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THE ORDINARY DIFFERENTIAL OPERATOR AND SOME OF ITS APPLICATIONS TO P-VALENTLY ANALYTIC FUNCTIONS

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Abstract. In this paper, several applications of the well-known ordinary differential operator to certain p-valent functions are first determined and some of useful results of them, which are the results relating to both certain complex equations, complex inequalities and $p-$valently analytic functions, are then pointed out.

1. Introduction, definitions, notations and motivation

As is known from the literature, an ordinary differential equation is a equation consisting of only one (independent) variable and also one or more of derivatives of another (dependent) variable. Of course, these can be either real or complex variables. Let z and w be independent complex variable and dependent complex variable, respectively. Usually, an equation of z and $w(= w(z))$ in the form:

$$\sum_{j=0}^{m} \sum_{l=0}^{k} \sum_{i=0}^{n} \psi_i(z)\ w^{(i)}(z)\ \big[w(z)\big]^j \big[w^{(i)}(z)\big]^l \ = \ \phi(z) \tag{1}$$

is known as (linear or non-linear differential) equation with complex variable of order n, *where* $\psi_i(z)$ and $\phi(z)$ are continous functions in the certain domains of the complex plane and also n, m and k are positive integers. Of course, the function $w(z)$ is a differentiable function in any domain of complex plane. Addition, in (1), when $\phi(z) = 0$ and $\phi(z) \neq 0$, it can be received two non-linear equations which, in the literature, are also known as homogeneous and non-homogeneous differential equation with complex variable, respectively.

As is well known, ordinary differential equations are used in many different fields of sciences and, occasionally, the solutions of them will be very important for each one of their fields. In this work, without getting to find any solutions of the certain types of the (non-linear *or* linear) differential equations given by (1), some useful

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relationships between those differential equations with complex variable and certain $p-$valently analytic functions are only revealed. For these, we need to introduce certain information which are below.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of positive integers, $\mathbb{C}$ be the set of complex numbers, $\mathbb{U} := \big\{z \in \mathbb{C} : |z| < 1\big\}$ be the open unit disk and let $\mathcal{P}_n$ denote the family of the functions of the following form:

$$f(z) = 1 + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \quad \big(n \in \mathbb{N}\big)$$

which are anlytic in $\mathbb{U}$ and also let $\mathcal{T}_n(p)$ denote the family consisting of functions $f(z)$ in the form:

$$f(z) = z^p + a_{n+p}z^{n+p} + a_{n+p+1}z^{n+p+1} + \cdots \quad \big(n, p \in \mathbb{N}\big)$$

which are analytic and p-valent in $\mathbb{U}$.

To prove the main results relating to certain order (differential) equations with one complex variable and certain $p-$valently analytic functions, the following well-known result which is Lemma 1 below (cf., [6] and see also [7]) will be needed.

Lemma 1. Let $w(z)$ be non-constant and regular function in $\mathbb{U}$ with $w(0) = 0$. If there exists a point z_0 on the circle:

$$\Big\{z \in \mathbb{C} : |z| = r \quad \big(0 < r < 1\big)\Big\}$$

such that

$$|w(z_0)| = \max\Big\{|w(z)| : |z| \leq |z_0| \text{ and } z \in \mathbb{U}\Big\},$$

then

$$\frac{zw'(z)}{w(z)}\bigg|_{z=z_0} = c \quad \big(1 \leq c < \infty\big) \tag{2}$$

and

$$\Re e\left(1 + \frac{zw''(z)}{w'(z)}\bigg|_{z=z_0}\right) \geq c \quad \big(1 \leq c < \infty\big). \tag{3}$$

2. The main results and certain consequences

In this section, there is a need to introduce some definitions and also remind some information for the main results. For those, firstly, we shall look over the following.

Before stating and then proving the main results, it needs first to articulate an operator, which is qth order ordinary differential operator, in the form:

$$\mathcal{D}_z^q\{f\} := \mathcal{D}_z^q\{f(z)\} = \left(\frac{p!}{(p-q)!}z^p + \sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}a_k z^k\right)z^{-q}, \tag{4}$$

where $p \in \mathbb{N}$, $n \in \mathbb{N}$, $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $p > q$ and $f(z) \in \mathcal{T}_n(p)$.

Recently, by using certain (differential) operators which are similar to the definition given by (4), some researchers focused on the functions $f(z)$ in $\mathcal{T}_n(p)$ and obtained some useful results of those functions satisfying the following inequalities:

$$\Re e\left(\frac{z\mathcal{D}_z^{1+q}\{f\}}{\mathcal{D}_z^q\{f\}}\right) > \alpha \quad \Big(q \in \mathbb{N}; \mathcal{D}_z^q\{f\} \neq 0\Big) \tag{5}$$

and

$$\Re e\left(\frac{\mathcal{D}_z^q\{f\}}{z^{p-q}}\right) > \alpha, \tag{6}$$

where $0 \leq \alpha < p-q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $p > q$ and $z \in \mathbb{U}$. Clearly, the above rational type functions, expressed by (5) and (6), are both p-valently analytic in $\mathbb{U}$ and $p-$valently meromorphic in $\mathbb{U}-\{0\}$. Indeed, in statements like (5) and (6), in which the functions quotients involved have removable singularities at the point $z=0$, it is tacitly assumed that such function quotients have had these singularities removed. Thereby, the investigations concerning the inequalities in (5) and also (6) are quite important for their analytical and geometrical properties. For their details, See [1], [2] and [8]. Furthermore, for some of the results related to the similar definitions as above, the papers in [3], [4] and also [5] can be also checked. Particularly, as it is indicated just above, the specific types of the related inequalities play a significant role in univalent function theory. Indeed, a function $f(z) \in \mathcal{T}_n(p)$ satisfying the inequality (5) together with the case $q := 0$ is known $p-$valently starlike function of order α $(0 \leq \alpha < p; p \in \mathbb{N})$ (*or* $p-$valently starlike function wrt origin *when* $\alpha := 0$) in $\mathbb{U}$, a function $f(z) \in \mathcal{T}_n(p)$ satisfying the inequality (5) along with the case $q := 1$ is known $p-$valently convex function of order α $(0 \leq \alpha < p; p \in \mathbb{N})$ (*or* $p-$valently convex function wrt origin *when* $\alpha := 0$) in $\mathbb{U}$, a function $f(z) \in \mathcal{T}_n(p)$ satisfying the inequality (6) together with the case $q := 0$ is known $p-$valently colose-to-starlike function of order α $(0 \leq \alpha < p; p \in \mathbb{N})$ (*or* $p-$valently close-to-starlike wrt origin *when* $\alpha := 0$) in $\mathbb{U}$ and also a function $f(z) \in \mathcal{T}_n(p)$ satisfying the inequality (5) along with the case $q := 1$ is known $p-$valently close-to-convex function of order α $(0 \leq \alpha < p; p \in \mathbb{N})$ (*or* $p-$valently close-to-convex function wrt origin *when* $\alpha := 0$) in $\mathbb{U}$. For both their details and also certain special results of them, one can look over the works given by [1], [2] and (also) [8].

We can now begin to present and then prove our main results. The first is in the following form.

Theorem 1. Let the function $\psi(z) \in \mathcal{P}_n$ satisfy the inequality

$$2\,\Re e\big\{\psi(z)\big\} < 1 \qquad \big(z \in \mathbb{U}\big) \tag{7}$$

and also let the function $\omega(z) \in \mathcal{T}_n(p)$ be a solution for the following complex equation:

$$z\,\mathcal{D}_z^q\{\omega(z)\}\mathcal{D}_z^{2+q}\{\omega(z)\} - \mathcal{D}_z^{1+q}\{\omega(z)\}\Big[z\mathcal{D}_z^{1+q}\{\omega(z)\} + \Big(\psi(z)-1\Big)\mathcal{D}_z^q\{\omega(z)\}\Big] = 0. \tag{8}$$

Then, the inequality:

$$\Big|z\,\mathcal{D}_z^{1+q}\{\omega(z)\} - (p-q)\mathcal{D}_z^q\{\omega(z)\}\Big| < (p-q)\Big|\mathcal{D}_z^q\{\omega(z)\}\Big| \tag{9}$$

holds, *where* $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Proof. Let the function $\omega := \omega(z) \in \mathcal{T}_n(p)$ be the solution of the complex equation given by (8). In view of the definition in (4), define a function $w(z)$ as

$$\frac{z\mathcal{D}_z^{1+q}\{\omega\}}{\mathcal{D}_z^q\{\omega\}} = (p-q)\big[1 + w(z)\big] \tag{10}$$

where $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $z \in \mathbb{U}$ and, of course, $\mathcal{D}_z^q\{\omega\} \neq 0$ for all $q \in \mathbb{N}$. Clearly, the function $w(z)$ both belongs to $\mathcal{T}_n(1)$ and is analytic in $\mathbb{U}$ with $w(0) = 0$. Then, the statement (10) gives us

$$\frac{z\left(\frac{z\mathcal{D}_z^{1+q}\{\omega\}}{\mathcal{D}_z^q\{\omega\}}\right)'}{\frac{z\mathcal{D}_z^{1+q}\{\omega\}}{\mathcal{D}_z^q\{\omega\}}} = \frac{zw'(z)}{1+w(z)} \qquad \big(z \in \mathbb{U}\big)$$

or, equivalently,

$$\frac{zw'(z)}{1+w(z)} = 1 + z\left(\frac{\mathcal{D}_z^{2+q}\{\omega\}}{\mathcal{D}_z^{1+q}\{\omega\}} - \frac{\mathcal{D}_z^{1+q}\{\omega\}}{\mathcal{D}_z^q\{\omega\}}\right) \quad \big(q \in \mathbb{N}_0; z \in \mathbb{U}\big). \tag{11}$$

Clearly, the identity in (11) shows that it is equal to the function $\psi(z)$ belonging to $\mathcal{P}_n$ and also an unique solution for the complex equation given by (8).

Suppose now that there exists a point $z_0 \in \mathbb{U}$ such that

$$\big|w(z_0)\big| = \max\Big\{\big|w(z)\big| \ : \ |z| \leq |z_0| \ \ \big(z \in \mathbb{U}\big)\Big\} = 1.$$

By means of Lemma 1, using the assertion (2), we then obtain from (11) that

$$\Re e\Big(\psi(z)\big|_{z=z_0}\Big) = \Re e\left(\frac{zw'(z)}{1+w(z)}\bigg|_{z=z_0}\right) = \frac{c}{2} \geq \frac{1}{2} \quad \big(since\ c \geq 1\big),$$

which obviously contradicts the assumption in (7) of Theorem 2.1. Hence, $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus, the equality in (10) immediately yields that

$$\left|\frac{z\mathcal{D}_z^{1+q}\{\omega\}}{\mathcal{D}_z^q\{\omega\}} - (p-q)\right| = (p-q)|w(z)| < p-q$$

$$\big(p > q\ ;\ p \in \mathbb{N}\ ;\ q \in \mathbb{N}_0\ ;\ z \in \mathbb{U}\big),$$

which is the inequality given by (9). This completes the desired proof.

Theorem 2. Let the function $\psi(z) \in \mathcal{P}_n$ satisfy the inequality in (7) and also let the function $\omega(z) \in \mathcal{T}_n(p)$ be the solution for the complex equation:

$$z\,\mathcal{D}_z^{1+q}\{\omega(z)\} - \big(\psi(z) + p - q\big)\mathcal{D}_z^q\{\omega(z)\} = 0. \tag{12}$$

Then, the inequality:

$$\Big|\mathcal{D}_z^q\{\omega(z)\} - p\,z^{p-q}\Big| < p\,|z|^{p-q}$$

is true, *where* $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Proof. Let the function $\omega := \omega(z) \in \mathcal{T}_n(p)$ be the solution of the complex equation given by (12). By the help of the definition of the operator in (4) and the (assertion (2) or assertion (3) of) Lemma 1, if one defines a function $w(z)$ as

$$\frac{\mathcal{D}_z^q\{\omega\}}{z^{p-q}} = \frac{p!}{(p-q)!}\big[1 + w(z)\big]$$

$$\big(p > q\ ;\ p \in \mathbb{N}\ ;\ q \in \mathbb{N}_0\ ;\ z \in \mathbb{U}\big),$$

and then follows the same steps used in the proof of Theorem 1, it can be easily proven. The detail is here omitted.

Despite the main results include several interesting and important results relating to certain equations and complex functions, it is only just to present some of them, which are associated with certain relations between certain types of the (non-linear (or linear) and (or) homogenous) certain order differential equations with complex variable and $p-$valently analytic functions in the domain $\mathbb{U}$. All right, it is impossible to determine all of them. Some of these are here omitted and the others appertaining to the information given at the beginning of this section are also given.

The following corollary (Corollary 1 below) is the first consequence consisting of $p-$valently starlikeness of a function $\omega(z) \in \mathcal{T}_n(p)$ can be easily determined by setting $q := 0$ in Theorem 1.

Corollary 1. Let the function $\psi(z) \in \mathcal{P}_n$ satisfy the inequality in (7) and also let the function $\omega(z) \in \mathcal{T}_n(p)$ be a solution for the complex differential equation:

$$z\omega(z)\omega''(z) - \omega'(z)\Big[z\omega'(z) + \Big(\psi(z) - 1\Big)\omega(z)\Big] = 0.$$

Then, the inequality:

$$\Big|z\omega'(z) - p\omega(z)\Big| < p\Big|\omega(z)\Big| \quad \big(z \in \mathbb{U}\big)$$

is true and it yields the inequality (5) when $f(z) := \omega(z)$, $\alpha := 0$ and $q := 0$, which implies that $\omega(z) \in \mathcal{T}_n(p)$ is a $p-$valently starlike function (wrt origin) in $\mathbb{U}$.

The following corollary (below) is the second consequence concerning $p-$valently convexity of a function $\omega(z) \in \mathcal{T}_n(p)$ can be next derived by letting $q := 1$ in Theorem 1.

Corollary 2. Let the function $\psi(z) \in \mathcal{P}_n$ satisfy the inequality in (4) and also let the function $\omega(z) \in \mathcal{T}_n(p)$ be the solution for the complex differential equation:

$$z\omega'(z)\omega'''(z) - \omega''(z)\Big[z\omega'(z) + \Big(\psi(z) - 1\Big)\omega'(z)\Big] = 0.$$

Then, the following inequality

$$\Big|z\omega''(z) - (p-1)\omega'(z)\Big| < p\Big|\omega'(z)\Big| \quad \big(z \in \mathbb{U}\big)$$

holds true and it yields the inequality (5) with $f(z) := \omega(z)$, $\alpha := 0$ and $q := 1$, which implies that $\omega(z) \in \mathcal{T}_n(p)$ is a $p-$valently convex function (wrt origin) in $\mathbb{U}$.

The following result (Corollary 3 below) is the third consequence including $p-$valently close-to-starlikeness of a function $\omega(z) \in \mathcal{T}_n(p)$ can be then determined by putting $q := 0$ in Theorem 2.

Corollary 3. Let the function $\psi(z) \in \mathcal{P}_n$ satisfy the inequality in (4) and also let the function $\omega(z) \in \mathcal{T}_n(p)$ be the solution for the complex differential equation:

$$z\omega'(z) - \big(p + \psi(z)\big)\omega(z) = 0.$$

Then, the inequality:

$$\Big|\omega(z)z^{-p} - 1\Big| < p \quad \big(z \in \mathbb{U}\big)$$

is true and it yields the inequality (6) with $f(z) := \omega(z)$, $\alpha := 0$ and $q := 0$, which implies that $\omega(z) \in \mathcal{T}_n(p)$ is a $p-$valently close-to-starlike function (wrt origin) in $\mathbb{U}$.

The following corollary (below) is the last consequence involving $p-$valently close-to-convexity of a function $\omega(z) \in \mathcal{T}_n(p)$ can be also determined by setting $q := 1$ in Theorem 2.

Corollary 4. Let the function $\psi(z) \in \mathcal{P}_n$ satisfy the inequality in (4) and also let the function $\omega(z) \in \mathcal{T}_n(p)$ be the solution for the complex differential equation:

$$z\omega''(z) - \big(p + \psi(z) - 1\big)\omega'(z) = 0.$$

Then, the following inequality

$$\left|\omega'(z)z^{1-p} - p\right| < p \quad \big(p \in \mathbb{N}; z \in \mathbb{U}\big)$$

holds true and it yields the inequality (6) with $f(z) := \omega(z)$, $\alpha := 0$ and $q := 1$, which implies that $\omega(z) \in \mathcal{T}_n(p)$ is a $p-$valently close-to-convex function (wrt origin) in $\mathbb{U}$.

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