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A MATRIX ITERATIVE TECHNIQUE FOR THE SOLUTION OF FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

E. S. SHOUKRALLA, S. A. EL-SERAFI AND NERMEIN A. SABER

Abstract. A Matrix Iterative Algorithm is given for the approximate solution of Fredholm integral equations of the second kind. The Algorithm modifies the ideas of iterated kernels via Hilbert matrix. Thus reducing the required solution so that only one coefficient matrix is computed. Concluded results are observed during the solution of some numerical examples.

1. Introduction

Integral equation is encountered in a variety of applications in potential theory, geophysics, electricity and magnetism, radiation, and control systems [1]. Many methods of solving Fredholm integral equation of the second kind have been developed in recent years [1, 3, 6, 7, 12], such as quadrature method, collocation method and Galerkin method, expansion method, product-integration method, deferred correction method, graded mesh method, and Petrov-Galerkin method. In addition, the iterated kernel method is a Traditional method for solving the second kind. However, this method also requires a huge size of calculations. The objective of this paper is to establish a promising iterative algorithm that can be easily programmed.

The given iterative Algorithm is presented for the approximate solution of integral equations. Consideration is limited to linear non -homogeneous Fredholm integral equations of the second kind. The given procedure beginning by replacing the kernel of an integral equation approximately by a degenerate kernel [4, 5, 8] in a matrix form using Maclaurin polynomial of degree **n**, whereas the data function is approximated by Maclaurin polynomial of the same degree **n** [2]. Owing to the simplicity of some operational matrix of integration, the iterated kernels is represented in a very simple form via Hilbert matrix. This simplifies the present iterative Algorithm and reduces the problem of computing iterative solutions to the computation of only one matrix.

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Despite of the advantages of methods [4, 5] there was apparent higher cost comparing with the present method, that minimizes the computational effort and smooth the round-off errors out.

Due to the simple form of the obtained iterated kernels, which is straightforward and convenient for computation, The present method may be generalized to solve both second kind and well-posed singular integral equations of the first kind [9, 10].

2. Iterative Algorithm

Consider the Fredholm integral equation of the second kind

$$\phi(x) = \int_{\alpha}^{\beta} K(x,y)\phi(y)dy + f(x) \quad \forall q \geq 1, \tag{1}$$

where the function $f(x)$ and the kernel $K(x,y)$ are given. The kernel $K(x,y)$ is defined in the square $\Omega = \{\alpha \leq x \leq \beta, \alpha \leq y \leq \beta\}$ in the **xy**-plane. The function $\phi(x)$ is the unknown required solution.

The given iterated Algorithm starts with an initial approximation $\phi^{(0)}(x)$ to the solution $\phi(x)$ of integral equation 1 and then generates a sequence of solutions $\{\phi^{(q)}(x)\}_{q=0}^{\infty}$ that converges to $\phi(x)$ such that

$$\|\tfrac{\phi^{(q)}(x)-\phi^{(q-1)}(x)}{\phi^{(q)}(x)}\| < \delta : \delta > 0$$

After the initial solution $\phi^{(0)}(x)$ is chosen, the sequences of approximate solutions can be generated by computing

$$\phi^{(q)}(x) = f(x) + \int_{\alpha}^{\beta} K(x,y)\phi^{(q-1)}(x)dy \quad \forall q \geq 1. \tag{2}$$

If we begin by $\phi^{(0)}(x) = 0$ then we get

$$\phi^{(q)}(x) = f(x) + \sum_{s=1}^{q} \int_{\alpha}^{\beta} K^{(s)}(x,y)f(x)dy \quad \forall q \geq 1, \tag{3}$$

where the iterated kernels $K^{(s)}(x,y)$ can be found by the recurrence form

$$K^{(s)}(x,y) = \int_{\alpha}^{\beta} K(x,z)K^{(s-1)}(z,y)dz; \quad s \geq 2 \quad \forall q \geq 1, \tag{4}$$

where $K^{(1)}(x,y) = K(x,y)$. If the given kernel $K(x,y)$ can be approximated using Maclaurin polynomial of degree $\mathbf{n}$, then we can put it in the matrix form

$$K(x,y) = P^t(y)L(n)KL(n)P(x), \tag{5}$$

where $K = (k_{ij})$ is an $(n+1) \times (n+1)$ coefficients matrix whose entries are defined by

$$k_{ij}=\begin{cases}\frac{\partial^i\partial^i K(x,y)}{\partial x^i\partial x^j}|_{(x,y)=(0,0)} & i+j\leq n\\ 0 & i+j>n\end{cases} \tag{6}$$

and the matrix $L(n)$ is an $(n+1)\times(n+1)$ matrix defined by

$$L(n)=diag(\frac{1}{0!}\quad \frac{1}{1!}\quad ..\quad \frac{1}{n!}) \tag{7}$$

The matrix $P(x)$ of order $(n+1)\times(n+1)$ is defined to be

$$P^t(x)=[p_0(x)\quad p_1(x)\quad ..\quad p_n(x)] \tag{8}$$

where $p_i(x)=x^i \quad for \quad i=\overline{0;n}$.

Now, we define the Maclaurin operational integration matrix ,H, to be

$$H=\int_\alpha^\beta P(y)P^t(y)dy=\beta B(\beta)HB(\beta)-\alpha B(\alpha)HB(\alpha) \tag{9}$$

where

$$B(\alpha)=diag(1\quad \alpha\quad \alpha^2\quad ...\alpha^n)\Rightarrow B(\beta)=diag(1\quad \beta\quad \beta^2\quad ...\beta^n) \tag{10}$$

Here **H** is the well - known Hilbert matrix of order $(n+1)\times(n+1)$ with elements $H_{ij}=(i+j-1)^{-1}$. Substituting 5 into 4 and by virtue of 9 the iterated kernels $K^{(s)}(x,y)$ become

$$K^{(s)}(x,y)=P^t(x)[(L(n)KL(n))^tH]^{(s-1)}(L(n)KL(n))^tP(y) \tag{11}$$

Also, approximating the data function $f(x)$ in Maclourin polynomial of degree **n** yields

$$f(x)=P^t(x)F \quad ; \quad F^t=[f_0\quad f_1\quad ...\quad f_n] \tag{12}$$

where $f_i=\frac{1}{i!}\{\frac{d^if(x)}{dx^i}\}_{x=0};\quad i=\overline{0;n}$.

Now, Substituting 11,12 into 3 we find that

$$\phi^{(q)}(x)=P^t(x)F+\sum_{s=1}^q\int_\alpha^\beta P^t(x)[\tilde{K}^tH]^{s-1}\tilde{K}^tP(y)[P^t(y)F]dy \tag{13}$$

Again, using the operational matrix given by 9, we get

$$\phi^{(q)}(x)=P^t(x)[I_{n+1}+\sum_{s=1}^q[\tilde{K}^tH]^s]F_{\forall q\geq 1} \tag{14}$$

where $\tilde{K}=L(n)KL(n)$.

Let $Q^{(q)}$ is the $(n+1)\times 1$ iterative coefficients vector defined by

$$Q^{(q)}(a_{ij})=[I_{n+1}+\sum_{s=1}^q[\tilde{K}H]^s] \tag{15}$$

then we have $\phi^{(q)}(x)=P^t(x)Q^{(q)}F$

If $C=\tilde{K}H$,then the approximate solution $\phi^{(q)}(x)$ converges to the exact $\phi(x)$ if one of the following three conditions is satisfied

$$\|C\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |C_{ij}| < 1$$
$$\|C\|_\infty = \max_{1\le j\le n} \sum_{i=1}^{n} |C_{ij}| < 1$$
$$|C\|_2 = (\rho(CC^t))^2 < 1.$$

Where $\rho(CC^t)$ is the spectral radius of CC^t.

That is, to find the iterative solutions $\phi^{(q)}(x)$ of integral equation 1 it is required only to compute the matrix K given by 6.

3. Computational Results

Example 1

Consider the Integral Equation

$$\phi(x) = 1 + \int_0^1 xy^2\phi(y)dy$$

whose exact solution [11] is given by $\phi(x) = 1 + \frac{4}{9}x$.
Take **n=3**, then we get

$$\tilde{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and since **H** is 4×4 anf $F^t = [1 \quad 0 \quad 0 \quad 0]$, then we get table (1). From which it is observed that the 7 th iteration, $\phi^{(7)}(x) = 1 + 0.4444x$, is identical to the exact solution.

Example 2

Consider the Integral Equation

$$\phi(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 (xy)\phi(y)dy$$

whose exact solution [11] is given by $\phi(x) = x$.
Take **n=2**, then we get

$$\tilde{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

and since **H** is 3×3 and $F^t = [0 \quad \frac{5}{6} \quad 0]$ then we get table (2), from which it is observed that the fifth iteration , $\phi^{(5)}(x) = x$, is identical to the exact solution.

Table (1): represent the coefficient matrix $Q(q)$ given by equation 15 and a_{ij} is the ij-entry of the $Q(q)$ matrix of example (1)

a_{ij}	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$	$Q^{(5)}$	$Q^{(6)}$	$Q^{(7)}$
a_{11}	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
a_{12}	0.3333	0.4167	0.4375	0.4427	0.4440	0.4443	0.4444
a_{13}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
a_{14}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table (2): represent the coefficient matrix $Q(q)$ given by equation 15 and a_{ij} is the ij-entry of the $Q(q)$ matrix of example (2)

a_{ij}	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$	$Q^{(5)}$
a_{11}	0.0000	0.0000	0.0000	0.0000	0.0000
a_{12}	0.9722	0.9954	0.9992	0.9999	1.0000
a_{13}	0.0000	0.0000	0.0000	0.0000	0.0000

4. Conclusion

A simple Iterative Algorithm for the solution of Fredholm Integral Equations of the second kind has been presented. The given method gives a very simple form for the iterated kernels via the well - known Hilbert matrix. Thus, the iterative solutions of an integral equation of the second kind can be reduced to the solution of a matrix equation, whereas only one coefficient matrix is required to be computed. Therefore, computational complexity can be considerably reduced and much computational time can be saved. The new proposed approach needs a small number of iterations to provide an exact result, that proofs the power of the presented Algorithm, and stimulates to find out the relation between the integral equations and Hilbert Matrix.

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Shoukralla, E. S.

Dept. Math. Faculty of Eng. Future University, Egypt

E-mail address: shokralh@fue.edu.eg, shoukrala@hotmail.com

EL-Serafi, S. A., Nermein A. Saber

Dept Math. Faculty od Elec. Eng. Menofia University, Egypt

E-mail address: engnermeinn@hotmail.com