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EXISTENCE AND HYERS-ULAM STABILITY OF NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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ABSTRACT. In this article, we study the nonlinear impulsive differential equations with nonlocal conditions in β-normed spaces. We have approached the new concepts of β-Ulam's type stability. Also we present sufficient conditions for the existence of solutions for impulsive Cauchy problem. Then we obtain generalized β-Ulam-Hyers-Rassias stability results for the impulsive problems on a compact interval with nonlocal conditions.

1. Introduction

In the past decades, many researchers studied differential equations with instantaneous impulses of the type

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \in J' := J \setminus \{t_1, \dots, t_m\},\ J := [0, T], \\ x(t_k^+) &= x(t_k^-) + I_k(x(t_k^-)), \quad k = 1, 2, \dots, m. \end{aligned} \tag{1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ and $I_k : \mathbb{R} \to \mathbb{R}$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. Here, I_k is a sequence of instantaneously impulse operators and have been used to describe abrupt changes such as shocks, harvesting, and natural disasters. For more existence, stability and periodic solutions on (1) and other impulsive models, one can read the monographs of [8, 11, 26]. Furthermore, the result of Hyers-Ulam stability for linear differential equations have been generalized in [1, 2, 3, 4, 5].

In pharmacotherapy, the above instantaneous impulses can not describe the certain dynamics of evolution processes. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. So we do not expect to use (1) to describe such process. In fact, the above situation should be shown by a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. From the viewpoint of general

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theories, Hernández and O'Regan [16] initially offered to study a new class of abstract semilinear impulsive differential equations with not instantaneous impulses in a PC-normed Banach space. Meanwhile, Pierri et al. [23] continue the work in a PC_α-normed Banach space and develop the results in [16].

Motivated by [16, 23, 25, 28], we continue to study existence and uniqueness of solutions to differential equations with not instantaneous impulses in a $P\beta$-normed Banach space (see Section 2) of the form

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t), \mathcal{T}u(t), \mathcal{S}u(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m, \\ & u(0) + g(u) = u_0, \\ u(t) &= \zeta_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \end{aligned} \tag{2}$$

in a Banach space $\mathbb{X}$, where A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)|t \geq 0\}$ and t_i, s_i are pre-fixed numbers satisfying $0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < s_{m-1} \leq t_m \leq s_m \leq t_{m+1} = T$, $f : [0, T] \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous, $g \in \mathcal{PC}(J, \mathbb{X})$ and $\zeta_i : [t_i, s_i] \times \mathbb{X} \to \mathbb{X}$ is continuous for all $i = 1, 2, \ldots, m$,

$$\mathcal{T}u(t) = \int_0^t \mathcal{K}(t, s)u(s)ds,\ \mathcal{K} \in \mathbb{C}[\mathcal{D}, \mathcal{R}^+],$$
$$\mathcal{S}u(t) = \int_0^t \mathcal{H}(t, s)u(s)ds,\ \mathcal{H} \in \mathbb{C}[\mathcal{N}, \mathcal{R}^+],$$

where $\mathcal{D} = \{(t, s) \in \mathcal{R}^2\} : 0 \leq s \leq t \leq T$, $\mathcal{N} = \{(t, s) \in \mathcal{R}^2\} : 0 \leq s \leq t \leq T$ and $\mathcal{PC}(J, \mathbb{X})$ consist of a function u that are a map from J into $\mathbb{X}$ such that $u(t)$ is continuous.

It is remarkable that Ulam type stability problems [27] have attracted many famous researchers. The readers can refer to monographs of Hyers [17], [18], Rassias [24] and other recent works [6, 7, 13, 14, 15, 19, 20, 21, 22] in standard normed spaces and [12, 29] in β-normed spaces.

We introduce some auxiliary facts and offer four new concepts of β-Ulam's type stability for (2) (see Definitions 2.3–2.6). This is our main original contribution of this paper. As a result, existence and uniqueness and a generalized β-Ulam's type stability result on a compact interval are established.

2. Preliminaries

Definition 2.1. ([9]) Suppose E is a vector space over $\mathbb{K}$. A function $\|\cdot\|_\beta$ $(0 < \beta \leq 1) : E \to [0, \infty)$ is called a β-norm if and only if it satisfies (i) $\|x\|_\beta = 0$ if and only if $x = 0$; (ii) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in E$; (iii) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$. The pair $(E, \|\cdot\|_\beta)$ is called a β-normed space. A β-Banach space is a complete β-normed space.

Throughout this paper, let $J = [0, T]$, $\beta \in (0, 1)$ be a fixed constant and $\mathcal{C}(J, \mathbb{X})$ be the Banach space of all continuous functions from J into $\mathbb{X}$ with the new norm $\|x\|_\beta := \max\{|x(t)|^\beta : t \in J\}$ for $x \in \mathcal{C}(J, \mathbb{X})$. We need the $P\beta$-Banach space $\mathcal{PC}(J, \mathbb{X}) := \{x : J \to \mathbb{R} : x \in \mathcal{C}((t_k, t_{k+1}], \mathbb{X}),\ k = 0, 1, \ldots, m$ and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1, \ldots, m$, with $x(t_k^-) = x(t_k)\}$ with the norm $\|x\|_{P\beta} := \sup\{|x(t)|^\beta : t \in J\}$. Meanwhile, we set $\mathcal{PC}^1(J, \mathbb{X}) := \{x \in \mathcal{PC}(J, \mathbb{X}) : x' \in \mathcal{PC}(J, \mathbb{X})\}$ with $\|x\|_{P\beta^1} := \max\{\|x\|_\beta, \|x'\|_\beta\}$. Clearly, $\mathcal{PC}^1(J, \mathbb{X})$ endowed with the norm $\|\cdot\|_{P\beta^1}$ is a $P\beta$-Banach space.

Definition 2.2 ([16])**.** A function $x \in \mathcal{PC}^1(J, \mathbb{X})$ is called a solution of the problem

$$\begin{gathered} u'(t) = Au(t) + f(t, u(t), \mathcal{T}u(t), \mathcal{S}u(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m, \\ u(t) = \zeta_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \\ u(0) + g(u) = u_0, \quad u_0 \in \mathbb{X}, \end{gathered} \tag{3}$$

if u satisfies

$$\begin{gathered} u(0) = x_0 - g(u); \\ u(t) = \zeta_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ u(t) = S(t)[u_0 - g(u)] + \int_0^t S(t-s) f(s, u(s), \mathcal{T}u(s), \mathcal{S}u(s)) ds, \quad t \in [0, t_1]; \\ u(t) = S(t)[\zeta_i(s_i, u(s_i)) - g(u)] + \int_{s_i}^t S(t-s) f(s, x(s), \mathcal{T}u(s), \mathcal{S}u(s)) ds, \\ t \in (s_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{gathered}$$

In general, we do not expect to get a precise solution of (2.1). However, we can try to get a function which satisfies some suitable approximation inequalities.

Let $0 < \beta < 1$, $\epsilon > 0$, $\psi \geq 0$ and $\varphi \in PC(J, \mathbb{R}_+)$. We consider the following inequalities:

$$\begin{gathered} |v'(t) - Av(t) - f(t, v(t), \mathcal{T}v(t), \mathcal{S}v(t))| \leq \epsilon, \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m, \\ |v(t) - \zeta_i(t, v(t)) + g(t)| \leq \epsilon, \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \end{gathered} \tag{4}$$

and

$$\begin{gathered} |v'(t) - Av(t) - f(t, v(t), \mathcal{T}v(t), \mathcal{S}v(t))| \leq \varphi(t), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m, \\ |v(t) - \zeta_i(t, v(t)) + g(t)| \leq \psi, \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \end{gathered} \tag{5}$$

and

$$\begin{gathered} |v'(t) - Av(t) - f(t, v(t), \mathcal{T}v(t), \mathcal{S}v(t))| \leq \epsilon\varphi(t), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m, \\ |v(t) - \zeta_i(t, v(t)) + g(t)| \leq \epsilon\psi, \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m. \end{gathered} \tag{6}$$

Next, our aim is to find a solution $v(\cdot)$ close to the measured output $u(\cdot)$ and whose closeness is defined in the sense of β-Ulam's type stabilities.

Definition 2.3. Equation (2) is β-Ulam-Hyers stable if there exists a real number $c_{f_j, \mathcal{M}, \mathcal{G}, \beta, \zeta_i, \varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{X})$ of (4) there exists a solution $x \in PC^1(J, \mathbb{X})$ of (2) with

$$|v(t) - u(t)|^\beta \leq c_{f_j, \mathcal{M}, \mathcal{G}, \beta, \zeta_i, \varphi} \epsilon^\beta, \quad t \in J.$$

Definition 2.4. *Equation (2) is generalized β-Ulam-Hyers stable if there exists $\theta_{f_j, \mathcal{M}, \mathcal{G}, \beta, \zeta_i, \varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{f_j, \mathcal{M}, \mathcal{G}, \beta, \zeta_i, \varphi}(0) = 0$ such that for each solution $y \in PC^1(J, \mathbb{X})$ of (4) there exists a solution $x \in PC^1(J, \mathbb{X})$ of (2) with*

$$|v(t) - u(t)|^\beta \leq \theta_{f_j, \mathcal{M}, \mathcal{G}, \beta, \zeta_i, \varphi}(\epsilon^\beta),\ t \in J.$$

Definition 2.5. Equation (2) is β-Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f_j, \mathcal{M}, \mathcal{G}, \beta, \zeta_i, \varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{X})$ of (6) there exists a solution $x \in PC^1(J, \mathbb{X})$ of (2) with

$$|v(t) - u(t)|^\beta \leq c_{f_j, \mathcal{M}, \mathcal{G}, \beta, \zeta_i, \varphi} \epsilon^\beta (\psi^\beta + \varphi^\beta(t)),\ t \in J.$$

Definition 2.6. Equation (2) is generalized β-Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f_j,\mathcal{M},\mathcal{G},\beta,\zeta_i,\varphi} > 0$ such that for each solution $y \in PC^1(J, \mathbb{X})$ of (5) there exists a solution $x \in PC^1(J, \mathbb{X})$ of (2) with

$$|v(t) - u(t)|^\beta \le c_{f_j,\mathcal{M},\mathcal{G},\beta,\zeta_i,\varphi}(\psi^\beta + \varphi^\beta(t)), \quad t \in J.$$

Obviously, (i) Definition 2.3 implies Definition 2.4; (ii) Definition 2.5 implies Definition 2.6; (iii) Definition 2.5 for $\varphi(\cdot) = \psi = 1$ implies Definition 2.3; (iv) Definitions 2.3-2.6 become to Ulam's stability concepts in Wang et al. [28] when $\beta = 1$ and $s_i = t_i$.

Remark 2.1. A function $v \in PC^1(J, \mathbb{R})$ is a solution of (5) if and only if there is $G \in PC(J, \mathbb{R})$ and a sequence G_i, $i = 1, 2, \ldots, m$ (which depend on y) such that

(i) $|G(t)| \le \varphi(t)$, $t \in J$ and $|G_i| \le \psi$, $i = 1, 2, \ldots, m$;
(ii) $v'(t) = Av(t) + f(t, v(t)) + G(t)$, $t \in (s_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, m$;
(iii) $v(t) = g_i(t, v(t)) + G_i$, $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$.

By remark 2.1 we get the following results.

Remark 2.2. If $v \in PC^1(J, \mathbb{X})$ is a solution of (5) then v is a solution of the integral inequality

$$\begin{aligned} &|v(t) - S(t-s)[\zeta_i(t, v(t)) - g(t)]| \le \mathcal{M}\psi, \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ &\Big|v(t) - S(t-s)[v(0) - g(x)] \int_0^t S(t-s) f(s, v(s), \mathcal{T}v(s), \mathcal{S}v(s)) ds\Big| \\ &\qquad \le \int_0^t \|S(t-s)\| \varphi(s) ds, \quad t \in [0, t_1]; \end{aligned} \tag{7}$$

$$\begin{aligned} &\Big|v(t) - S(t-s)[\zeta_i(s_i, v(s_i)) - g(x)] - \int_{s_i}^t S(t-s) f(s, v(s)\mathcal{T}v(s), \mathcal{S}v(s)) ds\Big| \\ &\qquad \le \|S(t-s)\|\psi + \int_{s_i}^t \|S(t-s)\| \varphi(s) ds, t \in [s_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{aligned}$$

We can give similar remarks for the solutions of the inequalities (4) and (6). To study Ulam's type stability, we need the following integral inequality results (see [10, Theorem 16.4]).

Lemma 2.1. *(i) Let the following inequality holds*

$$u(t) \le a(t) + \int_0^t b(s) u(s) ds, \quad t \ge 0,$$

where u, $a, \in PC(\mathbb{R}_+, \mathbb{R}_+)$, a is nondecreasing and $b(t) > 0$. Then, for $t \in \mathbb{R}_+$,

$$u(t) \le a(t) \exp\Big(\int_0^t b(s) ds\Big).$$

(ii) Assume

$$u(t) \le a(t) + \delta u(t) + \int_0^t b(s) u(s) ds + \sum_{0<t_k<t} \beta_k u(t_k^-), \quad t \ge 0,$$

where $u,\ \delta,\ a, b \in PC(\mathbb{R}_+, \mathbb{R}_+)$, $a,\ \delta$ *is nondecreasing and* $b(t) > 0$, $\beta_k > 0$, $k \in \{1, \ldots, m\}$. *Then, for* $t \in \mathbb{R}_+$,

$$u(t) \leq a(t)(1+\beta+\delta)^k \exp\Big(\int_0^t b(s)ds\Big), \quad t \in (t_k, t_{k+1}],\ k \in \{1, \ldots, m\},$$

where $\beta = \sup_{k\in\{1,\ldots,m\}}\{\beta_k\}$.

3. Main results

We use the following assumptions:

(H1) A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, whose domain $D(A)$ is dense in H such that $\|S(t)\| \leq \mathcal{M}$, for all $t \in J$.

(H2) $f \in C(J\times\mathbb{X}\times\mathbb{X}\times\mathbb{X} \to \mathbb{X})$, $g :\to \mathbb{X}$ and there exists constants $\mathcal{L}_{f_1}, \mathcal{L}_{f_2}, \mathcal{L}_{f_3} \geq 0$, $\mathcal{G} \geq 0$ such that

$$|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq \mathcal{L}_{f_1}|x_1 - y_1| + \mathcal{L}_{f_2}|x_2 - y_2| + \mathcal{L}_{f_3}|x_3 - y_3|,$$

$t \in J,\ x_j, y_j \in \mathbb{X}, j = 1, 2, 3.$

$$|g(\eta_1) - g(\eta_2) \leq \mathcal{G}|\eta_1 - \eta_2|, \quad \eta_1, \eta_2 \in \mathcal{PC}(J, \mathbb{X}).$$

(H3) Denote $M_f = max\{M_{f_1}, M_{f_2}, M_{f_2}\}$, $\mathcal{K}^* = \sup\limits_{t\in J}\displaystyle\int_0^t |\mathcal{K}(t,s)|dt \leq \infty$,

$\mathcal{H}^* = \sup\limits_{t\in J}\displaystyle\int_0^t |\mathcal{H}(t,s)|dt \leq \infty.$

(H4) $\zeta_i \in C([t_i, s_i] \times \mathbb{X}, \mathbb{X})$ and there are positive constants L_{ζ_i}, $i = 1, 2, \ldots, m$ such that

$$|\zeta_i(t, u_1) - \zeta_i(t, u_2)| \leq L_{\zeta_i}|u_1 - u_2|,$$

for each $t \in [t_i, s_i]$ and all $u_1, u_2 \in \mathbb{R}$.

(H5) : Let $\varphi \in C(J, \mathbb{R}_+)$ be a nondecreasing function. There exists $c_\varphi > 0$ such that

$$\int_0^t \varphi(s)ds \leq c_\varphi\varphi(t),$$

for each $t \in J$.

Concerning the existence and uniqueness result for the solutions to (3), we give the following theorem.

Theorem 3.1. *Assume that* (H1)–(H4) *are satisfied. Then* (3) *has a unique solution* x *provided that*

$$\Omega := max\Big\{\mathcal{M}\big[\mathcal{G}^\beta + L^\beta_{\zeta_i} + L^\beta_{f_1}(t_{i+1} - s_i) + L^\beta_{f_2}(t_{i+1} - s_i)\mathcal{K}^* + L^\beta_{f_3}(t_{i+1} - s_i)\mathcal{H}^*\big], \\ \big[\mathcal{M}\mathcal{G}^\beta + \mathcal{M}L^\beta_{f_1}t_1^\beta + \mathcal{M}\mathcal{K}^*L^\beta_{f_2}t_1^\beta + \mathcal{M}\mathcal{H}^*L^\beta_{f_3}t_1^\beta\big] : i = 1, 2, \ldots, m\Big\} < 1. \tag{8}$$

Proof. Consider a mapping $\mathcal{F} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$(\mathcal{F}u)(0) = u_0 - g(u);$$
$$(\mathcal{F}u)(t) = \zeta_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m;$$
$$(\mathcal{F}u)(t) = S(t-s)[u_0 - g(u)] + \int_0^t S(t-s) f(s, x(s), \mathcal{T}u(s), \mathcal{S}u(s)) ds, \quad t \in [0, t_1];$$
$$(\mathcal{F}u)(t) = S(t-s)[\zeta_i(s_i, u(s_i)) - g(u)] + \int_{s_i}^t S(t-s) f(s, u(s), \mathcal{T}u(s), \mathcal{S}u(s)) ds,$$
$$t \in (s_i, t_{i+1}],\ i = 1, 2, \ldots, m.$$

Obviously, $\mathcal{F}$ is well defined.

For any $u, v \in PC(J,\mathbb{R})$ and $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \le{}& \mathcal{M}\mathcal{G}|u(s) - v(s)| + \mathcal{M}L_{\zeta_i}|u(s_i) - v(s_i)| + \mathcal{M}L_{f_1}\int_{s_i}^t |u(s) - v(s)| ds \\ &+ \mathcal{M}L_{f_2}\int_{s_i}^t |Tu(s) - Tv(s)| ds + \mathcal{M}L_{f_3}\int_{s_i}^t |Su(s) - Sv(s)| ds.\end{aligned}$$

Now,

$$\begin{aligned}L_{f_2}\int_{s_i}^t |Tu(s) - Tv(s)| ds \le{}& L_{f_2}\int_{s_i}^t \int_0^s |K(s,\tau)||u(\tau) - v(\tau)| d\tau ds \\ \le{}& L_{f_2}\int_{s_i}^t |u(\tau) - v(\tau)| \int_0^t |K(s,\tau)| d\tau ds \\ \le{}& L_{f_2}\int_{s_i}^t \max_{t\in[s_i,t_{i+1}]} |u(\tau) - v(\tau)| \int_0^t |K(s,\tau)| d\tau ds \\ \le{}& L_{f_2}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}}\mathcal{K}^*.\end{aligned}$$

Similarly,

$$L_{f_3}\int_{s_i}^t |Su(s) - Sv(s)| ds \le L_{f_3}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}}\mathcal{H}^*.$$

Substitute the equation (2.3) and (2.4) into equation (2.2), we have

$$\begin{aligned}&|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\ &\le \mathcal{M}\mathcal{G}|u - v|_C + \mathcal{M}L_{\zeta_i}\|u - v\|_C + \mathcal{M}L_{f_1}\int_{s_i}^t \max_{t\in[s_i,t_{i+1}]} |u(s) - v(s)| ds \\ &\quad + \mathcal{M}L_{f_2}\int_{s_i}^t \max_{t\in[s_i,t_{i+1}]} |Tu(s) - Tv(s)| ds + \mathcal{M}L_{f_3}\int_{s_i}^t \max_{t\in[s_i,t_{i+1}]} |Su(s) - Sv(s)| ds \\ &\le \mathcal{M}\mathcal{G}|u - v|_C + \mathcal{M}L_{\zeta_i}\|u - v\|_C + \mathcal{M}L_{f_1}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}} \\ &\quad + \mathcal{M}L_{f_2}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}}\mathcal{K}^* + \mathcal{M}L_{f_3}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}}\mathcal{H}^* \\ &\le \mathcal{M}\big[\mathcal{G}|u - v|_C + L_{\zeta_i}\|u - v\|_C\big] + \mathcal{M}L_{f_1}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}} \\ &\quad + \mathcal{M}L_{f_2}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}}\mathcal{K}^* + \mathcal{M}L_{f_3}(t_{i+1} - s_i)\|u - v\|_{\mathcal{PC}}\mathcal{H}^* \\ &\le \mathcal{M}\big[\mathcal{G}^\beta + L_{\zeta_i}^\beta\big]\|u - v\|_{\mathcal{C}} + \mathcal{M}\big[L_{f_1}^\beta(t_{i+1} - s_i) + L_{f_2}^\beta(t_{i+1} - s_i)\mathcal{K}^* \\ &\quad + L_{f_3}^\beta(t_{i+1} - s_i)\mathcal{H}^*\big]\|u - v\|_{\mathcal{PC}}.\end{aligned}$$

which implies

$$|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)|^{\beta}\le \mathcal{M}\big[\mathcal{G}^{\beta}+L^{\beta}_{\zeta_i}\big]\|u-v\|_{\mathcal{P}\beta}+\mathcal{M}\big[L^{\beta}_{f_1}(t_{i+1}-s_i)+L^{\beta}_{f_2}(t_{i+1}-s_i)\mathcal{K}^{*} +L^{\beta}_{f_3}(t_{i+1}-s_i)\mathcal{H}^{*}\big]\|u-v\|_{\mathcal{P}\beta}.$$

This reduces to

$$\|\mathcal{F}u-\mathcal{F}v\|_{P\beta}\le \mathcal{M}\big[\mathcal{G}^{\beta}+L^{\beta}_{\zeta_i}+L^{\beta}_{f_1}(t_{i+1}-s_i)+L^{\beta}_{f_2}(t_{i+1}-s_i)\mathcal{K}^{*} + L^{\beta}_{f_3}(t_{i+1}-s_i)\mathcal{H}^{*}\big]\|u-v\|_{\mathcal{P}\beta},\quad t\in(s_i,t_{i+1}].$$

Proceeding as above, we obtain that

$$\|\mathcal{F}u-\mathcal{F}v\|_{P\beta}\le\big[\mathcal{M}\mathcal{G}^{\beta}+\mathcal{M}L^{\beta}_{f_1}t_1^{\beta}+\mathcal{M}\mathcal{K}^{*}L^{\beta}_{f_2}t_1^{\beta}+\mathcal{M}\mathcal{H}^{*}L^{\beta}_{f_3}t_1^{\beta}\big]\|u-v\|_{P\beta},\quad t\in[0,t_1],$$

$$\|\mathcal{F}u-\mathcal{F}v\|_{P\beta}\le \mathcal{M}\mathcal{G}^{\beta}+\mathcal{M}L^{\beta}_{\zeta_i}\|u-v\|_{P\beta},\quad t\in(t_i,s_i],\ i=1,2,\dots,m.$$

From the above facts, we have

$$\|\mathcal{F}u-\mathcal{F}v\|_{P\beta}\le\Omega\|u-v\|_{P\beta},$$

where Ω is defined in (8). Finally, we can deduce that $\mathcal{F}$ is a contraction mapping. Then, one can derive the result immediately. □

Next, we discuss hte stability of (2) by using the concept of generalized β-Ulam-Hyers-Rassias in the above section.

Theorem 3.2. *Assume that* (H1)-(H5) *and* (8) *are satisfied. Then* (2) *is generalized β-Ulam-Hyers-Rassias stable with respect to* (φ,ψ).

Proof. Let $v\in PC^1(J,\mathbb{R})$ be a solution of (5). Denote by u the unique solution of the impulsive Cauchy problem

$$\begin{aligned} u'(t)&=f(t,u(t),Tu(t),Su(t)),\quad t\in(s_i,t_{i+1}],\ i=0,1,2,\dots,m,\\ u(t)&=\zeta_i(t,u(t)),\quad t\in(t_i,s_i],\ i=1,2,\dots,m,\\ u(0)&=u_0-g(u).\end{aligned}\tag{9}$$

Then we obtain

$$u(t)=\begin{cases} S(t-s)[\zeta_i(t,u(t))-g(t)],\quad t\in(t_i,s_i],\ i=1,2,\dots,m;\\ S(t-s)[u(0)-g(u)]+\displaystyle\int_0^t S(t-s)f(s,u(s),\mathcal{T}u(s),\mathcal{S}u(s))ds,\quad t\in[0,t_1];\\ S(t-s)[\zeta_i(s_i,y(s_i))-g(x)]+\displaystyle\int_{s_i}^t S(t-s)f(s,y(s)\mathcal{T}u(s),\mathcal{S}u(s))ds,\\ \qquad\qquad t\in[s_i,t_{i+1}],\ i=1,2,\dots,m.\end{cases}$$

Keeping in mind (7), for each $t\in(s_i,t_{i+1}]$, $i=1,2,\dots,m$, we have

$$\begin{aligned}\Big|v(t)-S(t-s)[\zeta_i(s_i,v(s_i))-g(v)]-&\int_{s_i}^t S(t-s)f(s,v(s)\mathcal{T}v(s),\mathcal{S}v(s))ds\Big|\\ &\le\|S(t-s)\|\psi+\int_{s_i}^t\|S(t-s)\|\varphi(s)ds\\ &\le\mathcal{M}\psi+\mathcal{M}c_{\varphi}\varphi(t),\end{aligned}$$

and for $t\in(t_i,s_i]$, $i=1,2,\dots,m$, we have

$$|v(t)-S(t-s)[\zeta_i(t,v(t))-g(v)]|\le\mathcal{M}\psi,$$

and for $t \in [0, t_1]$, we have

$$\Big|v(t) - S(t-s)[v(0) - g(v)]\int_0^t S(t-s)f(s, v(s), \mathcal{T}v(s), \mathcal{S}v(s))ds\Big| \leq \mathcal{M}c_\varphi \varphi(t).$$

Hence, for each $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned}
&|v(t) - u(t)| \\
&= \Big|v(t) - \mathcal{M}\zeta_i(s_i, x(s_i)) + \mathcal{M}g(u) - \mathcal{M}\int_{s_i}^t f(s, u(s), Tu(s), Su(s))ds\Big| \\
&\leq \Big|v(t) - \mathcal{M}\zeta_i(s_i, v(s_i)) + \mathcal{M}g(v) - \mathcal{M}\int_{s_i}^t f(s, v(s), Tv(s), Sv(s))ds\Big| \\
&\quad + \mathcal{M}\big|\big[g(v) - g(u)\big]\big| + \mathcal{M}\Big|\zeta_i(s_i, v(s_i)) - \zeta_i(s_i, u(s_i))\Big| \\
&\quad + \mathcal{M}\Big(\Big[\int_{s_i}^t |f(s, v(s), Tv(s), Sv(s)) - f(s, u(s), Tu(s), Su(s))|ds\Big]\Big) \\
&\leq \mathcal{M}(1 + c_\varphi)[\psi + \varphi(t)] + \mathcal{M}\mathcal{G}|v(s) - u(s)| + \mathcal{M}L_{\zeta_i}|v(s_i) - u(s_i)| \\
&\quad + \mathcal{M}\int_{s_i}^t \Big[L_{f_1}|v(s) - u(s)| + L_{f_2}|Tv(s) - Tu(s)|ds + L_{f_3}|Sv(s) - Su(s)|\Big]ds \\
&\leq \mathcal{M}(1 + c_\varphi)[\psi + \varphi(t)] + \mathcal{M}\mathcal{G}|v(s) - u(s)| + \mathcal{M}\sum_{0<s_i<t} L_{\zeta_i}|v(s_i) - u(s_i)| \\
&\quad + \mathcal{M}\int_0^t \Big[L_{f_1}|v(s) - u(s)| + L_{f_2}\mathcal{K}^*|v(s) - u(s)|ds + L_{f_3}\mathcal{H}^*|v(s) - u(s)|\Big]ds \\
&\leq \mathcal{M}(1 + c_\varphi)[\psi + \varphi(t)] + \mathcal{G}|v(s) - u(s)| + \mathcal{M}\sum_{0<s_i<t} L_{\zeta_i}|v(s_i) - u(s_i)| \\
&\quad + \mathcal{M}\int_0^t \Big[L_{f_j}\big(1 + \mathcal{K}^* + \mathcal{H}^*\big)\Big]|v(s) - u(s)|ds.
\end{aligned}$$

Clearly, $a(t) := \mathcal{M}(1 + c_\varphi)[\psi + \varphi(t)]$, $t \in (s_i, t_{i+1}]$, is nondecreasing and $a \in PC(\mathbb{R}_+, \mathbb{R}_+)$. By Lemma 2.1 (ii), we obtain

$$\begin{aligned}
|v(t) - u(t)| &\leq \mathcal{M}(1 + c_\varphi)[\psi + \varphi(t)](1 + \mathcal{G} + L_\zeta)^i \exp\Big(\int_0^t L_{f_j}\big(1 + \mathcal{K}^* + \mathcal{H}^*\big)ds\Big) \\
&\leq \mathcal{M}(1 + c_\varphi)[\psi + \varphi(t)](1 + \mathcal{G} + L_\zeta)^i \exp\big(L_{f_j}\big(1 + \mathcal{K}^* + \mathcal{H}^*\big)t_{i+1}\big),
\end{aligned}$$

where $L_\zeta = \max\{L_{\zeta_1}, L_{g\zeta_2}, \dots, L_{\zeta_m}\}$. Thus,

$$\begin{aligned}
|v(t) - u(t)|^\beta &\leq \big[\mathcal{M}(1 + c_\varphi)[\psi + \varphi(t)](1 + \mathcal{G} + L_g)^i \exp\big(L_{f_j}\big(1 + \mathcal{K}^* + \mathcal{H}^*\big)t_{i+1}\big)\big]^\beta \\
&\leq \big[\mathcal{M}(1 + c_\varphi)(1 + \mathcal{G}L_\eta)^i \exp\big(L_{f_j}\big(1 + \mathcal{K}^* + \mathcal{H}^*\big)t_{i+1}\big)\big]^\beta[\psi + \varphi(t)]^\beta \\
&\leq \big[\mathcal{M}(1 + c_\varphi)(1 + \mathcal{G} + L_\zeta)^i \exp\big(L_{f_j}\big(1 + \mathcal{K}^* + \mathcal{H}^*\big)t_{i+1}\big)\big]^\beta(\psi^\beta + \varphi(t)^\beta),
\end{aligned} \tag{10}$$

for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, m$.

Further, for $t \in (t_i, s_i]$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned}
|v(t) - u(t)|^\beta &\leq \mathcal{M}|v(t) - g_i(t, u(t)) + \mathcal{M}g(u)|^\beta \\
&\leq \mathcal{M}|v(t) - g_i(t, v(t))|^\beta + \mathcal{M}|g(v) - g(u)| + \mathcal{M}|g_i(t, v(t)) - g_i(t, u(t))|^\beta \\
&\leq \mathcal{M}\psi^\beta + \mathcal{M}\mathcal{G}^\beta + \mathcal{M}L_{\zeta_i}^\beta|v(t) - u(t)|^\beta,
\end{aligned}$$

which yields

$$|v(t)-u(t)|^{\beta} \le \frac{1}{1-\mathcal{M}\mathcal{G}^{\beta}-\mathcal{M}L_{\zeta_i}^{\beta}}\psi^{\beta}. \quad ((8) \text{ implies } \mathcal{M}L_{\zeta_i}^{\beta}+\mathcal{M}\mathcal{G}^{\beta}<1) \tag{11}$$

Moreover, for $t \in [0,t_1]$, we have

$$\begin{aligned} |v(t)-u(t)| &= \big|v(t)-v(0)+g(u)-\int_0^t f(s,u(s),\mathcal{T}u(s),\mathcal{S}u(s))ds\big| \\ &\le \big|v(t)-v(0)+g(v)-\int_0^t f(s,v(s),\mathcal{T}v(s),\mathcal{S}v(s))ds\big| + \mathcal{M}|g(v)-g(u)| \\ &+ \Big(\int_0^t |f(s,v(s),\mathcal{T}v(s),\mathcal{S}v(s))-f(s,x(s),\mathcal{T}u(s),\mathcal{S}u(s))|ds\Big) \\ &\le \mathcal{M}c_{\varphi}\varphi(t)+\mathcal{M}\mathcal{G}|g(v)-g(u)|+\mathcal{M}\int_0^t L_{f_j}(1+\mathcal{K}^*+\mathcal{H}^*)|v(s)-u(s)|ds. \end{aligned}$$

By Lemma 2.1 (i), we obtain

$$\begin{aligned} |v(t)-u(t)| &\le \mathcal{M}(1+\mathcal{G})c_{\varphi}\varphi(t)\exp\Big(\int_0^t L_{f_j}(1+\mathcal{K}^*+\mathcal{H}^*)ds\Big) \\ &\le \mathcal{M}(1+\mathcal{G})c_{\varphi}\varphi(t)\exp\big(L_{f_j}(1+\mathcal{K}^*+\mathcal{H}^*)t_1\big). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} |v(t)-u(t)|^{\beta} &\le \big[\mathcal{M}(1+\mathcal{G})c_{\varphi}\varphi(t)\exp\big(L_{f_j}(1+\mathcal{K}^*+\mathcal{H}^*)t_1\big)\big]^{\beta} \\ &\le \big[\mathcal{M}(1+\mathcal{G})c_{\varphi}\exp\big(L_{f_j}(1+\mathcal{K}^*+\mathcal{H}^*)t_1\big)\big]^{\beta}\varphi(t)^{\beta}, \quad t \in [0,t_1]. \end{aligned} \tag{12}$$

Summarizing, we combine (10), (11) and (12) and derive that

$$\begin{aligned} &|v(t)-u(t)|^{\beta} \\ &\le \Big(\big[\mathcal{M}(1+c_{\varphi})(1+\mathcal{G}+L_{\zeta})^{i}\exp\big(L_{f_j}\big(1+\mathcal{K}^*+\mathcal{H}^*)t_{i+1}\big)\big]^{\beta} \\ &\quad + \frac{1}{1-\mathcal{M}\mathcal{G}^{\beta}-\mathcal{M}L_{\zeta_i}^{\beta}}\psi^{\beta} + \big[\mathcal{M}(1+\mathcal{G})c_{\varphi}\exp\big(L_{f_j}(1+\mathcal{K}^*+\mathcal{H}^*)t_1\big)\big]^{\beta}\varphi(t)^{\beta}\Big)(\psi^{\beta}+\varphi^{\beta}(t)) \\ &= c_{f_j,\mathcal{M},\mathcal{G},\beta,\zeta_i,\varphi}(\psi^{\beta}+\varphi^{\beta}(t)), \quad t \in J, \end{aligned}$$

which implies that (2) is generalized β-Ulam-Hyers-Rassias stable with respect to (φ,ψ). The proof is complete. □

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