Electronic Journal of Mathematical Analysis and Applications
Vol. 4(1) Jan. 2016, pp. 168-174.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

STRONG CONVERGENCE OF JUNGCK ITERATIVE SCHEME IN HILBERT SPACE

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Abstract. In this paper we establish the strong convergence of Jungck-Ishikawa and Jungck-Mann iterative process using two self maps with E. A. property. We employing a general contractive condition and obtained unique coincidence fixed point in Hilbert space. Our result generalize , extend and improve the result of Qihou [13] , R. A. Rashwan and A. M. Saddeek [14].

1. Introduction

In 1974, Ishikawa [8] proved the convergence of two step iteration scheme of a Lipschitzian pseudo contractive map on a Hilbert space.

Ishikawa [8] iteration process $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Ty_n, \qquad n \geq 0,$$
$$y_n = (1 - \beta_n)\, x_n + \beta_n\, Tx_n, \qquad n \geq 0,$$

where x_0 in C is arbitrary , $\{\alpha_n\}$ and $\{\beta_n\}$ being sequences of real numbers in $[0, 1]$.

Let C be a nonempty closed convex subset of a Hilbert space X and $T : C \to C$ a selfmap. For arbitrary x_0 in C, we define Mann[9] iteration process $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Tx_n, \qquad n \geq 0.$$

Recently many author work on Jungck-Noor, Jungck-Ishikawa and Jungck-Mann iteration on different spaces. For more literature of fixed point on Hilbert space we refer to [4] ,[5] ,[8] , [15] and [16] .

In 2005, Singh et. al [17] proved the stability of Jungck type iterative procedure.

Let C be a nonempty closed convex subset of a Hilbert space X and S, T be two selfmap on C. An Jungck- Mann scheme for S and T is defined by

$$Sx_{n+1} = (1 - \alpha_n)\, Sx_n + \alpha_n\, Tx_n, \qquad n \geq 0, \tag{1}$$

where x_0 in C is arbitrary , $\{\alpha_n\}$ being sequence of real numbers in $[0, 1]$.

Convergence results are also proved by many authors (see [2], [3], [6] , [7] and [12]).

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Hilbert space , Jungck-Mann , Jungck-Ishikawa, E. A. property.
Submitted July 16, 2015.

In 2008, Olatinwo and Imoru [11] work on convergence results of Jungck-Mann and Jungck-Ishikawa iterations.

Let C be a nonempty closed convex subset of a Hilbert space X and S, T be two self maps on C. An Jungck-Ishikawa scheme for S and T is defined by

$$\begin{aligned} &x_0 \in C, \\ Sy_n &= (1-\beta_n)\, Sx_n + \beta_n\, Tx_n, \qquad n \geq 0, \\ Sx_{n+1} &= (1-\alpha_n)\, Sx_n + \alpha_n\, Ty_n, \qquad n \geq 0, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ being sequences of real numbers in $[0, 1]$.

In 2008, Olatinwo [10] proved convergence results using Jungck-Noor three step iteration process.

Let C be a nonempty closed convex subset of a Hilbert space X and S, T be two self maps on C. An Jungck-Noor scheme for S and T is defined by

$$\begin{aligned} &x_0 \in C, \\ Sx_{n+1} &= (1-\alpha_n)\, Sx_n + \alpha_n\, Ty_n, \qquad n \geq 0, \\ Sy_n &= (1-\beta_n)\, Sx_n + \beta_n\, Tz_n, \qquad n \geq 0, \\ Sz_n &= (1-\gamma_n)\, Sx_n + \gamma_n\, Tx_n, \qquad n \geq 0, \end{aligned} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ being sequences of real numbers in $[0, 1]$.

In order to prove our results we need the following lemmas:-

Lemma 1.1([13]) Let a real sequence $\{x_n\}_{n=1}^{\infty}$ satisfy the following conditions

$$x_{n+1} \leq \alpha x_n + \beta_n,$$

where $x_n \geq 0, \beta_n \geq 0, 0 \leq \alpha < 1$ and $\lim_n \beta_n = 0$, then $\lim_n x_n = 0$.

Lemma 1.2([8]) If X is a real Hilbert space , then the following inequality holds

$$\|\lambda x + (1-\lambda)\, y - z\|^2 = \lambda \|x - z\|^2 + (1-\lambda)\, \|y - z\|^2 - \lambda\,(1-\lambda)\, \|x - y\|^2,$$

for all $x, y, z \in X$ and $\lambda \in [0, 1]$.

Lemma 1.3([16]) Let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences such that

(i) $0 \leq \alpha_n, \beta_n < 1$,
(ii) $\beta_n \to 0$ as $n \to \infty$ and
(iii) $\sum \alpha_n \beta_n = \infty$.

Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum \alpha_n \beta_n (1-\beta_n)\, \gamma_n$ is bounded. Then γ_n has a subsequence which converges to zero.

On contractive conditions there exist an extensive literature on fixed point. We are using the condition

$$\|Tx - Ty\| \leq \max\{\|Sx - Tx\|, a\,\|Sx - Sy\|\} \qquad \textit{where } \frac{1}{4} \leq a < 1. \tag{4}$$

Definition 1.4 [1] Two self maps S and T satisfy E. A. property if there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

Example 1.1 Let $(R_+, |.|)$ and define S and T by $Sx = x^2$ and $Tx = x + 2$ we have that $Sx = Tx \Leftrightarrow x = 2$. Let $\{x_n\} \in X$ given by $x_n = 2 + \frac{1}{n}$, $n \geq 1$ then $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 4$, so S and T satisfy E. A. property.

Example 1.2 Let $X = [0,1]$ define T and S by

$$Tx = \begin{cases} 0 & x = [0,1) \\ \frac{1}{2} & x = 1 \end{cases}, Sx = x^2,$$

It is clear, it satisfy E.A. property at 0 and condition (4).
Example 1.3 Let $X = [0,1]$. Define T and S by

$$T\ x = \frac{x}{9}\ , Sx = \frac{x}{3}\ ,$$

It is clear it satisfy E.A. property at 0 and condition(4).
Example 1.4 Let $X = [0,1]$. Define T and S by

$$Tx = \frac{1}{2}(\frac{1}{2} + x), Sx = 1 - x,$$

It is clear, it satisfy E. A. property at $\frac{1}{2}$ and condition(4).

The main purpose of this paper is to prove the strong convergence results for two self maps by Jungck-Mann and Jungck-Ishikawa iteration.

2. Main result

Theorem 2.1 Let S and T are two self mappings in a bounded closed convex subset C of a Hilbert space X such that T and S satisfy E. A. property and condition (4) then

(i) S and T have unique coincidence point and

(ii) If Jungck-Mann iteration (1) with $\{\alpha_n\} \in [0,1]$ then Jungck-Mann iteration $\{Sx_n\}_{n=0}^{\infty}$ strongly converges to p.

Proof : Since S and T satisfy E. A. property there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = p \quad ,$$

since $S(C)$ is a closed subset of C therefore

$$\lim_{n\to\infty} Sx_n = p \in S(C),$$

and hence forth there exist a point $z \in X$ such that $Sz = p$.

Assert that $Sz = Tz$. By condition (4)

$$\begin{aligned} \|Sx_n - Tz\| &\leq \max\{\|Sz - Tz\|, a\|Sz - Sx_n\|\} \\ &\leq a\|p - Tz\|. \end{aligned}$$

Hence $Tz = Sz = p$ is a coincidence point of T and S.
Unique coincidence point. Let $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$ then

$$\begin{aligned} \|p_1 - p_2\| = \|Tz_1 - Tz_2\| &\leq \max\{\|Sz_1 - Tz_1\|, a\|Sz_1 - Sz_2\|\} \\ &\leq a\|p_1 - p_2\|. \end{aligned}$$

Therefore p is the unique coincidence fixed point of S and T.

For the strong convergence result we have

$$\|Sx_{n+1} - p\|^2 = \|(1-\alpha_n)Sx_n + \alpha_n Tx_n - p\|^2$$

$$\leq \alpha_n\|Tx_n - p\|^2 + (1-\alpha_n)\|Sx_n - p\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - Sx_n\|^2. \quad (5)$$

For the estimate of $\|Tx_n - p\|^2$ in (5), we get

$$\|Tx_n - p\|^2 \leq a^2\|Sx_n - p\|^2. \quad (6)$$

Substitute (6) in (5), we get

$$\|Sx_{n+1}-p\|^2 \leq a^2\alpha_n\|Sx_n-p\|^2+(1-\alpha_n)\|Sx_n-p\|^2-\alpha_n(1-\alpha_n)\|Tx_n-Sx_n\|^2$$
$$\leq [1-\alpha_n(1-k)]\|Sx_n-p\|^2-\alpha_n(1-\alpha_n)\|Tx_n-Sx_n\|^2.$$

Say $a^2=k$. Since $\{\alpha_n\}\in[0,1]$ and $\frac{1}{4}\leq a<1$ then $\frac{1}{4}\leq k<1$. Therefore $[1-\alpha_n(1-k)]>0$.

It follows from the boundedness of the C that $\|Tx_n-Sx_n\|$ is bounded. Thus

$$\lim_{n\to\infty}\alpha_n(1-\alpha_n)\|Tx_n-Sx_n\|^2=0.$$

It follows from Lemma 1.1 that

$$\lim_n Sx_n=p.$$

Therefore $\{Sx_n\}_{n=0}^{\infty}$ strongly converges to p.

Theorem 2.2 Let S and T are two be self mappings in a bounded closed convex subset C of a Hilbert space X such that T and S satisfy E. A. property and condition (4) then

(i) S and T have unique coincidence point and
(ii) If Jungck-Ishikawa iteration (2) with
  (a) $0\leq\alpha_n,\beta_n<1$,
  (b) $\beta_n\to 0$ as $n\to\infty$ and
  (c) $\sum\alpha_n\beta_n=\infty$.

Then Jungck-Ishikawa iteration $\{Sx_n\}_{n=0}^{\infty}$ strongly converges to p.

Proof :- The proof of (i) is the same as Theorem 2.1 we shall prove for the Ishikawa iteration $\{Sx_n\}_{n=0}^{\infty}$ strongly converges to p.

$$\|Sx_{n+1}-p\|^2=\|(1-\alpha_n)Sx_n+\alpha_nTy_n-p\|^2$$
$$\leq\alpha_n\|Ty_n-p\|^2+(1-\alpha_n)\|Sx_n-p\|^2-\alpha_n(1-\alpha_n)\|Ty_n-Sx_n\|^2. \quad (7)$$

For the estimate of $\|Ty_n-p\|^2$ in (7), we get

$$\|Ty_n-p\|^2\leq a^2\|Sy_n-p\|^2. \quad (8)$$

For the estimate of $\|Sy_n-p\|^2$ in (8), we get

$$\|Sy_n-p\|^2=\|(1-\beta_n)Sx_n+\beta_nTx_n-p\|^2$$
$$\leq\beta_n\|Tx_n-p\|^2+(1-\beta_n)\|Sx_n-p\|^2-\beta_n(1-\beta_n)\|Tx_n-Sx_n\|^2. \quad (9)$$

For the estimate of $\|Tx_n-p\|^2$ in (9), we get

$$\|Tx_n-p\|^2\leq a^2\|Sx_n-p\|^2. \quad (10)$$

Using condition (8), (9) and (10) in (7) we get

$$\|Sx_{n+1}-p\|^2\leq\left[1-\alpha_n+a^2\alpha_n-a^2\alpha_n\beta_n\right]\|Sx_n-p\|^2+a^4\alpha_n\beta_n\|Sx_n-p\|^2$$
$$-a^2\alpha_n\beta_n(1-\beta_n)\|Tx_n-Sx_n\|^2-\alpha_n(1-\alpha_n)\|Ty_n-Sx_n\|^2$$
$$\leq\left[1-\alpha_n+a^2\alpha_n-a^2\alpha_n\beta_n+a^4\alpha_n\beta_n\right]\|Sx_n-p\|^2$$
$$-a^2\alpha_n\beta_n(1-\beta_n)\|Tx_n-Sx_n\|^2-\alpha_n(1-\alpha_n)\|Ty_n-Sx_n\|^2.$$

Say $a^2=k$ since $\frac{1}{4}\leq a<1$ then $\frac{1}{4}\leq k<1$

$$\|Sx_{n+1}-p\|^2\leq[1-(1-k)\alpha_n-(1-k)k\alpha_n\beta_n]\|Sx_n-p\|^2$$

$$-k\alpha_n\beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2 - \alpha_n(1-\alpha_n)\|Ty_n - Sx_n\|^2,$$

since $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, $\beta_n \to 0$ as $n \to \infty$ and $\sum \alpha_n\beta_n = \infty$ then

$$[1-(1-k)\alpha_n - (1-k)k\alpha_n\beta_n] = \hbar > 0.$$

$$\|Sx_{n+1} - p\|^2 \le \hbar\|Sx_n - p\|^2 - k\alpha_n\beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2.$$

It follows from the boundedness of the C that $\|Tx_n - Sx_n\|$ is bounded . Thus

$$\lim_{n\to\infty} k\alpha_n\beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2 = 0.$$

It follows from Lemma 1.3 that

$$\lim_n Sx_n = p.$$

Therefore $\{Sx_n\}_{n=0}^{\infty}$ is strongly converges to p.
Remark: If we put $S = I$ and $T = S$ in Theorem 2.1 we get the Theorem of paper Qihou[13] and if replace $S = I$ and $T = S$ in Theorem 2.2 we get the Theorem of R. A. Rashwan and A. M. Saddeek[14].

3. Application

Solution of the equation 2x−cosx−3=0.

To solve this equation, we rearrange it as $Sx = Tx$, with S, T defined by $Sx = x$ and $Tx = \frac{cosx+3}{2}$

$x_0 = 1.5$ and $\alpha_n = \beta_n = \frac{1}{\sqrt{1+n}}$. For diagram and data we are using MATLAB programing

Table 1: Jungck Mann iteration

No of iteration	Sx_{n+1}	Tx_n
1	1.5250	1.5354
2	1.5238	1.5229
3	1.5236	1.5235
4	1.5236	1.5236
5	1.5236	1.5236
6	1.5236	1.5236
7	1.5236	1.5236
8	1.5236	1.5236
9	1.5236	1.5236
10	1.5236	1.5236

Table 2: Jungck ishikawa iteration

No of iteration	Sy_n	Sx_{n+1}
1	1.5250	1.5162
2	1.5226	1.5207
3	1.5229	1.5223
4	1.5232	1.5230
5	1.5234	1.5233
6	1.5235	1.5234
7	1.5235	1.5235
8	1.5235	1.5235
9	1.5236	1.5236
10	1.5236	1.5236

4 Observation

We observe that Jungck Mann iteration scheme converges faster than Jungck Ishikawa iteration scheme

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