.5506710426989
1	1.63212055882856	1.91250000000	1.6321177527222	1.6321204954801

Table 9 Comparison between absolute error functions obtained by present and other existed method for$x(t)$of Example 3 for *N=6, 8*

t_i	**Transform method**[9] $e_{x,6}(t_i)$	**Present method**	
		$e_{x,6}(t_i)$	$e_{x,8}(t_i)$
0	0	0	0
0.2	1.5575e-002	1.9935e-005	1.2334e-008
0.4	5.1209e-002	1.4856e-005	1.0663e-008
0.6	1.0150e-001	1.2232e-005	8.6308e-009
0.8	1.6630e-001	1.039e-005	6.8161e-009
1	2.3351e-001	2.8061e-006	6.3349e-008

Table 10 Comparison between absolute error functions obtained by present and other existed method for $y(t)$of Example 3 for *N=6, 8*

t_i	**Transform method**[9] $e_{y,6}(t_i)$	**Present method**	
		$e_{y,6}(t_i)$	$e_{y,8}(t_i)$
0	0	0	0
0.2	2.3262e-003	1.9935e-005	1.2334e-008
0.4	1.6867e-002	1.4856e-005	1.0663e-008
0.6	5.4499e-002	1.2232e-005	8.6308e-009
0.8	1.3194e-001	1.039e-005	6.8161e-009
1	2.8038e-001	2.8061e-006	6.3349e-008

Example 5

Consider the fourth order linear system differential equation

$$\begin{array}{l} x^{(4)}(t)+3y^{(3)}(t)+x''(t)+3y'-\frac{1}{1+t}x(t)+y(t)=0 \\ \frac{1}{5}(1+t)^3y^{(4)}(t)+x^{(3)}(t)-12x''(t)+3y''(t)+6y'(t)=0 \end{array}$$

with conditions

$$x(0)=1, x(1)=\frac{1}{8}, x'(0)=-3, x''(0)=12, y(0)=1, y(1)=\frac{1}{16}, y'(0)=-4, y''(0)=20.$$

Following the procedure in Section 7, we find the rational Chebyshev coefficients matrix for N =4 by using rational Chebyshev collocation method as:

$$\mathrm{A}=\left[\begin{array}{cccccccccc} \frac{5}{16} & -\frac{15}{32} & \frac{3}{16} & -\frac{1}{32} & 0 & \frac{35}{128} & -\frac{7}{16} & \frac{7}{32} & -\frac{1}{16} & \frac{1}{128} \end{array}\right]^T$$

Thus the solution of Example 5 becomes

$$x(t)=\frac{1}{(t+1)^3}, y(t)=\frac{1}{(t+1)^4}$$

which is exact solution of this example

8. Conclusion

The rational Chebyshev collocation for solving systems of high-order linear ordinary differential equations with variable coefficients numerically is presented. A considerable advantage of the method is that the rational Chebyshev coefficients of the solution are found very easily by using computer programs. For this reason, this process is faster than the other methods. Also, an interesting feature of this method is that when a differential system has linearly independent polynomial solution of degree N or less than N. In addition, an interesting feature of this method is to find the analytical solutions if the system has an exact solutions that are a rational functions. Besides, we see that there exists a solution which is closer to the exact solution if the truncation limit N is increased. The method can also be extended to system of linear integral and integro differential equations. In addition, it can be applied to systems of partial differential equations. Illustrative examples with the satisfactory results are used to demonstrate the application, effectiveness and accuracy of this method.

References

[1] S. Abbasbandy, H. Ghehsareh and I. Hashim, *An approximate solution of the MHD flow over a non-linear stretching sheet by rational Chebyshev collocation method*, U.P.B. Series A, Vol.74, Iss. 4, 2012.

[2] A. Akyüz and M. Sezer, *Chebyshev polynomial solutions of systems of high-order linear differential equations with variable coefficients*, Applied Mathematics and Computation 144 (2003) 237–247.

[3] S.L. Campbell, *Least squares completions for nonlinear differential algebraic equations*, Numerische Mathematik 65 (1993), 77–94.

[4] A. Davies, D. Crann, *The solution of differential equations using numerical Laplace transforms*, Int. J. Math. Educ. Sci. Technol. 30 (1) (1999) 65–79.

[5] E. Gökmen, M. Sezer, *Taylor collocation method for systems of high-order linear differential–difference equations with variable coefficients*, Ain Shams Engineering Journal (2013) 4, 117–125.

[6] J.C. Mason, D.C. Handscomb, *Chebyshev polynomials*, A CRC Press Company, New York (2003).

[7] K. Parand and M. Razzaghi, *Rational Chebyshev tau method for solving Volterra population model,* Applied Mathematics and Computation 149 (2004) 893–900.

[8] M. A. Ramadan, K. R. Raslan and M. A. Nassar, *An approximate analytical solution of higher-order linear differential equations with variable coefficients using improved rational Chebyshev collocation method*, Applied and Computational Mathematics; 3:6, (2014),315-322.

[9] M. Thangmoon and S. Pusjuso, *The numerical solutions of differential transform method and Laplace transform method for a system of differential equations*, Nonlinear Anal: Hybrid Syst., (2010), 4:425 – 31.

[10] S. Yalçinbaş ,N. Özsoy and M. Sezer, *Approximate solution of higher 0rder linear differential equations by means of anew rational Chebyshev collocation method*, Mathematical and Computational Applications, Vol. 15, No. 1, pp. 45-56, 2010.

[11] G. Yüksel, M. Gulsu and M. Sezer, *A Chebyshev Plynomial Approach for Higher – Order Linear Fredholm- VolterraIntegro-Differential Equations*, Gazi University Journal of Science, 25(2), 393 – 401, 2012.

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