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GENERALIZATION OF TITCHMARSH'S THEOREM FOR THE JACOBI-DUNKL TRANSFORM

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Abstract. In this paper, using a generalized Jacobi-Dunkl translation operator, we prove an analog of Titchmarsh's Theorem for functions satisfying the Jacobi-Dunkl Lipschitz condition in $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$.

1. INTRODUCTION

Titchmarsh's [[10],Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz Condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1 [[10]] Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then, the following are equivalents:
(a) $\|f(t+h) - f(t)\| = 0(h^\alpha), \quad \text{as} \quad h \to 0,$
(b) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}) \quad \text{as} \quad r \to \infty,$
where $\widehat{f}$ stand for the Fourier transform of f.

In this paper, we prove in analog of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the Jacobi-Dunkl Lipschitz condition in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$. For this purpose, we use the generalized translation operator. similar results have been established in the context of non compact rank one Riemannian symetric spaces [[9]].

In section 2 below, we recapitulate from [[1],[2],[3],[5]] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$.
Section 3 is devoted to the main result after defining the class $Lip(\psi, 2, \alpha, \beta)$ of functions in $L^2_{\alpha,\beta}(\mathbb{R})$ satisfying the ψ-Jacobi-Dunkl Lipschitz condition correspondent to the generalized Jacobi-Dunkl translation.

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2. NOTATION AND PRELIMINARIES

The Jacobi-Dunkl function with parameters $(\alpha,\beta), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, defined by the formula

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) & \text{if } \lambda \in \mathbb{C}\backslash\{0\} \\ 1 & \text{if } \lambda = 0 \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh(x))^2\right),$$

F is the Gausse hypergeometric function (see [[1],[6],[7]]).
$\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U} & ,\lambda \in \mathbb{C} \\ \mathcal{U}(0) = 1 \end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}.$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by

$$D\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'(x)}{A(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive. The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2+\rho^2}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form above (see [[2]])

$$\psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R}.$$

Denote $L^2_{\alpha,\beta}(\mathbb{R}) = L^2_{\alpha,\beta}(\mathbb{R}, A_{\alpha,\beta}(t)dt)$ the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^2_{\alpha,\beta}(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(t)|^2 A_{\alpha,\beta}(t)dt\right)^{1/2} < +\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels , we define the Jacobi-Dunkl transform of a function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ by

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_{\mathbb{R}} f(t)\psi_\lambda^{\alpha,\beta}(t)A_{\alpha,\beta}(t)dt, \quad \lambda \in \mathbb{R},$$

and the inversion formula

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}f(\lambda)\psi_{-\lambda}^{\alpha,\beta}(t)d\sigma(\lambda),$$

where

$$d\sigma(\lambda)=\frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}1_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu)=\frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))},\quad \mu\in\mathbb{C}\backslash(i\mathbb{N})$$

and $1_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
Denote $L^2_\sigma(\mathbb{R})=L^2(\mathbb{R},d\sigma(\lambda))$. The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2_\sigma(\mathbb{R})$, i.e.

$$\|f\|:=\|f\|_{L^2_{\alpha,\beta}(\mathbb{R})}=\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2_\sigma(\mathbb{R})}. \tag{1}$$

The operator of Jacobi-Dunkl translation is defined by

$$T_xf(y)=\int_{\mathbb{R}}f(z)d\nu^{\alpha,\beta}_{x,y}(z),\quad \forall x,y\in\mathbb{R}$$

where $\nu^{\alpha,\beta}_{x,y}(z),\quad x,y\in\mathbb{R}$ are the signed measures given by

$$d\nu^{\alpha,\beta}_{x,y}(z)=\begin{cases}K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{if } x,y\in\mathbb{R}^*\\ \\ \delta_x & \text{if } y=0\\ \delta_y & \text{if } x=0\end{cases}$$

Here, δ_x is the Dirac measure at x. And,

$$\begin{aligned}K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}1_{I_{x,y}}\times\int_0^\pi\rho_\theta(x,y,z)\\ &\times (g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta\\ I_{x,y} &= [-|x|-|y|,-||x|-|y||]\cup[||x|-|y||,|x|+|y|]\\ \rho_\theta(x,y,z) &= 1-\sigma^\theta_{x,y,z}+\sigma^\theta_{z,x,y}+\sigma^\theta_{z,y,x}\end{aligned}$$

$$\forall z\in\mathbb{R},\theta\in[0,\pi],\sigma^\theta_{x,y,z}=\begin{cases}\frac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)} & ,\text{if } xy\neq 0\\ \\ 0 & ,\text{if } xy=0\end{cases}$$

$$g_\theta(x,y,z)=1-\cosh^2(x)-\cosh^2(y)-\cosh^2(z)+2\cosh(x)\cosh(y)\cosh(z)\cos\theta$$

$$t_+=\begin{cases}t & ,\text{if } t>0\\ \\ 0 & ,\text{if } t\leq 0\end{cases}$$

and,

$$M_{\alpha,\beta}=\begin{cases}\frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & ,\text{if } \alpha>\beta\\ \\ 0 & ,\text{if } \alpha=\beta\end{cases}$$

In [[2]], we have

$$\mathcal{F}_{\alpha,\beta}(T_hf)(\lambda)=\psi^{\alpha,\beta}_\lambda(h)\mathcal{F}_{\alpha,\beta}(f)(\lambda);\quad \lambda,h\in\mathbb{R}. \tag{2}$$

For $\alpha\geq\frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^\infty\frac{(-1)^n(\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)},\quad z\in\mathbb{C}.$$

Moreover, we see that

$$\lim_{z\to 0}\frac{j_\alpha(z)-1}{z^2}\neq 0,$$

by consequence , there exists $C_1>0$ and $\eta>0$ satisfying

$$|z|\leq\eta\Rightarrow|j_\alpha(z)-1|\geq C_1|z|^2 \tag{3}$$

Lemma 2.1. The following inequalities are valids for Jacobi functions $\varphi_\mu^{\alpha,\beta}(t)$:
(c) $|\varphi_\mu^{\alpha,\beta}(t)|\leq 1$,
(d) $|1-\varphi_\mu^{\alpha,\beta}(t)|\leq t^2(\mu^2+\rho^2)$.
Proof. (See[[8]],Lemma 3.1,Lemma 3.2).

Lemma 2.2. Let $\alpha\geq\beta\geq\frac{-1}{2},\alpha\neq\frac{-1}{2}$. Then for $|\nu|\leq\rho$, there exists a positive constant C_2 such that

$$|1-\varphi_{\mu+i\nu}^{\alpha,\beta}(t)|\geq C_2|1-j_\alpha(\mu t)|.$$

Proof. (See[[4]],Lemma 9).

3. MAIN RESULT

In this section we give the main result of this paper. We need first to define the ψ-Jacobi-Dunkl Lipschitz class.

Definition 3.1. A function $f\in L^2_{\alpha,\beta}(\mathbb{R})$ is said to be in the ψ-Jacobi-Dunkl Lipschitz class, denoted by $Lip(\psi,2,\alpha,\beta)$, if

$$\|N_hf\|=O(\psi(h)),\quad as\quad h\to 0,$$

where $N_h=T_h+T_{-h}-2I$, I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$ and ψ is a continuous increasing function on $[0,\infty),\psi(0)=0$, $\psi(ts)=\psi(t)\psi(s)$ for all $t,s\in[0,\infty)$ and this function verify

$$\int_0^{1/h}s\psi(s^{-2})ds=O(\frac{1}{h^2}\psi(h^2))\quad\text{as}\quad h\to 0.$$

Lemma 3.2. For $f\in L^2_{\alpha,\beta}(\mathbb{R})$, then

$$\|N_hf\|^2=4\int_{\mathbb{R}}|\varphi_\mu^{\alpha,\beta}(h)-1|^2|\mathcal{F}_{\alpha,\beta}f(\lambda)|^2d\sigma(\lambda).$$

Proof. We us formula (2), we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_hf)(\lambda)=(\psi_\lambda^{\alpha,\beta}(h)+\psi_\lambda^{\alpha,\beta}(-h)-2)\mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi_\lambda^{\alpha,\beta}(h)=\varphi_\mu^{\alpha,\beta}(h)+i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{\alpha+1,\beta+1}(h),$$

$$\psi_\lambda^{\alpha,\beta}(-h)=\varphi_\mu^{\alpha,\beta}(-h)-i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{\alpha+1,\beta+1}(-h),$$

and $\varphi_\mu^{\alpha,\beta}$ is even (see [[2]]), then

$$\mathcal{F}_{\alpha,\beta}(N_hf)(\lambda)=2(\varphi_\mu^{\alpha,\beta}(h)-1)\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By Parseval's identity (formula (1)), we have the result.

Theorem 3.3. Let $f \in L^2_{\alpha,\beta}(\mathbb{R})$. Then the following are equivalents

(i) $f \in Lip(\psi, 2, \alpha, \beta)$,

(ii) $\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(\psi(r^{-2})), \quad as \quad r \to \infty.$

Proof. $(i) \Rightarrow (ii)$. Assume that $f \in Lip(\psi, 2, \alpha, \beta)$, then we have

$$\|N_h f\| = O(\psi(h)), \quad as \quad h \to 0.$$

From lemma 3.2, we have

$$\|N_h f\|^2 = 4 \int_{\mathbb{R}} |1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)$$

By (3) and lemma 2.2, we get

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |1-\varphi^{\alpha,\beta}_\mu(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2 C_2^2 \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\mu h|^4 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

From $\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}$ we have

$$\begin{aligned} \left(\frac{\eta}{2h}\right)^2 \quad &- \quad \rho^2 \leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2 \\ &\Rightarrow \quad \mu^2 h^2 \geq \frac{\eta^2}{4} - \rho^2 h^2. \end{aligned}$$

Take $h \leq \frac{\eta}{3\rho}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\eta)$.
So,

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |1-\varphi^{\alpha,\beta}_\mu(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2 C_2^2 C_3^2 \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

There exists then a positive constant C such that

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C \int_{\mathbb{R}} |1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C\psi(h^2).$$

For all $0 < h < \frac{\eta}{3\rho}$. Then we have ,

$$\int_{r\leq|\lambda|\leq 2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C\psi(2^{-2}\eta r^{-2}), \quad r \to \infty.$$

Thus there exists $K > 0$ such that

$$\int_{r\leq|\lambda|\leq 2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq K\psi(r^{-2}), \quad r \to \infty.$$

Furthermore , we obtain

$$\begin{aligned} \int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \quad &= \quad \sum_{i=0}^{\infty} \int_{2^i r\leq|\lambda|\leq 2^{i+1}r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\ &= \quad O(\psi(r^{-2}) + \psi(2^{-2}r^{-2}) + ...) \\ &= \quad O(\psi(r^{-2}) + \psi(r^{-2}) + ...) \\ &= \quad O(\psi(r^{-2})). \end{aligned}$$

This proves that

$$\int_{|\lambda|\geq r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)=O(\psi(r^{-2}))),\quad as\quad r\to\infty.$$

$(ii)\Rightarrow(i)$. Suppose now that

$$\int_{|\lambda|\geq r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)=O(\psi(r^{-2})),\quad as\quad r\to\infty,$$

and write

$$\begin{aligned}\int_{\mathbb{R}}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\ &+ \int_{|\lambda|\geq\frac{1}{h}}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda).\end{aligned}$$

Using the inequality (c) of lemma 2.1, we get

$$\int_{|\lambda|\geq\frac{1}{h}}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\leq 4\int_{|\lambda|\geq\frac{1}{h}}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda).$$

Then

$$\int_{|\lambda|\geq\frac{1}{h}}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)=O(\psi(h^2)),\quad as\quad h\to 0. \tag{4}$$

Set

$$\phi(\lambda)=\int_\lambda^\infty|\mathcal{F}_{\alpha,\beta}(f)(x)|^2d\sigma(x).$$

An integration by parts gives

$$\begin{aligned}\int_0^{\frac{1}{h}}\lambda^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &= \int_0^{\frac{1}{h}}-\lambda^2\phi'(\lambda)d\lambda\\ &= -\frac{1}{h^2}\phi(\frac{1}{h})+2\int_0^{\frac{1}{h}}\lambda\phi(\lambda)d\lambda\\ &\leq 2\int_0^{\frac{1}{h}}\lambda\psi(\lambda^{-2})d\lambda\\ &= O(\frac{1}{h^2}\psi(h^2)).\end{aligned}$$

From lemma 2.1 , we get

$$\begin{aligned}\int_{|\lambda|\leq\frac{1}{h}}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &\leq \int_{|\lambda|\leq\frac{1}{h}}|1-\varphi_\mu^{\alpha,\beta}(h)||\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\ &\leq \int_{|\lambda|\leq\frac{1}{h}}(\mu^2+\rho^2)h^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\ &\leq h^2\int_{|\lambda|\leq\frac{1}{h}}\lambda^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\ &= O(h^2\frac{1}{h^2}\psi(h^2))\\ &= O(\psi(h^2)).\end{aligned}$$

Hence,

$$\int_{|\lambda|\leq\frac{1}{h}}|1-\varphi_{\mu}^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)=O(\psi(h^2)). \tag{5}$$

Finally, we conclude from (4) and (5) that

$$\begin{aligned}\int_{\mathbb{R}}|1-\varphi_{\mu}^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}}+\int_{|\lambda|\geq\frac{1}{h}}\\ &= O(\psi(h^2))+O(\psi(h^2))\\ &= O(\psi(h^2)).\end{aligned}$$

And this ends the proof.

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