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ON CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY CONVOLUTION

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ABSTRACT. In this paper we introduce three subclasses of T; $S_s^*T(g;\alpha,\beta)$, $S_c^*T(g;\ \alpha,\beta)$ and $S_{sc}^*T(g;\alpha,\beta)$; consisting of analytic functions with negative coefficient define using convolution, and are respectively, starlike with respect to symmetric points,starlike with respect to conjugate points and starlike with respect to symmetric conjugate points. Several properties like, coefficient bounds, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are investigated.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic and univalent in the open unite disk $U = \{z : |z| < 1\}$. Let S^* be the subclass of S consisting of starlike functions in U. It well know that$f \in S^*$ if and only if $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in U)$.

Let S_s^* be in the subclass of S consisting of functions of the form (1) satisfying

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \qquad (z \in U). \tag{2}$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [8] (see also Robertson [7], Stankiewicz [10], Wu [13] and Owa et al. [6]). In [5], El-Ashwah and Thomas, introduced and studied two other classes namely the class S_c^* consisting of functions starlike with respect to conjugate points and S_{Sc}^* consisting of functions starlike with respect to symmetric conjugate points. In [12], Sudharsan et al. introduced the class $S_S^*(\alpha,\beta)$ of the functions $f(z) \in S$ and satisfying the following condition (see also [9]):

$$\left|\frac{zf'(z)}{f(z)-f(-z)} - 1\right| < \beta\left|\alpha\frac{zf'(z)}{f(z)-f(-z)} + 1\right| \tag{3}$$

for some $0 \le \alpha \le 1,\ 0 < \beta \le 1$ and $z \in U$.

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Let T denote the subclass of S consisting of the functions of the form :

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{4}$$

For $f \in S$ be given by (1) and $g \in S$ given by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \qquad (b_k \geq 0) \tag{5}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{6}$$

Definition 1. Let the function $f(z)$ of the form (4). Then $f(z)$ is said to be in the class $S_s^* T(g, \alpha, \beta)$ if it satisfies the following condition:

$$\left| \frac{z(f*g)'(z)}{(f*g)(z) - (f*g)(-z)} - 1 \right| < \beta \left| \alpha \frac{z(f*g)'(z)}{(f*g)(z) - (f*g)(-z)} + 1 \right|, \tag{7}$$

where $0 \leq \alpha \leq 1, 0 < \beta \leq 1, \grave{0} \leq \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$.

Definition 2. Let the function $f(z)$ of the form (4). Then $f(z)$ is said to be in the class $S_c^* T(g, \alpha, \beta)$ if it satisfies the following condition :

$$\left| \frac{z(f*g)'(z)}{(f*g)(z) + \overline{(f*g)\overline{(z)}}} - 1 \right| < \beta \left| \alpha \frac{z(f*g)'(z)}{(f*g)(z) + \overline{(f*g)\overline{(z)}}} + 1 \right|, \tag{8}$$

where $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$.

Definition 3. Let the function $f(z)$ of the form (4). Then $f(z)$ is said to be in the class $S_{sc}^* T(g, \alpha, \beta)$ if it satisfies the following condition :

$$\left| \frac{z(f*g)'(z)}{(f*g)(z) - \overline{(f*g)\overline{(-z)}}} - 1 \right| < \beta \left| \alpha \frac{z(f*g)'(z)}{(f*g)(z) - \overline{(f*g)\overline{(-z)}}} + 1 \right|, \tag{9}$$

where $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$.

Specializing the parameters α, β and the function g we have the following classes studied earlier :

(i) $S_s^*(\frac{z}{(1-z)}; 1, 1) = S_s^*$ (see [8]);

(ii) $S_s^*(\frac{z}{(1-z)}; \alpha, \beta\ ) = S_s^*(\alpha, \beta\ )$ (see[12]);

$(iii) S_s^*(z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta) = S_{s,n}^*(\alpha, \beta)$ (Aouf et al [1]).

Also we can obtain the following new classes for different choices of the function g :

$$\begin{aligned}(i)\ S_s^*(z+\sum_{k=2}^{\infty}\Gamma_k[a_1;b_1]z^k;\alpha,\beta) &= S_{q,s}(a_1,b_1,\alpha,\beta)\\ &= \left\{f\in A:\Re\left\{\frac{z(H_{q,s}[a_1;b_1]\ f(z))'}{(H_{q,s}[a_1;b_1]\ f(z)-(H_{q,s}[a_1;b_1]\ f(-z)}-1\right\}\right.\\ &< \left.\beta\left|\alpha\frac{z(H_{q,s}[a_1;b_1]\ f(z))'}{(H_{q,s}[a_1;b_1]\ f(z)-(H_{q,s}[a_1;b_1]\ f(-z)}+1\right|\right\} \qquad (10)\end{aligned}$$

where

$$\Gamma_k[a_1;b_1]=\frac{(a_1)_{k-1}\cdots(a_q)_{k-1}}{(b_1)_{k-1}\dots(b_s)_{k-1}(k-1)!}$$

and $H_{q,s}[a_1;b_1]$ is the Dziok-Srivastava operator (see [4]);

$$\begin{aligned}(ii)\ S_s^*(z+\sum_{k=2}^{\infty}\left[\frac{1+\ell+\mu(k-1)}{1+\ell}\right]^m z^k;\alpha,\beta) &= S(\mu;\alpha,\beta)\\ &= \left\{f\in A:\Re\left\{\frac{z(I^m(\mu,\ell)\ f(z))'}{I^m(\mu,\ell)\ f(z)-I^m(\mu,\ell)\ f(-z)}-1\right\}\right.\\ &< \left.\beta\left|\alpha\frac{z(I^m(\mu,\ell)\ f\ (z))'}{I^m(\mu,\ell)\ f(z)-I^m(\mu,\ell)\ f(-z)}+1\right|\right\} \qquad (11)\end{aligned}$$

where $m\in N_0,\mu,\ell\geq 0,z\in U$ and $I^m(\mu,\ell)$ is extended multiplier transformations operator(see[2]);

$$\begin{aligned}(iii) S_s^*(z+\sum_{k=2}^{\infty}C_k(b,s)z^k;\alpha,\beta) &= S_b^s(\alpha,\beta)\\ &= \left\{f\in A:\Re\left\{\frac{z(\jmath_b^S\ f(\ z))'}{\jmath_b^s\ f(z)-\jmath_b^s\ f(-z)}-1\right\}\right.\\ &\left.<\beta\left|\alpha\frac{z(\jmath_b^s\ f\ (z))'}{\jmath_b^s\ f(z)-\jmath_b^s\ f(-z)}+1\right|\right\}\end{aligned}$$

where

$$C_k(b,s)=\left|\left(\frac{1+b}{k+b}\right)^s\right|\left(b\in\mathbb{C}\setminus\mathbb{Z}^-,\mathbb{Z}^-=\mathbb{Z}\setminus\mathbb{N}_0;s\in\mathbb{C};z\in U\right) \qquad (12)$$

and $\jmath_b^s$ is the Srivastava-Attyia operator(see[11]).

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2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that 0 $\leq\alpha\leq 1,\ 0<\beta\leq 1,\ 0\leq\frac{2(1-\beta)}{1+\alpha\beta}<1$ and $z\in U$. We shall use the technique of Dziok] to prove the following theorems.

Theorem 1. Let the function $f(z)$ be defined by (4) and $(f*g)(z)-(f*g)(-z)$ $\neq 0$ for $z\neq 0$. Then $f(z)\in S_s^*T(g;\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\left\{(1+\alpha\beta)\,k-(1-\beta)[1-(-1)^k]\right\}a_kb_k\leq\beta(2+\alpha)-1. \qquad (13)$$

Proof. Let $|z| = 1$. Then we have

$$
\begin{aligned}
& |z\ (f*g)'(z) - (f*g)(z) + (f*g)(-z)| - \beta\,|\alpha z\ (f*g)'(z) + (f*g)(z) - (f*g)(-z)| \\
= & \left| z + \sum_{k=2}^{\infty}[k-1+(-1)^k]a_k b_k z^k \right| - \beta \left| (\alpha+2)z - \sum_{k=2}^{\infty}[\alpha k + 1 - (-1)^k]a_k b_k z^k \right| \\
& \leq \sum_{k=2}^{\infty}\left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\} a_k b_k - [\beta(\alpha+2)-1] \leq 0.
\end{aligned}
$$

Hence, by the maximum modulus theorem, we have $(f*g)(z) \in S_s^*T(g;\alpha,\beta)$. For the converse, assume that

$$
\begin{aligned}
& \left| \frac{\frac{z(f*g)'(z)}{(f*g)(z)-(f*g)(-z)} - 1}{\alpha\frac{z(f*g)'(z)}{(f*g)(z)-(f*g)(-z)} + 1} \right| \\
= & \left| \frac{-z - \sum\limits_{k=2}^{\infty}[k-1+(-1)^k]a_k b_k z^k}{(\alpha+2)z - \sum\limits_{k=2}^{\infty}[\alpha k+1-(-1)^k]a_k b_k z^k} \right| < \beta.
\end{aligned}
$$

Since $|\Re z| \leq |z|$ for all z, we have

$$
\Re\left\{ \frac{z + \sum\limits_{k=2}^{\infty}[k-1+(-1)^k]a_k b_k z^k}{(\alpha+2)z - \sum\limits_{k=2}^{\infty}[\alpha k+1-(-1)^k]a_k b_k z^k} \right\} < \beta. \tag{14}
$$

Choose values of z on the real axis so that $\frac{z(f*g)'(z)}{(f*g)(z)-(f*g)(-z)}$ is real and $(f*g)(z) - (f*g)(-z) \neq 0$ for $z \neq 0$. Upon clearing the denominator in (14) and letting $z \to 1^-$ through real valuse, we obtain

$$
1 + \sum_{k=2}^{\infty}[k-1+(-1)^k]a_k b_k \leq \beta(\alpha+2) - \beta\sum_{k=2}^{\infty}[\alpha k+1-(-1)^k]a_k b_k.
$$

These gives the required condition

Corollary 1. Let the function $f(z)$ defined by (4) be in the class $S_s^*T(g;\alpha,\beta)$. Then we have

$$
a_k \leq \frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\}\, b_k} \qquad (k \geq 2). \tag{15}
$$

The equality in (15) is attained the function $f(z)$ given by

$$
f(z) = z - \frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\}\, b_k} z^k \qquad (k \geq 2). \tag{16}
$$

Theorem 2. Let the function $f(z)$ be defined by (4). Then $f\ (z) \in S_c^*T(g;\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\{(1+\alpha\beta)k-2(1-\beta)\}\,a_kb_k \leq \beta(2+\alpha)-1. \tag{17}$$

Corollary 2. Let the function $f(z)$ defined by (4) be in the class $S_c^*T(g;\alpha,\beta)$. Then we have

$$a_k \leq \frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k-2(1-\beta)\}\,b_k} \qquad (k \geq 2). \tag{18}$$

The equality in (18) is attained for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k-2(1-\beta)\}\,b_k}z^k \qquad (k \geq 2). \tag{19}$$

Theorem 3. Let the function $f(z)$ be defined by (4). Then $f(z) \in S_{sc}^*T(g;\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\}a_kb_k \leq \beta(2+\alpha)-1. \tag{20}$$

Corollary 3. Let the function $f(z)$ defined by (4) be in the class $S_{sc}^*T(g;\alpha,\beta)$. Then we have

$$a_k \leq \frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)-(1-\beta)[1-(-1)^k\}\,b_k} \qquad (k \geq 2). \tag{21}$$

The equality in (21) is attained for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\}\,b_k}z^k \qquad (k \geq 2). \tag{22}$$

By taking $\alpha = 0$ and $\beta = 1-\gamma \quad (0 < \gamma < \frac{1}{2})$ in Theorem 1, we have the following corollary :

Corollary 4. Let the function $f(z)$ defined by (4). Then $f(z) \in S_s^*(\gamma)$ if and only if

$$\sum_{k=2}^{\infty}\left\{k-\gamma[(1-(-1)^k]\right\}a_kb_k \leq 1-2\gamma. \tag{23}$$

3. Distortion Theorems

Theorem 4. Let the function $f(z)$ defined by (4) be in the class $S_s^*T(g;\alpha,\beta)$.Then for $|z| = r < 1$, we have

$$|f(z)| \geq r - \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)b_2}r^2 \tag{24}$$

and

$$|\ f(z)| \leq r + \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)b_2}r^2, \tag{25}$$

provided that $b_{k+1} \geq b_k > 0 \quad (k \geq 2)$. The equalities in (24),(25) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)b_2} z^2 \tag{26}$$

at $z = r$ and $z = r^{i(2\tau+1)\pi}$ $(\tau \in z)$.

Proof. Since for $k \geq 2$,

$$[2(1+\alpha\beta)b_2] \leq \left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\} b_k,$$

using Theorem 1, we have

$$\begin{aligned} & [2(1+\alpha\beta)b_2] \sum_{k=2}^{\infty} a_k \\ \leq & \sum_{k=2}^{\infty} \left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\} a_k b_k \leq \beta(2+\alpha) - 1 \end{aligned} \tag{27}$$

that is, that

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)b_2}. \tag{28}$$

It follows from (4) and (28), we have

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)b_2} r^2$$

and

$$|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)b_2} r^2.$$

This completes the proof of Theorem 4.

Theorem 5. Let the function $f(z)$ defined by (4) be in the class $S_s^* T(g; \alpha, \beta)$.Then for $|z| = r < 1$, we have

$$|f'(z)| \geq 1 - \frac{[\beta(2+\alpha)-1][2-\beta(1+\alpha)]}{(1+\alpha\beta)^2 b_2} r \tag{29}$$

and

$$|f'(z)| \leq 1 + \frac{[\beta(2+\alpha)-1][2-\beta(1+\alpha)]}{(1+\alpha\beta)^2 b_2} r, \tag{30}$$

provided that $b_{k+1} \geq b_k > 0$ $(k \geq 2)$. The result is sharp for the function $f(z)$ given by (26).

Proof. From Theorem 1, we have

$$(1+\alpha\beta)b_2 \sum_{k=2}^{\infty} k a_k < (1-\beta) \sum_{k=2}^{\infty} a_k b_k + \beta[(2+\alpha)-1] \tag{31}$$

and

$$\sum_{k=2}^{\infty} a_k b_k < \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)}. \tag{32}$$

using (31) and (32), we have

$$\sum_{k=2}^{\infty} k a_k < \frac{[\beta(2+\alpha)-1][2-\beta(1+\alpha)]}{(1+\alpha\beta)^2 b_2},$$

and the remaning part of the proof is similar to the proof of Theorem 5.

Theorem 6. Let the function $f(z)$ defined by (4) be in the class $S_c^*T(g;\alpha,\beta)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq r - \frac{\beta(2+\alpha)-1}{2\beta(1+\alpha)b_2} r^2 \tag{33}$$

and

$$|f(z)| \geq r + \frac{\beta(2+\alpha)-1}{2\beta(1+\alpha)b_2} r^2, \tag{34}$$

prvided that $b_{k+1} \geq b_k > 0 \quad (k \geq 2)$. The equalities in (31),(32) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(2+\alpha)-1}{2\beta(1+\alpha)b_2} z^2 \tag{35}$$

at $z = r$ and $z = r^{i(2\tau+1)\pi}$ $(\tau \in z)$.

Proof. The proof is similar to the proof of Theorem 4.

Theorem 7. Let the function $f(z)$ defined by (4) be in the class $S_c^*T(g;\alpha,\beta)$.Then for $|z| = r < 1$, we have

$$|f'(z)| \geq 1 - \frac{\beta(2+\alpha)-1}{\beta(\alpha+1)b_2} r \tag{36}$$

and

$$|f'(z)| \leq 1 + \frac{\beta(2+\alpha)-1}{\beta(\alpha+1)b_2} r, \tag{37}$$

provided that $b_{k+1} \geq b_k > 0$ $(k \geq 2)$. The result is sharp for the function $f(z)$ given by (33).

Proof. The proof is similar to the proof of Theorem 5.

4. Extreme points

Theorem 8. The class $S_s^*T(g;\alpha,\beta)$ is closed under convex linear combination.

Proof. Let the functions $f_j(z) = z - \sum\limits_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0)$ be in the class $S_s^*T(g;\alpha,\beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) (0 \leq \lambda \leq 1) \tag{38}$$

is in the class $S_s^*T(g;\alpha,\beta)$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{k=2}^{\infty} [\lambda a_{k,1} + (1-\lambda) a_{k,2}] z^k,$$

with the aid of Theorem 1, we have

$$\sum_{k=2}^{\infty}\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\}b_k[\lambda a_{k,1}+(1-\lambda)a_{k,2}]\leq[\beta(2+\alpha)-1],$$

which implies that $h(z)\in S_s^*T(g;\alpha,\beta)$.

Theorem 9. Let $f_1(z)=z$ and

$$f_k(z)=z-\frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\}\,b_k}z^k \qquad (k\geq 2) \tag{39}$$

for $0\leq\alpha\leq 1$, $0<\beta\leq 1$. Then $f(z)$ is in the class $S_s^*T(g;\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z) \tag{40}$$

where $\lambda_k\geq 0$, $(k\geq 1)$ and $\sum\limits_{k=1}^{\infty}\lambda_k=1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\lambda_k f_k(z)\\ &= z-\sum_{k=2}^{\infty}\frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\}\,b_k}\lambda^k z^k. \end{aligned} \tag{41}$$

Then we get

$$\frac{\sum\limits_{k=2}^{\infty}\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\}b_k}{\beta(2+\alpha)-1}\cdot\frac{\beta(2+\alpha)-1}{\sum\limits_{k=2}^{\infty}\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\}b_k}\lambda_k$$

$$=\sum_{k=1}^{\infty}\lambda_k=1-\lambda_1\leq 1. \tag{42}$$

By virtue of Theorem 1, this shows that $f(z)\in S_s^*T(g;\alpha,\beta)$. On the other hand, suppose that the function $f(z)$ defined by (4) is in the class $S_s^*T(g;\alpha,\beta)$. Again, by using Theorem 1, we can show that

$$a_k\leq\frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k\}\,b_k} \qquad (k\geq 2). \tag{43}$$

Setting

$$\lambda_k=\frac{\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k\right\}a_kb_k}{\beta(2+\alpha)-1} \qquad (k\geq 2), \tag{44}$$

and

$$\lambda_1=1-\sum_{k=2}^{\infty}\lambda_k \tag{45}$$

we can see that $f(z)$ can be expressed in the form (40). This completes the proof of Theorem 9.

Corollary 5. The extreme points of the class $S_s^*T(g;\alpha,\beta)$ are the functions $f_k(z)$ $(k \geq 1)$ given by Theorem 9. Similarly we can prove the following results.

Theorem 10. Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{\beta(2+\alpha)-1}{\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\}\, b_k} z^k \qquad (k \geq 2)$$

for $0 \leq \alpha \leq 1, 0 < \beta \leq 1$.Then $f(z)$ is in the class $S_{sc}^*T(g;\alpha,\beta)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$, where $\lambda_k \geq 0$ $(k \geq 0)$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Corollary 6. The extreme points of the class $S_{sc}^*T(g;\alpha,\beta)$ are the functions $f_k(z)$ $(k \geq 1)$ given by Theorem 10.

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 11. Let the function $f(z)$ defined by (4) be in the class $S_s^*T(g;\alpha,\beta)$, then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in $|z| < r_1$, where

$$r_1 = \inf_k \left\{ \frac{(1-\delta)\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k\right\} b_k}{k\ [\beta(2+\alpha)-1]} \right\}^{\frac{1}{k-1}} \qquad (k \geq 2). \tag{46}$$

The result is sharp with extremal function given by (16).

Proof. For close-to-convexity it is sufficient to show that $|f'(z)-1| \leq 1-\delta$ for $|z| < r_1$. We have

$$|f'(z)-1| \leq \sum_{k=2}^{\infty} k a_k \left|z\right|^{k-1}.$$

Thus $|f'(z)-1| \leq 1-\delta$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\delta}\right) a_k \left|z\right|^{k-1} \leq 1. \tag{47}$$

According to Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\}}{\beta(2+\alpha)-1} a_k b_k \leq 1. \tag{48}$$

Hence (47) will be true if

$$\left(\frac{k}{1-\delta}\right) \left|z\right|^{k-1} \leq \frac{\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\}}{\beta(2+\alpha)-1} b_k$$

or if

$$|z| \leq \left\{ \frac{(1-\delta)\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\} b_k}{k[\beta(2+\alpha)-1]} \right\}^{\frac{1}{k-1}} \qquad (k \geq 2). \tag{49}$$

Theorem 12. Let the function $f(z)$ defined by (4) be in the class $S_s^*T(g;\alpha,\beta)$, then $f(z)$ is starlike of the order δ $(0 \le \delta < 1)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(1-\delta)\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\} b_k}{(k-\delta)[\beta(2+\alpha)-1]} \right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{50}$$

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1-\delta \quad for \ \ |z| < r_2.$$

We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k |z|^{k-1}}{1-\sum\limits_{k=2}^{\infty} a_k |z|^{k-1}}.$$

Thus $\left| \frac{zf'(z)}{f\ (z)} - 1 \right| \le 1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{(k-\delta)a_k |z|^{k-1}}{(1-\delta)} \le 1. \tag{51}$$

Hence, by using (48), (51) will be true if

$$\frac{(k-\delta)|z|^{k-1}}{(1-\delta)} \le \frac{\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\} b_k}{\beta(2+\alpha)-1}$$

or if

$$|z| \le \left\{ \frac{(1-\delta)\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\} b_k}{(k-\delta)\beta(2+\alpha)-1} \right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{52}$$

Theorem 12, follows easily from (52).

Corollary 7. Let the function $f(z)$ defined by (4) be in the class $S_s^*T(g;\alpha,\beta)$, then f(z) is convex of order δ $(0 \le \delta < 1)$ in $|z| < r_3$, where

$$r_3 = \inf_k \left\{ \frac{(1-\delta)\left\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\right\} b_k}{k(k-\delta)[\beta(2+\alpha)-1]} \right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{53}$$

The result is sharp with the extremal function given by (16).

6. Integral Operators.

Theorem 13. Let the function $f(z)$ be in the class $S_s^*T(g;\alpha,\beta)$ and c be a real number such that $c > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f\ (t)\ dt \tag{54}$$

also in the class $S_s^*T(g;\alpha,\beta)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} d_k z^k \tag{55}$$

where

$$d_k = \left(\frac{c+1}{c+k}\right) a_k b_k \ . \tag{56}$$

Therefore

$$\begin{aligned} & \sum_{k=2}^{\infty} \left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\} d_k \\ = \ & \sum_{k=2}^{\infty} \left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\} \left(\frac{c+1}{c+k}\right) a_k b_k \\ \leq \ & \sum_{k=2}^{\infty} \left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\} a_k b_k \\ \leq \ & \beta(2+\alpha) - 1. \end{aligned} \tag{57}$$

Since $f(z) \in S_s^*T(g;\alpha,\beta)$. Hence, by Theorem 1, $F(z) \in S_s^*T(g;\alpha,\beta)$.

Theorem 14. Let c be a real number such that $c > -1$. If $F(z) \in S_s^*T(g;\alpha,\beta)$. Then the function $f(z)$ defined by (54) is univalent in $|z| < r^*$,where

$$r^* = \inf_k \left\{ \frac{\left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k\right\}(c+1)b_k}{k\ [\beta(2+\alpha)-1](c+k)} \right\}^{\frac{1}{k-1}} \qquad (\mathrm{k} \geq 2). \tag{58}$$

The result is sharp.

Proof. Let $F(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k \quad (a_k\ , b_k \geq 0)$. It follows from (54) that

$$f(z) = z^{1-c}\frac{[z^c F(z)]'}{c+1} = z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k. \tag{59}$$

In order to obtain the required result it suffices in $|z| < r^*$. Now

$$|f'(z)\ -1| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k\, |z|^{k-1}.$$

Thus $|f'(z)\ -1| < 1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k\, |z|^{k-1} < 1. \tag{60}$$

Hence by using (48),(60) will be satisfied if

$$\frac{k(c+k)}{(c+1)}\, |z|^{k-1} < \frac{\left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\}}{[\beta(2+\alpha)-1]},$$

i.e, if

$$|z| < \left[\frac{\left\{(1+\alpha\beta)k - (1-\beta)[1-(-1)^k]\right\}(c+1)}{k(c+k)[\beta(2+\alpha)-1]} \right]^{\frac{1}{k-1}} \qquad (k \geq 2). \tag{61}$$

Therefore $F(z)$ is univalent in $|z| < r^*$. Sharpness follows if we take

$$f(z) = z - \frac{k(c+k)[\beta(2+\alpha)-1]}{\{(1+\alpha\beta)k-(1-\beta)[1-(-1)^k]\}\,(c+1)b_k} z^k \tag{62}$$

$(k \geq 2; c > -1)$.

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