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GENERALIZED FOURIER-DUNKL TRANSFORM OF (ψ, γ)-GENERALIZED DUNKL LIPSCHITZ FUNCTIONS

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Abstract. Using a generalized translation operator, we obtain an analog of Theorem 5.2 in Younis [[5]] for the generalized Fourier-Dunkl transform for functions satisfying the (ψ, γ)-generalized Dunkl Lipschitz condition in the space $L^2_{\alpha,n}$.

1. Introduction and Preliminaries

Theorem 5.2 in Younis [[5]] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

Theorem 1.1 [[5]] Let $f \in L^2(\mathbb{R})$. Then the following are equivalents

(a) $\quad \|f(x+h) - f(x)\| = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right), \quad$ as $\quad h \to 0, 0 < \delta < 1, \gamma \geq 0$

(b) $\quad \displaystyle\int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right), \quad$ as $\quad r \to \infty,$

where $\widehat{f}$ stand for the Fourier transform of f.

In this paper, we consider a first-order singular differential-difference operator Λ on $\mathbb{R}$ which generalizes the Dunkl operator Λ_α. We prove an analog of Theorem 1.1 in the generalized Fourier-Dunkl transform associated to Λ in $L^2_{\alpha,n}$. For this purpose, we use a generalized translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ. Further details can be found in [[1]] and [[6]]. In all what follows assume where $\alpha > -1/2$ and n a non-negative integer.

Consider the first-order singular differential-difference operator defined on $\mathbb{R}$ by

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x} - 2n \frac{f(-x)}{x}.$$

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For $n = 0$, we define the differential-difference operator Λ_α by

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl (see [[3]], [[4]]) in connection with a generalization of the classical theory of spherical harmonics.
Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ..$$

Let $L^p_{\alpha,n}$, $1 \leq p < \infty$,be the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1} f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx \right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2(\mathbb{R}, |x|^{2\alpha+1})$.
The one-dimensional Dunkl kernel is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), z \in \mathbb{C}, \tag{1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m! \Gamma(m+\alpha+1)}, z \in \mathbb{C},$$

is the normalized spherical Bessel function of index α. It is well-known that the functions e_α are the solutions of the differential-difference equation

$$\Lambda_\alpha u = \lambda u, u(0) = 1.$$

Lemma 1.2 [[2]] For $x \in \mathbb{R}$ the following inequalities are fulfilled
i) $|e_\alpha(ix)| \leq 1$,
ii) $|1 - e_\alpha(ix)| \leq |x|$,
iii) $|1 - e_\alpha(ix)| \geq c$ with $|x| \geq 1$, where $c > 0$ is a certain constant which depends only on α.

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} e_{\alpha+2n}(i\lambda x).$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (1).
Proposition 1.3
i) φ_λ satisfies the differential equation

$$\Lambda \varphi_\lambda = i\lambda \varphi_\lambda.$$

ii) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq |x|^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Dunkl transform we call the integral transform

$$\mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x)\varphi_{-\lambda}(x)|x|^{2\alpha+1}dx, \lambda \in \mathbb{R}, f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Lambda(f) \in L^1_{\alpha+2n} = L^1(\mathbb{R}, |x|^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Dunkl transform is given by the formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}.$$

Proposition 1.4
i) For every $f \in L^2_{\alpha,n}$,

$$\mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda\mathcal{F}_\Lambda(f)(\lambda).$$

ii) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

iii) The generalized Fourier-Dunkl transform $\mathcal{F}_\Lambda$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2(\mathbb{R}, \mu_{\alpha+2n})$.

The generalized translation operators τ^x, $x \in \mathbb{R}$, tied to Λ are defined by

$$\begin{aligned}\tau^x f(y) &= \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt \\ &+ \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(-\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt,\end{aligned}$$

where

$$A(t) = \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)}(1+t)(1-t^2)^{\alpha+2n-1/2}.$$

Proposition 1.5 Let $x \in \mathbb{R}$ and $f \in L^2_{\alpha,n}$. Then $\tau^x f \in L^2_{\alpha,n}$ and

$$\|\tau^x f\|_{2,\alpha,n} \leq 2x^{2n}\|f\|_{2,\alpha,n}.$$

Furthermore,

$$\mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n}e_{\alpha+2n}(i\lambda x)\mathcal{F}_\Lambda(f)(\lambda). \tag{2}$$

2. **Main Results**

In this section we give the main result of this paper. We need first to define (ψ,γ)-generalized Dunkl Lipschitz class.

Definition 2.1. Let $\gamma \geq 0$. A function $f \in L^2_{\alpha,n}$ is said to be in the (ψ,γ)-generalized Dunkl Lipschitz class, denoted by $DLip(\psi,\gamma,2)$, if

$$\|\tau^h f(x) - h^{2n} f(x)\|_{2,\alpha,n} = O\left(\frac{h^{2n}\psi(h)}{(\log\frac{1}{h})^{\gamma}}\right) \quad \text{as} \quad h \to 0.$$

where the following three conditions hold:
(a) ψ is a continuous increasing function on $[0,\infty)$,
(b) $\psi(0) = 0$, $\psi(ts) = \psi(t)\psi(s)$ for all $t,s \in [0,\infty)$,
(c) and

$$\int_0^{1/h} s\psi(s^{-2})(\log s)^{-2\gamma} ds = O\left(h^{-2}\psi(h^2)\left(\log\frac{1}{h}\right)^{-2\gamma}\right), \quad h \to 0.$$

Theorem 2.2. Let $f \in L^2_{\alpha,n}$. Then, the following statements are equivalent

(a) $f \in DLip(\psi,\gamma,2)$
(b) $\displaystyle\int_{|\lambda|\geq r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty.$

Proof. $(a) \Rightarrow (b)$.Let $f \in DLip(\psi,\gamma,2)$ .Then we have

$$\|\tau^h f(x) - h^{2n} f(x)\|_{2,\alpha,n} = O\left(\frac{h^{2n}\psi(h)}{(\log\frac{1}{h})^{\gamma}}\right) \quad \text{as} \quad h \to 0.$$

Formula (2) and Plancherel equality give

$$\|\tau^h f(x) - h^{2n} f(x)\|^2_{2,\alpha,n} = h^{4n}\int_{-\infty}^{+\infty} |e_{\alpha+2n}(i\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

If $|\lambda| \in [\frac{1}{h},\frac{2}{h}]$, then $|\lambda h| \geq 1$ and (iii) of Lemma 1.2 implies that $1 \leq \frac{1}{c^2}|e_{\alpha+2n}(i\lambda h) - 1|^2$. Then

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &\leq \frac{1}{c^2}\int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}} |e_{\alpha+2n}(i\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\
&\leq \frac{1}{c^2}\int_{-\infty}^{+\infty} |e_{\alpha+2n}(i\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\
&\leq \frac{h^{-4n}}{4c^2}\|\tau^h f(x) - h^{2n} f(x)\|^2_{2,\alpha,n} \\
&= O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right).
\end{aligned}$$

We obtain

$$\int_{r\leq|\lambda|\leq 2r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leq C\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}, \quad r \to \infty.$$

where C is a positive constant. Now,

$$\begin{aligned}\int_{|\lambda|\geq r}|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda) &= \sum_{i=0}^{\infty}\int_{2^ir\leq|\lambda|\leq 2^{i+1}r}|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda)\\ &\leq C\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}+\frac{\psi((2r)^{-2})}{(\log 2r)^{2\gamma}}+\frac{\psi((4r)^{-2})}{(\log 4r)^{2\gamma}}+\cdots\right)\\ &\leq C\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\left(1+\psi(2^{-2})+(\psi(2^{-2}))^2+(\psi(2^{-2}))^3+\cdots\right)\\ &\leq K_\delta\frac{\psi(r^{-2})}{(\log r)^{2\gamma}},\end{aligned}$$

where $K_\delta = C(1-\psi(2^{-2}))^{-1}$ since $\psi(2^{-2})<1$.
Consequently

$$\int_{|\lambda|\geq r}|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda) = O\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\right), \quad as \quad r\to\infty.$$

$(b)\Rightarrow(a)$. Suppose now that

$$\int_{|\lambda|\geq r}|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda) = O\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\right), \quad as \quad r\to\infty.$$

and write

$$\|\tau^h f(x)-h^{2n}f(x)\|^2_{2,\alpha,n} = h^{4n}(I_1+I_2),$$

where

$$I_1 = \int_{|\lambda|<\frac{1}{h}}|e_{\alpha+2n}(i\lambda h)-1|^2|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda),$$

and

$$I_2 = \int_{|\lambda|\geq\frac{1}{h}}|e_{\alpha+2n}(i\lambda h)-1|^2|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda).$$

Firstly, we use the formulas $|e_{\alpha+2n}(i\lambda h)|\leq 1$ and

$$I_2\leq 4\int_{|\lambda|\geq\frac{1}{h}}|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda) = \left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right), \quad as \quad h\to 0.$$

Set

$$\phi(x) = \int_x^{+\infty}|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda).$$

Integrating by parts we obtain

$$\begin{aligned}\int_0^x\lambda^2|\mathcal{F}_\Lambda f(\lambda)|^2d\mu_{\alpha+2n}(\lambda) &= \int_0^x-\lambda^2\phi'(\lambda)d\lambda = -x^2\phi(x)+2\int_0^x\lambda\phi(\lambda)d\lambda\\ &\leq C_1\int_0^x\lambda\psi(\lambda^{-2})(\log\lambda)^{-2\gamma}d\lambda = O(x^2\psi(x^{-2})(\log x)^{-2\gamma}),\end{aligned}$$

where C_1 is a positive constant.
Therefore, by using part (ii) of Lemma 1.2 we see that

$$\begin{aligned}\int_{-\infty}^{+\infty}|e_{\alpha+2n}(i\lambda h)-1|^2|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &= O\left(h^2\int_{|\lambda|<\frac{1}{h}}\lambda^2|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)\\ &+ \left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right)\\ &= O\left(h^2\frac{h^{-2}\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right)+O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right)\\ &= O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right),\end{aligned}$$

and this completes the proof.□

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